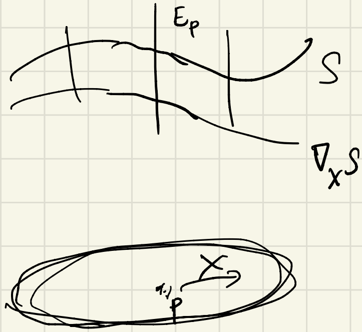
2021-9-15 Kähler geometry

Connections (covariant derivatives)

$E \xrightarrow{\pi} M$ C^∞ complex vector bundle.

$C^\infty(E) = \{ s : M \xrightarrow{C^\infty} E \mid \pi \circ s = id_M \}$



$$\begin{array}{c} E \\ \pi \big\downarrow \big\uparrow s \\ M \end{array} \qquad\qquad \begin{array}{c} s(p) \\ s \big\uparrow \big\downarrow \pi \\ p \end{array}$$

(M is a smooth mfd.)

$C^\infty_{\mathbb{C}}(M) = \{ \mathbb{C}\text{-valued smooth functions on } M \}$.

$f \in C^\infty_{\mathbb{C}}(M)$, $s \in C^\infty(E) \Longrightarrow fs \in C^\infty(E)$.

<u>Def</u> $\nabla : C^\infty(TM \otimes \mathbb{C}) \times C^\infty(E) \to C^\infty(E)$

$$(X, s) \longmapsto \nabla_X s$$

is called a (linear) connection if

(i) $\nabla_{fX} S = f \nabla_X S \qquad f \in C_c^\infty(M)$

(ii) $\nabla_X (fS) = (Xf) S + f \nabla_X S$

<u>Def</u> $\nabla_X S$ is called the covariant derivative of S in the direction of X.

<u>Exercise</u> Show using (i) that if $X_p = Y_p$ then $\nabla_X S = \nabla_Y S$ at p.

Def $e_1, \cdots, e_r \in C^\infty(E|_U)$ ($E|_U = \pi^{-1}(U)$)

U is an open set, is called a local <u>frame field</u> if

$e_1(p), \cdots, e_r(p)$ form a basis

of E_p for $\forall p \in U$.

$$E|_U \cong U \times \mathbb{R}^r \quad (\mathbb{C}^r)$$

$$e_i(p) = (p, e_i)$$

Def Given such a frame, define 1-forms $\theta^i{}_j$ by ~~θ^i_j~~ $\theta^i{}_j$

$$\nabla e_j = \theta^i{}_j \, e_i \quad \left(= \sum e_i \theta^i{}_j \right)$$

So $\nabla_X e_j = \sum_{i=1}^{r} \theta^i{}_j(X)\, e_i$.

Def $\theta = (\theta^i_j)$ is called the <u>connection form</u> or <u>connection matrix</u> of ∇ w.r.t. $e_1, \cdots, e_r$.

$$\nabla e = e\,\theta \quad (\text{similar to } T(e) = e\,A).$$

So if $s = s^1 e_1 + \cdots + s^r e_r$

$$\nabla_X s = (X s^1) e_1 + \cdots + (X s^r) e_r + s^1 \nabla_X e_1 + \cdots + s^r \nabla_X e_r$$

$$= (X s^i) e_i + s^j \theta^i_j(X) e_i$$

$$= (ds^i + \theta^i_j s^j)(X)\, e_i$$

$$\nabla \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix} \doteq d \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix} + \theta \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix}$$

Def A Riemannian metric $g \in C^\infty(T^*M \otimes T^*M)$ on M is the one satisfying

$$g : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(X, Y) \longmapsto g(X, Y)$$

is an inner product, i.e. symmetric, positive definite

Theorem-Definition (homework)

Given a Riemannian metric on M there is a unique connection of the tangent bundle TM satisfying

$$\begin{cases} (1) & X\, g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ (2) & \nabla_X Y - \nabla_Y X = [X, Y]. \end{cases}$$

(1) metric compatibility.

(2) torsion-free, symmetric

$$\begin{cases} (M, g) & \text{Riemannian mfd} \\ \nabla : & \text{Riemannian connection, Levi-Civita connection} \end{cases}$$

When ∇ is defined on TM, it also defines a connection on T^*M by duality:

$$\alpha \in C^\infty(T^*M), \quad X \in C^\infty(TM)$$

$$(*) \quad (\nabla_Y \alpha)(X) = Y(\alpha(X)) - \alpha(\nabla_Y X).$$

Further ∇ extends also to tensor products

$$\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right)$$

by derivation.

Convention : $f \in C^\infty_0(M)$, we set

$$\nabla f := df$$

$$\nabla_Y (\alpha \otimes X) = (\nabla_Y \alpha) \otimes X + \alpha \otimes \nabla_Y X$$

$c : TM \otimes T^*M \to C^\infty(M)$ evaluation

"contraction"

$$(*) \Leftrightarrow \nabla_X \left(c(\alpha \otimes X) \right) = c\left(\nabla_Y (\alpha \otimes X) \right)$$

" the contraction and the covariant derivative commute."

local tensor calcus.

choose a local coordinate system $(x^1, \cdots, x^n)$. Then

$s \in C^\infty\left((\overset{p}{\otimes} TM) \otimes (\overset{q}{\otimes} T^*M) \right)$ can be expressed as

$$s = s^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$$

$$e_i = \frac{\partial}{\partial x^i} \quad , \quad e^i = dx^i$$

$$\underset{\substack{\cap \\ C^\infty\left(\left(\bigotimes^p TM\right)\otimes\left(\bigotimes^q T^*M\right)\otimes T^*M\right)}}{\nabla s} = \nabla_k s^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q} \otimes dx^k$$

This is the definition of $\nabla_k s^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$.

$$\underset{\substack{\cap \\ C^\infty(T^*M \otimes T^*M)}}{g} = g_{ij}\, dx^i \otimes dx^j \qquad \text{metric } TM.$$

$$g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \qquad (g^{ij}) = (g_{ij})^{-1}$$

metric on T^*M

Using Riemannian metric one can define an isomorphism

$$\begin{array}{ccc} TM & \xrightarrow{\sim} & T^*M \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ X & \longmapsto & X^b \end{array}$$

by $(X^b, Y) \overset{\text{def}}{=} g(X, Y)$.

$$\begin{array}{c} (X^b, Y) \\ \| \\ X^b(Y) \end{array}$$

$$(X^\flat, \frac{\partial}{\partial x^i}) = g(X^j \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}) = g_{ij} X^j$$

$$\therefore X^\flat = g_{ij} X^j dx^i$$

$$\therefore (X^\flat)_i = g_{ij} X^j$$

$$X_i := g_{ij} X^j$$

$\flat$: musical isomorphism.

In the same way, for $\alpha = \alpha_i dx^i$

$$\alpha^\# = g^{ij} \alpha_i \frac{\partial}{\partial x^j} \in C^\infty(TM).$$

$$\alpha^j := g^{ij} \alpha_j$$

One may also define

$$\nabla^i \alpha_j := g^{ik} \nabla_k \alpha_j.$$

Then

$$\nabla^i (g_{jl} X^l) \overset{①}{=} \nabla^i X_j \overset{②}{=} g_{jk} (\underline{\nabla^i X^k})$$

two interpretations of $\nabla^i X_j$.

But these two are compatible because

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Lemma $\nabla g = 0$ "g is parallel"

☹ $(\nabla_X g)(Y, Z) = X(g(Y,Z))$

$$- g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

$$= 0 \quad \text{by } (1)$$

☺

We define

$$R \in C^\infty(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$$
$$= R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

by

$$R_{ijkl} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$= g\left(\frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}\right)$$

("Riemannian" curvature tensor)