Lecture 4. Quandles and almost complete knot invariant

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Contents

Colouring invariants

First, let us return to the simplest knot invariant; i.e., to the colouring invariant. Why is it possible to construct an invariant function by so simple means?

Even the fact that this invariant is related to maps from the knot group to the symmetric group S_3 does not tell us very much: an analogous construction with a greater number of colours does not work.

Let us now try to use a greater palette of colours. Let Γ be an arbitrary finite set (here the finiteness will be used in order to be able to count the number of colourings); all elements of Γ are to be called colours.

Suppose the set Γ is equipped with a binary operation $\alpha : \Gamma \times \Gamma \to \Gamma$; this operation will be denoted like this: $a \circ b \equiv \alpha(a, b)$.

Definition 1.1

By a proper colouring of a diagram D of an oriented link K by colours in Γ we mean a way of associating some colour with each arc of D in such a way that for each overcrossing arc (that has colour b), undercrossing arc lying on the left hand (colour a) and undercrossing lying on the right hand (colour c), the relation $a \circ b = c$ holds; see Fig. 1.

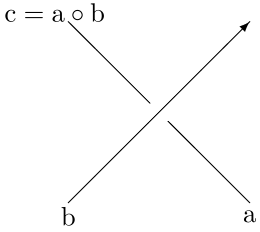

Figure 1: The rule of colourings

Conditions for $\circ$

Which should be the conditions for $\circ$ for the number of proper colourings to be invariant under Reidemeister moves?

Reiemeister move 1.
It is easy to show that the invariance of such a colouring function under Ω_1 implies the idempotence relation $a \circ a = a$ for all the elements $a \in \Gamma$ that can be associated to arcs and play the role of colour. However, in order to simplify the situation we shall not restrict ourselves only to this case, and require that $\forall a \in \Gamma : a \circ a = a$.

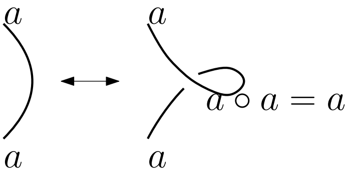

Figure 2: $a \circ a = a$

Conditions for $\circ$: continued

Reiemeister move 2.
Analogously, the invariance under Ω_2 requires the left invertibility of the operation $\circ$: for any a and b from Γ, the equation $x \circ a = b$ should have only the solution $x \in \Gamma$.

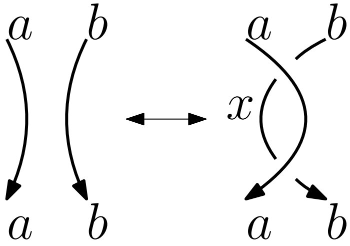

Figure 3: For all a, b, $\exists! x$ such that $x \circ a = b$

The inverse operation for $\circ$ is denoted by $/$. More precisely, the element b/a is defined to be the unique solution to the equation $x \circ a = b$.

Conditions for $\circ$: continued

Reiemeister move 3.

Finally, the invariance under Ω_3 implies right self distributivity of the operation $\circ$, which means that $\forall a, b, c \in \Gamma$ the equation $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ holds.

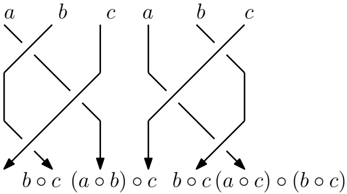

Figure 4: For all a, b, c, $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$.

In the sequel, each set with an operation $\circ$ satisfying the three properties described above, is called a quandle.

Each quandle generates a rule for proper colouring of link diagrams described above.
Thus, we conclude the following

Proposition 1.2

The number of proper colourings by elements of any quandle is a link invariant.

Remark 1.3

The quandle is discovered independently by S.V. Matveev [6] and D. Joyce [7]. In Matveev's work and in other works by Russian authors, this invariant is usually called the distributive groupoid; in Western literature it is usually called the quandle.

Quandle presentation

There is a common way for constructing quandles by using their presentations by generators and relations.

Let A be an alphabet consisting of letters. A word in the alphabet A is an arbitrary finite sequence of elements of A and symbols $(,), \circ, /$.

Now, let us define inductively the set $D(A)$ of admissible words according to the following rules:

1. For each $a \in A$, the word consisting of only the letter a is admissible.
2. If two words W_1, W_2 are admissible then the words $(W_1) \circ (W_2)$ and
   $(W_1)/(W_2)$ are admissible as well.
3. There are no other admissible words except for those obtained inductively by rules 1 and 2.

Sometimes we shall omit brackets when the situation is clear from the context. Thus, e.g. for letters a_1, a_2 we write the word $a_1 \circ a_2$ instead of $(a_1) \circ (a_2)$.

Quandle presentation : continued

Let R be a set of relations; i.e., identities of type $r_\alpha = s_\alpha$, where $r_\alpha, s_\alpha \in D(A)$ and α runs over some set X of indices. Let us introduce the equivalence relation for D(A), supposing $W_1 \equiv W_2$ if and only if there exists a finite chain of transformations starting from W_1 and finishing at W_2 according to the rules 1-5 described below:

1. $x \circ x \Longleftrightarrow x$;
2. $(x \circ y)/y \Longleftrightarrow x$;
3. $(x/y) \circ y \Longleftrightarrow x$;
4. $(x \circ y) \circ z \Longleftrightarrow (x \circ z) \circ (y \circ z)$;
5. $r_i \Longleftrightarrow s_i$.

The set of equivalence classes is denoted by $\Gamma\langle A|R\rangle$. It is easy to check that it is a quandle with respect to the operation $\circ$.

Remark 1.4

There is an analogous construction of biquandles which instead of edges deals with half-edges. By a half-edge we mean the result of breaking each edge at every overpass.
Hence, for each crossing, we have four elements of the biquandle, say, two incoming and two outgoing.
Then one can define the colourings of the two outgoing edges in terms of colourings of the two incoming edges. The invariance under Reidemeister moves will lead us to some conditions, we are not going to write directly.
Similarly to the quandle, for biquandles we have a fundamental biquandle and its realisations.
Quandles are partial cases of biquandles where the outgoing edge forming an overpass has the same colour as the incoming edge forming the overpass.

Geometric description of the knot quandle

Let K be an oriented knot in $\mathbb{R}^3$, and let N(K) be its small tubular neighbourhood. Let $E(K) = \overline{(\mathbb{R}^3 \backslash N(K))}$ be the complement to this neighbourhood. Fix a base point x_K on E(K). Denote by Γ_K the set of homotopy classes of paths in the space E(K) with fixed initial point at x_K and endpoint on $\partial N(K)$ (these conditions must be preserved during the homotopy). Note that the orientations of $\mathbb{R}^3$ and K define the orientation of the tubular neighbourhood of the knot (right screw rule). Let m_b be the oriented meridian hooking an arc b. Define $a \circ b = [b m_b b^{-1} a]$, where for $x \in \Gamma_K$ the letter x means a representative path, and square brackets denote the class that contains the path [x]; see Fig. 5.

Geometric description of the quandle : continued

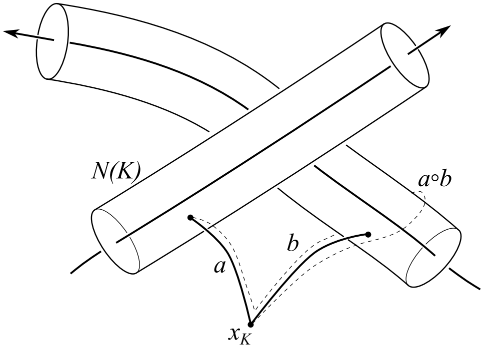

Figure 5: Intuitive description of the quandle operation.

The quandle axioms can also be checked straightforwardly. Also, one can easily check that the groupoids corresponding to different points x_K are isomorphic. This statement is left as an exercise.

Relation between knot quandles and knot groups

Here, we should emphasise that we first define our map from barely the set of elements of the quandle to the set of elements of the group. Then it will naturally yield a quandle operation on the group; this quandle operation will then lead us to further examples.
There is a natural map from the knot quandle $\Gamma(\mathrm{K})$ to the group $\pi_1(\mathbb{R}^3 \backslash \mathrm{E}(\mathrm{K}))$. Let us fix a point x outside the tubular neighbourhood. Now, with each element γ of the quandle (path from x to $\partial \mathrm{E}(\mathrm{K})$) we associate the loop $\gamma \mathrm{m} \gamma^{-1}$, where m is the meridian at the end point of γ.
This interpretation shows that the fundamental group can be constructed by the quandle: all meridians can play the role of generators for the fundamental groups, and all relations of type $\mathrm{a} \circ \mathrm{b} = \mathrm{c}$ have to be replaced with $\mathrm{bab}^{-1} = \mathrm{c}$.
Besides, the fundamental group has an obvious action on the quandle: for each loop g and element of the quandle γ, the path $\mathrm{g}\gamma$ is again an element of the quandle.

Quandles for manifolds with boundaries

Remark 1.5

Note that it is possible to apply "geometric description of knot quandles" to any oriented 3-dimensional manifolds M with non-empty boundaries $\partial(\mathrm{M})$. More precisely, we can consider the set of homotopy classes of paths in the space E(K) with fixed initial point at p and endpoint on $\partial(\mathrm{M})$. Then it must be an invariant under homeomorphisms on M.

Algebraic description of the knot quandle

Let D be a diagram of an oriented knot K. Denote the set of arcs of D by A_D. Let P be a crossing incident to two undercrossing arcs a and c and an overcrossing arc b. Let us write down the relation: $a \circ b = c$, where a is the arc lying on the left hand with respect to b and c is the arc lying on the right hand with respect to b. Denote the set of all relations for all crossings by R_D. Now, consider the quandle $\Gamma\langle A_D | R_D \rangle$, defined by generators A_D and relations R_D.

Theorem 1.6

Quandles Γ_K and $\Gamma\langle A_D|R_D\rangle$ are isomorphic.

Proof.
With each arc a of the projection D, we associate the path s_a in E(K) in such a way that

1. the path s_a connects the base point with a point of the part of the torus ∂N_K corresponding to the arc a;
2. at all points where the projection of s_a intersects that of D, the path s_a goes over the knot; see Fig. 6.

Proof : continued

Obviously, these conditions are sufficient for the definition of the homotopy class of $\mathrm{s_a}$.

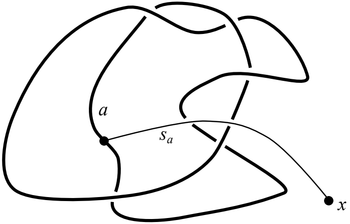

Figure 6: Defining the path $\mathrm{s_a}$.

Consequently, to each generator of $\Gamma = \langle \mathrm{A_D} | \mathrm{R_D} \rangle$, there corresponds an element of the quandle Γ_K. Thus we have defined the homomorphism $\phi : \Gamma\langle \mathrm{A_D} | \mathrm{R_D} \rangle \to \Gamma_\mathrm{K}$.

Proof : continued

In order to define the inverse homomorphism $\psi : \Gamma_K \to \Gamma\langle A_D | R_D \rangle$, let us fix $s \in \Gamma_K$. Then, the path representing s is constructed in such a way that the projection of the path intersects D transversely and contains no diagram crossing.

Denote by $a_n, a_{n-1}, \ldots, a_1$ those arcs of D going over the path s. Denote by a_0 the arc corresponding to the end of s. Now, for each $s \in \Gamma_K$, let us assign the element $((\ldots(a_0 \epsilon_1 a_1)\epsilon_2 \ldots a_{n-1})\epsilon_n a_n$ of the quandle $\Gamma\langle A_D | R_D \rangle$, where ϵ_i means / if s goes under a_i from the left to the right, or $\circ$, otherwise; see Fig. 7.

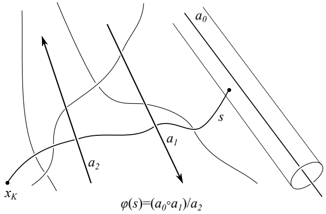

Figure 7: Constructing the map $\psi : \Gamma_K \to \Gamma\langle A_D, R_D \rangle$.

Proof : continued

It is easy to check that this map is well defined (i.e., it does not depend on the choice of representative s for the element of Γ_K) and that maps ϕ and ψ are inverse to each other. This completes the proof.

Completeness of the quandle

By Matveev, two (non–isotopic) knots are equivalent if one can be obtained from the other by changing both the orientation of the ambient space and that of the knot. For example, the right-handed and left-handed trefoils are equivalent.

Here by complete we mean that the quandle distinguishes knots up to equivalence up to both the orientation of the ambient space and that of the knot.

Roughly speaking, the quandle is an (almost) complete knot invariant because it contains the information about the fundamental group and "a bit more" – peripheral structures.

But, since knot quandles can distinguish knots up to equivalence up to both the orientation of the ambient space, the right-handed and the left-handed trefoils have isomorphic knot quandles. That is, the knot quandles cannot distinguish the right-handed and the left-handed trefoils. Such a situation is frequent.

Sketch of proof

The key points of the proof are the following:

1. For the unknot the situation is very clear: the fundamental group recognises it by Dehn's theorem (Theorem 2.3).
2. If a knot is not trivial then its complement is sufficiently large (by Dehn's theorem); it also has some evident properties to be defined.
3. For the class of manifolds satisfying this condition the fundamental group "plus a bit more" is a complete invariant up to equivalence defined above.
4. The knot quandle allows us to restore the fundamental group structure for the complement and "a bit more".

For the sake of simplicity, we shall work only with knots. The same results are true for the case of links.
It is easy to see that the fundamental group of the knot complement can be restored from the quandle (this will be shown a bit later).

Compressible surface

Let us first introduce some definitions.

Definition 2.1

A surface F in a manifold M is compressible in either of the following cases:

1. There is a non–contractible simple closed curve k in the interior of F and a disc D in M (whose interior lies in the interior of M) such that $D \cap F = \partial D = k$.
2. There is a ball E in M such that $E \cap F = \partial E$.

Otherwise the surface is called incompressible.

Irreducible 3–manifold

Definition 2.2

A 3–manifold M is called irreducible if any sphere $S^2 \subset M$ is compressible.
A 3–manifold M with boundary is called boundary–irreducible if its boundary ∂M is incompressible.

Let K be an oriented knot. Consider the fundamental group π of $\mathbb{R}^3 \backslash N(K)$ where N is a tubular neighbourhood of K. Obviously, $\partial N = T$ is a torus that has an oriented meridian m (a curve that has linking coefficient 1 with K). Now let us recall Dehn's theorem:

Theorem 2.3 (Dehn)

An m–component link L is trivial if and only if $\pi_1(\mathbb{R}^3 \backslash L)$ is isomorphic to the free group in m generators.

Sufficiently large and Haken manifolds

Definition 2.4

A manifold M is called sufficiently large if one can embed a handlebody (not the sphere) in M in such a way that the image map for the fundamental group has no kernel.

Definition 2.5

A Haken manifold is a compact, irreducible sufficiently large 3-manifold.

Remark 2.6

The classification of Haken manifolds can be done by "cut and paste" method, which is developed by W. Haken G. Hemion and S. Matveev, for more detail, see [10].
In particular, a knot complement is a Haken manifold!!!

If K is not trivial then the fundamental group $\pi(\mathrm{T})$ is embedded in π. This result follows from Dehn's theorem.

Definition 2.7

For a non-trivial knot K, the embedded system $\mathrm{m} \in \pi(\mathrm{T}) \subset \pi$ is called a peripheral system of K.

The Waldhausen theorem[1] says the following:

Theorem 2.8 ([4])

Let M, N be irreducible and boundary–irreducible 3–manifolds. Let M be sufficiently large and let $\psi : \pi_1(\mathrm{N}) \to \pi_1(\mathrm{M})$ be an isomorphism preserving the peripheral system. Then there exists a homeomorphism $\mathrm{f} : \mathrm{N} \to \mathrm{M}$, inducing ψ.

[1] We use the formulation taken from [6]

Proof of completeness

We are going to prove that the knot quandle is a complete knot invariant.

Now let K_1, K_2 be two knots. Suppose that ϕ is an isomorphism $\Gamma(K_1) \to \Gamma(K_2)$ of the quandles. Denote the complements to tubular neighbourhoods of K_1, K_2 by E_{K_1}, E_{K_2}, respectively.

Note that if K_1, K_2 are not trivial then the manifolds E_{K_1}, E_{K_2} are boundary–irreducible, sufficiently large and irreducible (by Dehn's lemma).

Proof of completeness : continued

Now, let us suppose that one of the two knots (say, K_1) is trivial. Then $\pi(K_1)$ is isomorphic to $\mathbb{Z}$. Since the knot group can be restored from the quandle, we have $\pi(K_2)$ is also $\mathbb{Z}$. Thus, K_2 is trivial.

Now consider the case when K_1, K_2 are non-trivial.

In this case, we know that the knot group can be restored from the quandle; besides, the meridian can also be obtained from the knot quandle: it can be chosen to be the image of any element of the quandle under the natural morphism.

Proof of completeness : continued

The normaliser of the meridian (as an element of the quandle representing the path from x to x_K) in the fundamental group consists precisely of the fundamental group of the tubular neighbourhood of the knot $\pi_1(T^2) = \mathbb{Z}^2$.

Indeed, each element of the group $\pi_1(T^2)$ is a path looking like ana^{-1} where a represents the meridian in the quandle (path from the initial point to the point on T^2), and n is a loop on the torus T^2. So, we have: $ana^{-1} \cdot a = an$ which is homotopic to a in the quandle.
Now, suppose that for some g we have: ga is homotopic to a. Then, there exists a path on the torus drawn by the endpoint while performing this homotopy. Denote this path by x. So, we have: $gaxa^{-1} = e$, so g is homotopic to $ax^{-1}a^{-1}$ that belongs to the fundamental group of T^2.

Proof of completeness : continued

The next step of the proof is the following. The quandle knows the peripheral system. Let K_1, K_2 be two non-trivial knots with the same peripheral structure. Consider an isomorphism of the knot groups. By the Waldhausen theorem, it generates a homeomorphism h between E_{K_1} and E_{K_2} and maps the meridian of the first one to a meridian of the second one. Thus, we have the same information on how to attach full tori N_1 and N_2 to E_{K_1} and E_{K_2} in order to obtain $\mathbb{R}^3$. Having a full torus N_i, $i = 1, 2$, in $\mathbb{R}^3$, we can contract its meridian to a point; hence, we get a curve λ_i which will be exactly the knot K_i. So, they are obviously isotopic. To perform all this, we must fix the orientation of E_{K_1} and E_{K_2}. Then we shall be able to choose the orientation of the meridian. If we choose the opposite orientation of them both, we shall obtain an equivalent knot.
However, if the orientation of the meridian is fixed, the knot can be uniquely restored.

Remark 2.9

In the proof of the completeness of knot quandles, it play an important role that the normaliser of the meridian in the fundamental group consists precisely of the fundamental group of the tubular neighbourhood of the knot $\pi_1(\mathrm{T}^2) = \mathbb{Z}^2$. It is worth to remind that a peripheral system is a meridian $\mathrm{m} \in \pi(\mathrm{T}) \subset \pi(\mathrm{K})$, and that is why we pay attention to the peripheral system.

Exercises

1. Let $(\Gamma, \circ)$ be a quandle and $/$ be the inverse operation of $\circ$. Show that $(\Gamma, /)$ is also a quandle. Furthermore, prove the following identities for Γ:
   $(a \circ b)/c = (a/c) \circ (b/c)$
   $(a/b) \circ c = (a \circ c)/(b \circ c)$.
2. Let $Q = \mathbb{Z}[t, t^{-1}]$. Let us define $\circ : Q \times Q \to Q$ by
   $$a \circ b = ta + (1-t)b.$$
   Prove that $(Q, \circ)$ satisfies the quandle axioms. This quandle is called an Alexander quandle.
3. Let Q be a group. Let us define $\circ : Q \times Q \to Q$ by
   $$a \circ b = bab^{-1}.$$
   Prove that $(Q, \circ)$ satisfies the quandle axioms. This quandle is called the conjugation quandle.

4. Let Q be a group. Let us define $\circ : Q \times Q \to Q$ by
$$a \circ b = b^n a b^{-n}.$$
Prove that $(Q, \circ)$ satisfies the quandle axioms.
5. Let Q be a group. Let us define $\circ : Q \times Q \to Q$ by
$$a \circ b = b a^{-1} b.$$
Prove that $(Q, \circ)$ satisfies the quandle axioms. This quandle is called a core quandle.
6. Let X be a group and $x \in X$ fixed. Let us define $\circ : X \times X \to X$ by
$$a \circ b = a b^{-1} x b.$$
   - Prove that $(X, \circ)$ satisfies the second and the third axioms of quandles, but does not satisfy the first axiom. That is $(X, \circ)$ is a rack.
   - Prove that the number of colouring of a diagram by $(X, \circ)$ is invariant under the move, shown in Fig. 8.

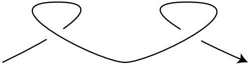

Figure 8:

7. Let X be a non-empty set and let $\circ, *$ be binary operations on X. Assume that the binary operations satisfy the following relations:

   - $x \circ x = x * x$
   - the maps $*x, \circ x : X \times X \to X$ and the map $S : X \times X \to X \times X$ defined by
   $$S(x, y) = (y * x, x \circ y),$$
   are bijective.
   - $(x \circ y) \circ (z \circ y) = (x \circ z) \circ (y * z)$
   - $(x \circ y) * (z \circ y) = (x * z) \circ (y * z)$
   - $(x * y) * (z * y) = (x * z) * (y \circ z)$

   Then $(X, \circ, *)$ is called a biquandle. Prove that the number of colorings of knots by a biquandle satisfying the rule described in Fig. 9 is a knot invariant.

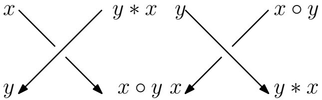

Figure 9: Biquandle colouring on crossings

Research problem 1

It is well known that topological groups, especially Lie group, give strong method to study manifold. On the other hands, in [8] R. L. Rubinsztein defined the topological quandle.

More precisely, let X be a topological space equipped with a continuous map $\mu : X \times X \to X$, denoted by $\mu(a, b) = a \circ b$, such that for every $b \in X$ the mapping $a \to a \circ b$ is a homeomorphism of X. The space X (together with the map μ) is called a topological quandle if it satisfies

1. $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$,
2. $a \circ a = a$,

for all $a, b, c \in X$. That is, we get a topological-valued invariant of knots.

Question
We remarked that a kind of quandle described geometrically can be defined for every manifolds with boundaries. How can we use the topological quandle not only for links, but also for arbitrarily manifolds?

Research problem 2

Knot quandles are an almost complete knot invariant and, naturally, many knot invariants are related with (or derived from) knot quandles, for example, Alexander polynomials, fundamental groups and so on. In particular, the Alexander polynomial can be categorified by the Floer homology and it gives more plentiful informations for knots than the Alexander polynomials.

Question
How can we categorify the quandles?

The Alexander polynomial can be defined not only by using quandles, but also by using Seifert surfaces and a skein relation (for more detail, see Chapter 5 in [1] and Chapter 6 in [2]). In particular, as asserted in Lecture 2, by using skein relation it might be possible to modify the Alexander polynomial.

Question
How can we construct invariants by modifying the skein relation for the Alexander polynomial?

Research problem 3

In [9] by V.O. Manturov an algebraic structure with two binary operations, so-called the long quandle, is defined and a colouring of (virtual) long knots by the long quandle is a (virtual) long knot invariant. Moreover, by using the long quandle, one can prove that the connected sum of virtual knots are non-commutative. For more details, please see [9].

Question
Try to define a quandle-like structure for knots or links, or higher dimensional knots.

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