Quantisation of Poisson structures

A quantization of a Poisson manifold $(M, \{\ \})$ is, by definition, a linear map

$$C^\infty(M) \overset{Q_\hbar}{\to} \mathcal{O}_\hbar,$$

where $\hbar = 6.6260701510^{-34} m^2 kg/s$ is the Planck constant, $\mathcal{O}_\hbar$ is a C*-algebra of operators on a "physical" Hilbert space and the quantization map $Q_\hbar$ satisfies

$$[Q_\hbar(f), Q_\hbar(f)] = i\hbar Q_\hbar(\{f, g\}).$$

Given a Hamiltonian system on M associated to a Hamiltonian function $H \in C^\infty(M)$, the time evolution of elements of $\mathcal{O}_\hbar$ is given by the equation

$$-i\hbar \frac{d}{dt} A = [Q_\hbar(H), A].$$

In reality the first item can be never satisfied except for a certan subclass of functions on M *and the second condition is a definition of time evolution.*

Hence this passage from classical to quantum system is, in practice, a bit of guesswork.

The formal deformation quantization was introduced to construct a consistent procedure, where the constant $\hbar$ is treated a formal variable and the basic commutator identity above holds asymptotically as $\hbar \to 0$. While mathematically very important, the asymptotics in $\hbar \to 0$ do not seem to have a lot to do with physics, since $\hbar$ can be just as well taken to be a unit of length i. e. equal 1.

A continuous version of asymptotic quantization requires that the family $\hbar \to \mathcal{O}_\hbar$ forms a continuous in norm family of C*-algebras convergent to the commutative algebra $C(M)$ such that

$$||[q_\hbar(f) q_\hbar(f)] - q_\hbar(fg)|| \overset{\hbar \to 0}{\longrightarrow} 0\ .$$

For quantization of smooth functions one would expect to have higher order estimates in $\hbar$. This appears in analysis in the disguise of the calculus pseudodifferential operators and in the study of C*-algebraic quantum groups and quantization of homogeneous spaces.

The geometric quantization procedure achieves it but, if it works, at the price of restricting to a discrete set of values of the parameter $\hbar$, determined by what is known as the Bohr-Sommerfeld condition. It is very useful in examples, especially for systems with finite number of degrees of freedom, and plays an important in the representation theory of Lie groups.

On the other hand, once one accepts the fact that the algebras $\mathcal{O}_\hbar$ are not isomorphic to the algebra of compact operators, there exists a wealth of examples where one can construct a continuous family of deformations. These are particularly important and, in fact, unavoidable when one studies extended systems, like the once encountered in (quantum) statistical physics or quantum field theory. Some more known examples are given by (fractional) quantum Hall effect and topological states of matter that has received a lot of attention in recent years.

The aim of this course is to explain various types of quantisation, their relation and, if time permits, some applications to the related study of perturbation expansions that one meets in algebraic quantum field theory.

The lectures will be based on original research papers and notes that will be developed during the course.