Constructible sheaves

In the last part of the course, we discuss recent approaches to mirror symmetry of toric varieties, using the "coherent-constructible correspondence".

Take a real manifold M. Recall that the locally constant sheaf $\underline{\mathbb{R}}_M$ is not an $\mathcal{O}_M$-module, and in particular not coherent. On the other hand, it is a first example of a constructible sheaf.

A (Whitney) stratification of M is a cover $M = \bigcup_i N_i$ by disjont sets N_i satisfying certain properties involving their closures and intersections.

Consider the category $\mathrm{Mod}(\mathbb{R}_M)$ of $\mathbb{R}_M$-modules.

<u>Def</u> A sheaf $E \in \mathrm{Mod}(\mathbb{R}_M)$ is called <u>constructible</u> if there exists a stratification such that $E|_{N_i}$ is locally constant of finite rank.

Removing the finiteness assumption, we have quasi-constructible sheaves.

Notation Write Con(M) and QCon(M) for the associated categories.

Ex For $M = \mathbb{R}$ and $N = [a,b] \subset \mathbb{R}$ with $i : N \hookrightarrow M$ then $E = i_* \mathbb{R}_N^{\oplus J}$ is quasi-constructible, and furthermore constructible for J finite.

Microsupport

There is an alternative, symplectic, characterization of quasi-constructibility, using the following (difficult) definition, refining the notion of support of a sheaf.

Def The microsupport $\mu Supp(E) \subset T^*M$ of $E \in Mod(\underline{\mathbb{R}}_M)$ is determined by the following:

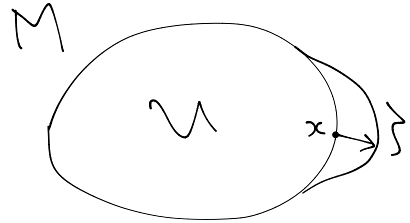
$(x, \xi) \notin \mu Supp(E)$ when there exists open $\mathcal{U} \ni (x, \xi)$ with projection $\pi(\mathcal{U})$ under $\pi : (x, \xi) \mapsto x$

such that $\boxed{H^*_S(E|_{\pi(\mathcal{U})})_y = 0}$

for any point $y \in \pi(\mathcal{U})$, where $S = \{z \mid \psi(z) \geq \psi(y)\}$ for any smooth $\psi : \pi(\mathcal{U}) \to \mathbb{R}$ with $Graph(d\psi|_{\pi(\mathcal{U})}) \subset \mathcal{U}$

<u>Rem</u> H^*_S here denotes "cohomology with support".

The idea of the definition is to measure the directions (x, ξ) in which when an open set U "grows" as shown, the cohomology $H^*(E|_U)$ changes

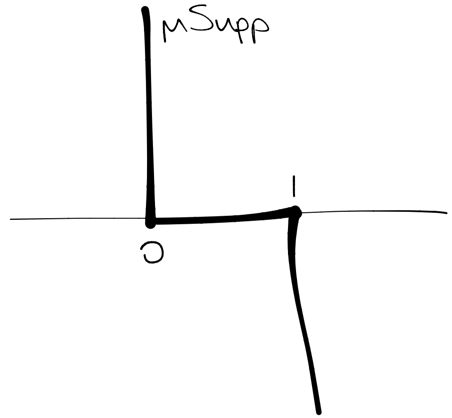


<u>Note</u> For alternative characterizations of μSupp, and the following statement, see Kashiwara–Shapira, "Sheaves on manifolds"

<u>Thm</u> ("Involutivity") For $E \in \mathrm{Mod}(\underline{\mathbb{R}}_M)$,

E quasi-constructible iff $\mu\mathrm{Supp}(E)$ Lagrangian.

<u>Ex</u> $E = i_* \underline{\mathbb{R}}_{[0,1]}$



Constructible sheaves and Fukaya categories

Above we saw a first relation between (quasi)constructible sheaves and (singular) Lagrangians. This can be extended to a relation between a category $\mathrm{Con}(M)$ and a Fukaya category of Lagrangians on T^*M.

The first general approach to this was given by Nadler and Zaslow, '06, and continued in further work.

<u>Thm</u> $D(Q\mathrm{Con}(M)) \xrightarrow{\sim} D\mathrm{Fuk}^{wr}(T^*M)$

The category $DFuk^{wr}$ is the "wrapped" Fukaya category. Its objects are Lagrangians, with extra structure. The terms "wrapped" refers to conditions on behaviour of Lagrangians in the limit $|\xi| \to \infty$ for $(m, \xi) \in T^*M$. Morphisms are defined similarly to HF^* above.

We indicate how the functor is defined on objects. Recall for $E \in QCon(M)$, $\mu Supp(E)$ is Lagrangian, but we saw it can be singular. This introduces difficulties in defining HF^*. On the other hand, we may obtain smooth Lagrangians as follows.

Take $E = i_* \mathbb{R}_U$ for $i : U \hookrightarrow M$ with open $U = M - Z$. Choose $z : M \to \mathbb{R}_{\geq 0}$ with $Z = \mathrm{Zeroes}(z)$, and let

$$L = \mathrm{Graph}(d\log z) \subset T^*M.$$

<u>Ex</u> $U = \mathbb{R} - \{0\} \subset \mathbb{R}$. Let $z : x \to x^2$.

Then $d\log z = 2/x$.