29章，16次课

1. lecture
2. notes
3. hw
4. references：推荐阅读
5. 研究

panacea

万金油

万灵丹

||

Fisher Randomization Test

(FRT)

# The Design of Experiments

By

Sir Ronald A. Fisher, Sc.D., F.R.S.

Honorary Research Fellow, Division of Mathematical Statistics, C.S.I.R.O., University of Adelaide; Foreign Associate, United States National Academy of Sciences; and Foreign Honorary Member, American Academy of Arts and Sciences; Foreign Member of the Swedish Royal Academy of Sciences, and the Royal Danish Academy of Sciences and Letters; Member of the Pontifical Academy; Member of the German Academy of Sciences (Leopoldina); formerly Galton Professor, University of London, and Arthur Balfour Professor of Genetics, University of Cambridge.



Fisher (1935)

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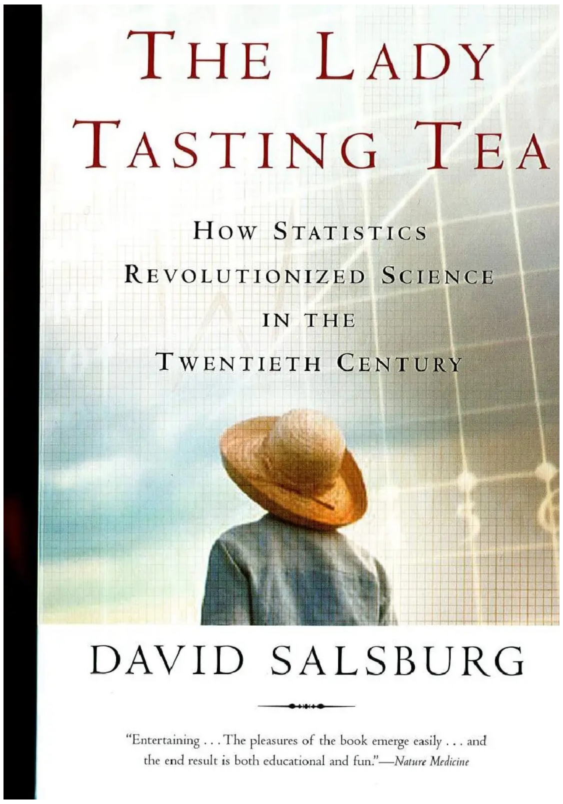



女士品茶

书 3.5.2

| | lady milk | lady tea | |
|---|---|---|---|
| Fisher milk first | X | 4-X | 4 |
| Fisher tea first | 4-X | X | 4 |
| | 4 | 4 | |

$$P_{FRT} = \frac{1}{\binom{8}{4}} = \frac{1}{70} = 0.014 < 0.05$$

CAPITAL OF STATISTICS
PROFESSION, HUMANITY & INTEGRITY



推荐文章

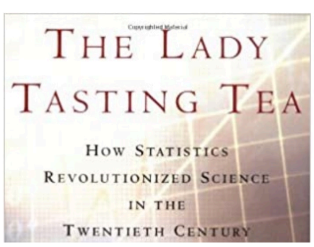
## 女士品茶的实验、假设和检验

丁鹏

关键词：实验设计; 假设检验

编辑：于淼、林枫 审稿：黄湘云、魏太云

R. A. Fisher 的名著《实验设计；第八版，1971 年》第二部分有十六页，仅仅讲了一个最简单的实验：女士品茶。这个故事非常有名，以至于 Salsburg 的统计学通俗读物就以它命名：《女士品茶：20 世纪统计怎样变革了科学》。



# Chapter 3 CRE FRT

CRE: completely randomized experiment

$n$ <
- $n_1$ treatment
- $n_0$ control

$n, n_1, n_0$ fixed

$$\vec{Z} = (Z_1, Z_2, \dots, Z_n)$$

$$\Pr(\vec{Z} = \vec{z}) = \frac{1}{\binom{n}{n_1}}$$

computationally: sample( $\vec{Z}$ )

R

$\vec{Z}$ ↓ $n_1$ 1's, $n_0$ 0's

$$\Pr(\vec{Z} \mid \vec{Y}(1), \vec{Y}(0)) = \Pr(\vec{Z}) = \text{uniform}$$

treatment assignment mechanism

处理分配机制

Rubin's terminology

$H_{0F}$ : $Y_i(1) = Y_i(0) \quad \forall\, i = 1 \cdots n$

sharp null

strong null

$\Rightarrow \quad Y_i(1) = Y_i(0) = Y_i = Z_i Y_i(1) + (1 - Z_i) Y_i(0)$

are all fixed

Data

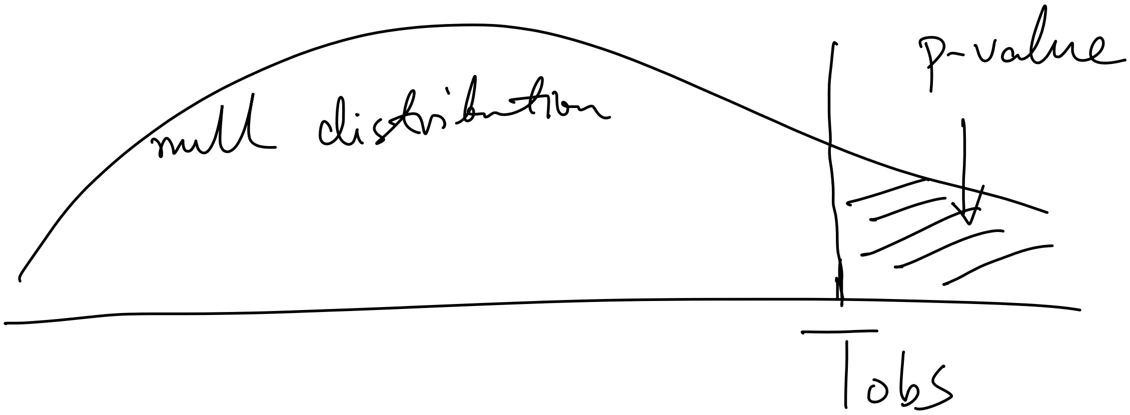
| $i$ | $Z$ | $Y$ |
|---|---|---|
| 1 | 1 | |
| 2 | $\vdots$ | |
| $\vdots$ | 1 | |
| | 0 | |
| | $\vdots$ | |
| $n$ | 0 | |

fixed under $H_{0F}$

random

检验统计量: $T = T(\vec{Z}, \vec{Y})$

measures deviation from $H_{0F}$



$$\vec{Z} \sim \text{Uniform}\left(\vec{z}^{\,1} \cdots \vec{z}^{\,M}\right)$$

$$M = \binom{n}{n_1}$$

$$\Rightarrow T(\vec{Z}, \vec{Y}) \sim \text{Uniform}\left(T(\vec{z}^{\,1}, \vec{Y}) \cdots T(\vec{z}^{\,M}, \vec{Y})\right)$$

$$p_{FRT} = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}\left\{T(\vec{z}^{\,m}, \vec{Y}) \geq T(\vec{Z}, \vec{Y})\right\}$$

$$\approx$$

$$\hat{p}_{FRT} = \frac{1}{R} \sum_{r=1}^{R} \mathbb{1}\left\{T(\vec{z}^{\,r}, \vec{Y}) \geq T(\vec{Z}, \vec{Y})\right\}$$

R次随机置换

sample $(\vec{Z})$

hw: $\text{var}\left(\hat{p}_{FRT}\right) \leq \frac{1}{4R}$

T 的选择：

e.g.1 $\hat{\tau} = \underbrace{\hat{Y}(1)}_{\frac{1}{n_1}\sum_{Z_i=1} Y_i} - \underbrace{\hat{Y}(0)}_{\frac{1}{n_0}\sum_{Z_i=0} Y_i}$

e.g.2 $$t_{unequal} = \frac{\hat{Y}(1) - \hat{Y}(0)}{\sqrt{\frac{\hat{S}^2(1)}{n_1} + \frac{\hat{S}^2(0)}{n_0}}}$$

样本方差

$$\frac{1}{n_1-1}\sum_{Z_i=1}\left(Y_i - \hat{Y}(1)\right)^2$$

R: t.test(.)

var.equal = FALSE

古典统计：Behrens-Fisher 问题

推荐：若用 FRT 检验 $\begin{cases} H_{0F}: \tau_i = 0 \\ H_{0N}: \tau = 0 \end{cases}$

那么 FRT + $t_{unequal}$ 更好

理论保证

e.g. 3 Wilcoxon 秩和

pool$(Y_1 \cdots Y_n)$

$\Longrightarrow$ rank$(Y_1 \cdots Y_n)$

秩 ranks: $R_1 \quad R_n$

$$R_i = \#\{j : Y_j \leq Y_i\}$$

$$W = \sum_{i=1}^{n} Z_i R_i$$

$$\Longleftrightarrow \quad \hat{\bar{R}}(1) - \hat{\bar{R}}(0)$$

e.g. 4 Kolmogorov – Smirnov 统计量

$$\hat{F}_1(y) = \frac{1}{n_1} \sum_{i=1}^{n} Z_i \, 1(Y_i \leq y)$$

$$\hat{F}_0(y) = \frac{1}{n_0} \sum_{i=1}^{n} (1 - Z_i) \, 1(Y_i \leq y)$$

$$D = \max_y \left| \hat{F}_1(y) - \hat{F}_0(y) \right|$$

eg 5 一般策略

① Neyman - Pearson : $H_0 \longleftrightarrow H_1$

$f_0(y) \qquad f_1(y)$

$$LRT = \frac{f_1(y)}{f_0(y)}$$

功效

power

② model - assisted

$lm(Y_i \sim Z_i) \Rightarrow \hat{\tau} = \hat{\bar{Y}}(1) - \hat{\bar{Y}}(0)$

$lm(Y_i \sim Z_i + X_i)$ 协变量

covariates

$\Downarrow$

$\hat{\tau}_F$ : Fisher's ANCOVA

$glm(Y_i \sim Z_i)$

$glm(Y_i \sim Z_i + X_i)$

推广 FRT: 固定

$H_0(\vec{\tau})$: $Y_i(1) - Y_i(0) = \tau_i, \ \forall i$

$+ \ (Z_i, Y_i)_{i=1}^n$

$\Longrightarrow (Z_i, Y_i(1), Y_i(0))_{i=1}^n$

计算 $p_{FRT}(\vec{\tau})$

"检验"的想法

$$\left\{ \tau = \frac{1}{n}\sum_{i=1}^n \tau_i : p_{FRT}(\vec{\tau}) > \alpha \right\}$$

构成了 $\tau$ 的 $(1-\alpha)$ 水平的置信集.

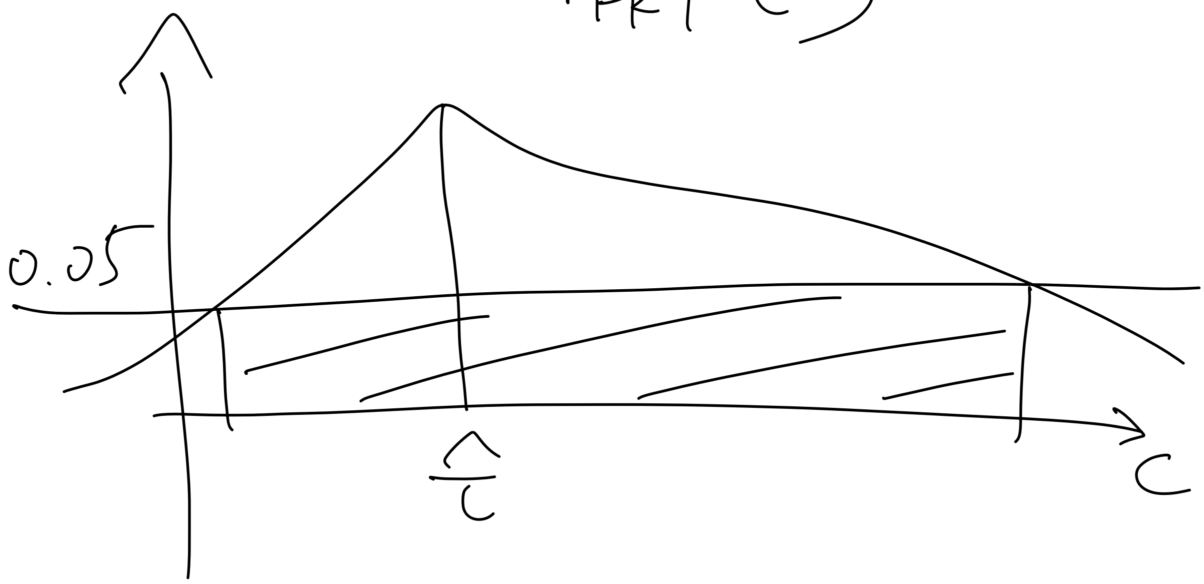
Li & Ding (2016): $Y$ 二值

$O(3^n) \longrightarrow O(n^2)$

Rosenbaum 特例： $H_0(c): Y_i(1) - Y_i(0) = c$

假设常数作用 　　 常数 ↓

$\Rightarrow p_{FRT}(c)$




testable implication:

$\hat{F}_1(y) \neq \hat{F}_0(y)$

Ding Feller Miratrix (2014 JRSSB)

chapter 4 Neyman (1923)

$(Y_i(1), Y_i(0))_{i=1}^{n} \Rightarrow \tau = \frac{1}{n}\sum_{i=1}^{n}(Y_i(1) - Y_i(0))$
$= \bar{Y}(1) - \bar{Y}(0)$

CRE: $\vec{Z} \sim$ Uniform

观测 $Y_i = Z_i Y_i(1) + (1 - Z_i) Y_i(0)$

无偏估计 $\tau$: $\hat{\tau} = \hat{\bar{Y}}(1) - \hat{\bar{Y}}(0)$

$\bar{Y}(1) = \frac{1}{n}\sum_{i=1}^{n} Y_i(1)$, $\bar{Y}(0) = \frac{1}{n}\sum_{i=1}^{n} Y_i(0)$

$\hat{\bar{Y}}(1) = \frac{1}{n_1}\sum_{Z_i=1} Y_i$, $\hat{\bar{Y}}(0) = \frac{1}{n_0}\sum_{Z_i=0} Y_i$

$E(\hat{\tau}) = \tau$

意思?

$Y_i(1), Y_i(0)$ 固定

$Z_i$ 随机

$$\hat{\tau} = \frac{1}{n_1}\sum_{i=1}^{n} Z_i Y_i - \frac{1}{n_0}\sum_{i=1}^{n}(1-Z_i)Y_i$$

$$= \frac{1}{n_1}\sum_{i=1}^{n} Z_i Y_i(1) - \frac{1}{n_0}\sum_{i=1}^{n}(1-Z_i)Y_i(0)$$

$$\Rightarrow E(\hat{\tau}) = \frac{1}{n_1}\sum_{i=1}^{n} \underbrace{E(Z_i)}_{\frac{n_1}{n}} Y_i(1) - \frac{1}{n_0}\sum_{i=1}^{n}\left(1-\underbrace{E(Z_i)}\right)Y_i(0)$$

$$= \frac{1}{n}\sum_{i=1}^{n} Y_i(1) - \frac{1}{n}\sum_{i=1}^{n} Y_i(0) = \tau$$

下次证明

方差: $$Var(\hat{\tau}) = \frac{S^2(1)}{n_1} + \frac{S^2(0)}{n_0} - \frac{S^2(\tau)}{n}$$

其中 $$S^2(\underset{0}{1}) = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i(\underset{0}{1}) - \bar{Y}(\underset{0}{1})\right)^2$$

$$S^2(\tau) = \frac{1}{n-1}\sum_{i=1}^{n}(\tau_i - \tau)^2$$

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回顾: $Y_i \mid Z_i = 1 \overset{iid}{\sim} [\mu_1, \sigma_1^2]$

$Y_i \mid Z_i = 0 \overset{iid}{\sim} [\mu_0, \sigma_0^2]$

$$\hat{\tau} = \hat{Y}(1) - \bar{Y}(0) \sim \left[\mu_1 - \mu_0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_0^2}{n_0}\right]$$

方差估计：$\hat{V} = \frac{\hat{S}^2(1)}{n_1} + \frac{\hat{S}^2(0)}{n_0}$

样本方差

保守估计：$E(\hat{V}) - var(\hat{\tau}) = \frac{S^2(\tau)}{n} \geq 0$

不保守：$\tau_i = \tau \quad \forall i$

区间估计：$\hat{\tau} \pm 1.96\sqrt{\hat{V}}$

正态

$$\begin{cases} ① \quad \hat{\tau} \overset{a}{\approx} N(\tau, var(\hat{\tau})) \\ ② \quad \hat{V} \approx \frac{S^2(1)}{n_1} + \frac{S^2(0)}{n_0} \end{cases}$$

随机性 ～ Z

Hoeffding's combinatorial CLT

阅读文献 $A = (a_{ij})$

$\pi : \{1 \cdots n\} \to \{1 \cdots n\}$

$$\sum_{i=1}^{n} a_{i\pi(i)}$$ 置换,

$\approx$ Normal

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Neyman's final results:

$\hat{\tau}$, $\hat{V}$