2020-12-1 Kähler geometry (Taxiong Liu)

$(f) = \sum d_E E. \leftarrow$ div. $d_E > 0$ if $E \subseteq \mathrm{zero}(f)$

$\mathrm{ord}_E f = d_E$

- Functions in $\mathcal{H}^{NA}$ can alternatively be described in terms of $\mathbb{C}^*$-inv. ideals on $X \times \mathbb{C}$ (called flag ideals in [Oda 13])

Each such ideal is of the form $\mathfrak{a} = \sum_{i=0}^{r} \tau^i \mathfrak{a}_i$ for a seq. of ideals $\mathfrak{a}_0 \subset \cdots \subseteq \mathfrak{a}_r \subseteq \mathcal{O}_X$ on X. define

$$\varphi_{\mathfrak{a}}(v) := -\sigma(v)(\mathfrak{a}) = \max_i \{-v(\mathfrak{a}_i) - i\}$$

($\forall f \in \mathfrak{a}$, $f = \sum_{i=0}^{r} f_i \tau^i$, $a_\lambda \in \mathbb{C}^*$. $\lambda = 0, \cdots r$

$a_\lambda \cdot f = \sum_{i=0}^{r} a_\lambda^{-i} f_i \tau^i$. take distinct a_λ.

$\Rightarrow f_i \tau^i = \sum_\lambda a_\lambda f \in \mathfrak{a}$ since the coeff. of system is Vandermonde matrix

$\mathfrak{a}_i = \{ f \in \mathcal{O}_X \mid f\tau^i \in \mathfrak{a} \} \subseteq \mathfrak{a}_{i+1}$)

*. $\varphi : X^{\mathrm{div}}_{\mathbb{Q}} \longrightarrow \mathbb{R}$ belong to $\mathcal{H}^{NA}$ iff. $\varphi = m^{-1} \varphi_{\mathfrak{a}} + c$. with $c \in \mathbb{Q}$, $m \in \mathbb{N}^*$ and $\mathfrak{a}$ is a $\mathbb{C}^*$-inv. ideal on $X \times \mathbb{C}$ cosupport on $X \times \{0\}$ ($\mathcal{O}_{X,x}/\mathfrak{a}_x \neq 0$ $x \in X \times \{0\}$)

and s.t. the sheaf $p_1^*(mL)\otimes\mathfrak{a}$ is globally generated on $X\times\mathbb{C}$

$$\mathfrak{a} \rightsquigarrow \mathfrak{X}_{\mathfrak{a}} = \widehat{Bl_{\mathfrak{a}}(X\times\mathbb{C})} \xrightarrow{\mu} X\times\mathbb{C} \to \mathbb{C}$$

normalization of Blow-up.

$$\mathcal{L}_{\mathfrak{a}} = \mu^* p_1^*(mL) - E \qquad E \text{ exceptional divisor}$$

$(\mathfrak{X},\mathcal{L})$. semiample T.C.

$$\begin{tikzcd} & \mathfrak{X}' \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ \mathfrak{X} & & X\times\mathbb{C} \end{tikzcd}$$

s.t. $\rho^*\mathcal{L} = \pi^* p_1^* L + D$.

$\rightsquigarrow \mathfrak{a}_{(m)} := \pi_*\mathcal{O}_{\mathfrak{X}'}(mD)$ is flag ideal

(c.f. section 2.6 of [BHJ].17).

<u>Lemma 3.4</u> $\forall \varphi \in \mathcal{H}^{NA}$ is bound, with $\sup_{X^{div}_{\mathbb{Q}}} \varphi = \varphi(v_{triv})$

(c.f. lemma 7.7 of [BHJ.17])

· <u>non-Archimedean functional</u>

$\forall \varphi = \varphi_{(\mathfrak{X},\mathcal{L})} \in \mathcal{H}^{NA}$, $(\overline{\mathfrak{X}},\overline{\mathcal{L}})$ is the compactification

$$E^{NA}(\varphi) := \frac{(\overline{\mathcal{L}}^{n+1})}{(n+1)V} \qquad V = \int_X \alpha^n$$

$$J^{NA}(\varphi) := \sup \varphi - E^{NA}(\varphi)$$

$$L_\theta^{NA}(\varphi) := \inf_{X^{div}_{\mathbb{Q}}} (A_\theta + \varphi), \quad D_\theta^{NA}(\varphi) = L_\theta^{NA}(\varphi) - E^{NA}(\varphi)$$

<u>Def</u>: (X, L) is Ding semistable (resp. uniformly Ding-stable) w.r.t. θ if $D_\theta^{NA} \geq 0$ on $\mathcal{H}^{NA}$.

(resp. $D_\theta^{NA} \geq \varepsilon J^{NA}$ on $\mathcal{H}^{NA}$, for $\varepsilon > 0$)

when $\theta = [\Delta]$ (X, Δ) klt

$$M_\Delta^{NA}(\phi) := DF_\Delta(\phi) + V^{-1}\big((\underbrace{\mathcal{X}_{0,red}}_{\sum_E E} - \underbrace{\mathcal{X}_0}_{\sum b_E E}) \cdot \mathcal{L}^n\big)$$

$$DF_\Delta(\phi) \underset{\Uparrow}{=} V^{-1}\big(K_{(\bar{\mathcal{X}}, \bar{\Delta})/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n\big) + \bar{S}_\Delta V^{-1} \frac{(\bar{\mathcal{L}}^{n+1})}{n+1}$$

Odaka - Xiaowei Wang intersection formula

<u>Def - Thm</u>

$((X, \Delta), L)$ is uniformly K-stable if $DF_\Delta \geq \delta J^{NA}$ on $\mathcal{H}^{NA}$ for some $\delta > 0$

$$\Longleftrightarrow M_\Delta^{NA} \geq \delta J^{NA} \text{ on } \mathcal{H}^{NA}$$

(c.f. <u>[BHJ]</u>)

Thm 3.7 ([BBJ.15 v1], Fujita 16)

(X,Δ) log Fano pair. $L = -K_X$. Then

(X,Δ) is Ding semistable $\Longleftrightarrow$ (X,Δ) is K-semistable

uniformly Ding stable $\Longleftrightarrow$ uniform K-stable

$\Delta = 0$. due to [BBJ.15.v1]

general case due to Fujita.

IV. Psh rays and Lelong number

Goal: $\varphi \in \mathcal{H}^{NA}$ $\longleftrightarrow$ psh ray with algebraic sing.

· Rays of linear growth

$U : \mathbb{R}_{\geq 0} \to PSH$, $\sup_X U_t$ is convex in t.

$$\lambda_{max} := \lim_{t \to \infty} t^{-1} \sup_X U_t \in \mathbb{R} \cup \{+\infty\}$$

$\lambda_{max} < +\infty$ iff. $\sup_X U_t = O(t)$, called U has linear growth

Prop 4.1. $U : \mathbb{R}_{\geq 0} \to \mathcal{E}^1$ has linear growth iff $d_1(U_t, 0) = O(t)$ as $t \to +\infty$. In particular, psh geodesic ray has linear growth.

Pf. By Prop 1.5, $E(u_t)$ is convex, hence admits a linear bound $E(u_t) \geq -Ct$ for $t \geq 1$.

Assume $\sup_X u_t = O(t)$, Pick $a > 0$ s.t. $u_t \leq at$

$d_1(u_t, at) = at - E(u_t) \leq C't$. and $d_1(at, 0) = at$

$(d_1(u, 0) \sim J(u))$ by the triangle ineq.

$$d_1(u_t, 0) \leq d_1(u_t, at) + d_1(at, 0) = O(t)$$

Assume $d_1(u_t, 0) = O(t)$.

$$\frac{1}{V}\int_X u_t \omega^n \leq \sup_X u_t \leq \frac{1}{V}\int_X u_t \omega^n + C$$

$$\Rightarrow \sup_X u_t = V^{-1}\int_X u_t \omega^n + O(1).$$

by $\left|\int_X u_t \omega^n\right| \leq C_n d_1(u_t, 0)$ $\square$

· <u>Lelong number.</u>

Psh ray u with linear growth, $u - at$ is bdd above as $t \to +\infty$ for $a \in \mathbb{R}$. Equivalently. S^1-inv. $p_1^* \omega_0$-psh function V on $X \times \mathbb{D}^*$. $V(x, \tau) := u_{-\log|\tau|}(x) + a \log|\tau|$

$t = -\log|\tau|$

is bounded above near $X \times \{0\}$

$\Rightarrow$ uniquely extends to a quasi-psh function on $X \times \mathbb{D}$.

for any divisorial valuation w on $X \times \mathbb{C}$, $w(V) > 0$ make sense as a generic Lelong number on a suitable blow up

$$w(U) := w(V) - a\,w(\tau)$$

this is indep. of the choice of a.

<u>Def 4.2</u>. Psh ray U. with linear growth, define

$$U_{NA} : X^{div}_{\mathbb{Q}} \longrightarrow \mathbb{R}.$$

$$U_{NA}(v) := -\sigma(v)(U)$$

<u>lemma 4.3</u>. $U_{NA}(v+tw) = \sup_{X^{div}_{\mathbb{Q}}} U_{NA} = \lambda_{max}$

· <u>Algebraic singularities</u>

choose a sm. hermitian metric h_0 on L. s.t. $Ric(h_0) = \omega_0$

$$\text{Psh ray } U : \mathbb{R}_{\geq 0} \longrightarrow PSH \overset{1:1}{\longleftrightarrow} S^1\text{-inv. psh. metrics}$$

$$e^{-2U} p_1^* h_0 \text{ on } (X \times \mathbb{D}^*, p_1^* L)$$

we say that <u>U induces a psh metric</u> on normal TC $(\mathcal{X}, \mathcal{L})$

if the corresponding psh metric on $(X\times\mathbb{D}^*, p_1^*L) \cong (\mathcal{X},\mathcal{L})|_{\mathbb{D}^*}$ extends to a psh metric on $(\mathcal{X},\mathcal{L})|_{\mathbb{D}}$

psh function on cpx space Y

restriction from ambient space.

locally, $Y \underset{\text{analytic}}{\subseteq} \Omega \overset{\text{open}}{\subseteq} \mathbb{C}^N$, $u = \varphi|_X$.

φ is psh on Ω

($\phi: X \to [-\infty, +\infty)$ is psh iff. $\forall$ hol map $f: D \to X$ s.t.

$u\circ f$ is subharmonic

or $u\circ f = -\infty$

- u is u.s.c.)

<u>Thm</u> (Demailly. Thm 1.7. Mesures de Monge-Ampère et caractérisction géométrique des variétés algébrique affines)

X locally irred, cpx space, $Y \subseteq X$ analytic subset in X

Let V be a psh on $X\setminus Y$, which is locally bdd above near Y.

So there is a psh V^* on X. which extends V uniques.

<u>lemma 4.4</u>. A psh way $u: \mathbb{R}_{\geq 0} \to \mathrm{PSH}$ and a normal T.C,

$(\mathcal{X}, \mathcal{L})$. TFAF:

(i). U induces a psh metric on $(\mathcal{X}, \mathcal{L})$

(ii). U has linear growth and $U_{NA} \leq \mathcal{G}_{(\mathcal{X}, \mathcal{L})}$

If the induced psh metric in (i) is locally bdd, then

$$U_{NA} = \mathcal{G}_{(\mathcal{X}, \mathcal{L})}.$$

Pf. Assume $\rho: \mathcal{X} \to X \times \mathbb{C}$, s.t. $\mathcal{L} = \rho^* p_1^* L + D$.

and pick $m > 0$ s.t. mD is Cartier.

Then (i) holds iff. $U + m^{-1} \log|f|$ is locally bdd above for any choice of local equation f of mD by Demailly's thm.

Since $D + a\mathcal{X}_0$ is effective for $a > 0$ large enough

$$f^{\frac{1}{m}} \tau^a = O(1) \quad \frac{1}{m} \log|f| + a \log|\tau| = O(1)$$

$$U_t \leq -\frac{1}{m} \log|f| + O(1) = a \log|\tau| + O(1) = -at + O(1)$$

which show that U has linear growth.

$\forall$ divisorial valuation w on $X \times \mathbb{C}$ with $w(\tau) > 0$.

$w(U + m^{-1} \log|f|) \geq 0$. by def of the Lelong number

$$\| $$

$$w(u) + m^{-1} w(\log|f|) = w(u) + m^{-1} w(mD)$$

$$= w(u) + w(D)$$

$$\Rightarrow w(u) \geq -w(D)$$

Let $w = \sigma(v)$. then $u_{NA}(v) \leq \sigma(v)(D) = \varphi_{(\mathcal{X}, \mathcal{L})}(v)$.

"$\Leftarrow$" Assume (ii). $u_{NA} \leq \varphi_{(\mathcal{X}, \mathcal{L})}$

$\forall\, v \in X^{div}_{\mathbb{Q}}$. $v = r(\mathrm{ord}_E)$. for some irred. component of $\mathcal{X}_0$.

$u_{NA}(v) \leq \varphi_{(\mathcal{X}, \mathcal{L})}(v)$. i.e. $\mathrm{ord}_E(u) \geq -\mathrm{ord}_E(D)$

$\mathrm{ord}_E(D + u) \geq 0$

i.e. $u + m^{-1}\log|f| \leq O(1)$ for any local equ. f of mD

$\square$

<u>Def 4.5</u>. Psh ray u has <u>algebraic singularities</u> if it induces a locally bdd psh. metric on some normal semiample T.C. $(\mathcal{X}, \mathcal{L})$

$\Rightarrow$ u has linear growth. $u_{NA} = \varphi_{(\mathcal{X}, \mathcal{L})}$.

<u>lemma 4.6</u>. $\forall\, \varphi \in \mathcal{H}^{NA}$. there exists a sm. psh ray U with algebraic singularities. s.t. $U_{NA} = \varphi$.

Further, $\forall$ psh ray. V. with $V_{NA} \leq \varphi$. $\Rightarrow V \leq U + O(1)$

<u>Pf</u>. By definition of $\mathcal{H}^{NA}$, pick. normal. semiample T.C. $(\mathcal{X}, \mathcal{L})$. with $\varphi = \varphi_{(\mathcal{X},\mathcal{L})}$.

Since $\mathcal{L}$ is semiample, it admits a smooth S^1-inv. psh metric.

$$\left(h := \frac{1}{2} \log\left(|s_0|^2 + \cdots + |s_N|^2 \right) \right)$$

it's locally bdd. which induce the desired U s.t

$$U_{NA} = \varphi_{(\mathcal{X},\mathcal{L})}$$

$\square$