2020-12-4 Kähler geometry (Yaxiong Liu)

V> Ding stability and twisted KE current

$(X.\omega_0)$ cpt. Kähler. L. ample $\mathbb{Q}$-l.b. $\omega_0 \in c_1(L)$

θ klt with $c_1(X.\theta) = c_1(L)$

Thm 5.1. If the D_θ is coercive on $\mathcal{E}^1$, then $(X.L)$ is uniformly Ding stable w.r.t θ

If θ is semiample, the converse also holds.

Cor 5.2. If $\theta \geq 0$, then

- if $c_1(L)$ contains a θ-twisted KE current, then $(X.L)$ is Ding-semistable w.r.t. θ
- If $(X.L)$ is uniformly Ding-stable, then $c_1(L)$ contains a θ-twisted KE current.

- slopes of functional

(c.f. Boucksom-Hisamoto-Jonsson.
uniform K-stability and asymptotics of energy func.
in Kähler geometry. [19]
)

Lemma 5.3 (Thm 3.6 of [BHJ. 19])

If $u:\mathbb{R}_{\geq 0}\to PSH$ is psh ray with algebraic singularities,

then : (i) $E(u_t)=tE^{NA}(u_{NA})+O(1)$

(ii). $J(u_t)=tJ^{NA}(u_{NA})+O(1)$.

$$F^{NA}(u_{NA})=\lim_{t\to+\infty}t^{-1}F(u_t)=:F'^{\infty}(u).$$

Thm 5.4. $\forall$ psh ray $u:\mathbb{R}_{\geq 0}\to\mathcal{E}^1$ with linear growth, $L_\theta^{NA}(u_{NA})$ is finite, and equal to the integrability threshold

$$\sup\left\{c\in\mathbb{R}\,\middle|\,\int_1^\infty e^{2(ct-L_\theta(u_t))}dt<+\infty\right\}$$

Rem. If $\theta\geq 0$, $L_\theta(u_t)$ is convex, the integrability threshold equal to the slope $\lim_{t\to\infty}t^{-1}L_\theta(u_t)$.

Pf of Thm 5.1

Assume $D_\theta\geq\varepsilon J-C$ on $\mathcal{E}^1$. for some $\varepsilon, C>0$.

claim: $D_\theta^{NA}\geq\varepsilon J^{NA}$ on $\mathcal{H}^{NA}$

By lemma 4.6, $\forall\varphi\in\mathcal{H}^{NA}$, $\varphi=u_{NA}$ for some psh ray $u:\mathbb{R}_{\geq 0}\to\mathcal{E}^1$ with algebraic singularities

By lemma 5.3, $E(U_t) = t\, E^{NA}(u_{NA}) + O(1)$

$$J(U_t) = t\, E^{NA}(u_{NA}) + O(1)$$

By Thm 5.4. $L_\theta^{NA}(\varphi) = \sup\{c \in \mathbb{R} \mid \int_1^\infty e^{2(ct - L_\theta^{NA}(U_t))} dt < \infty\}$

By coercivity. $L_\theta(U_t) \geq E(U_t) + \varepsilon J(U_t) - C$

$$= t(E^{NA}(\varphi) + \varepsilon J^{NA}(\varphi)) + O(1)$$

$$\Rightarrow \quad L_\theta^{NA}(u_{NA}) \geq E^{NA}(u_{NA}) + \varepsilon J^{NA}(u_{NA})$$

i.e. $D_\theta^{NA} \geq \varepsilon J^{NA}$ on $\mathcal{H}^{NA}$.

For the converse direction, we need some lemma.

Let $U: \mathbb{R}_{>0} \to \Sigma'$ be a psh ray with $U_t \leq O(1)$ at $t \to +\infty$.

So that U defines a quasi-psh function on $X \times \mathbb{D}$ with multiplier ideal sheaf $\mathfrak{a}_m := \mathcal{J}(mU)$ cosupported in the central $X \times \{0\}$

By S^1-invariance of U, then $\mathfrak{a}_m$ is S^1-inv. ($e^{i\theta} \curvearrowright \mathfrak{a}_m \Rightarrow \mathfrak{a}_m$)

and hence uniquely extends to a $\mathbb{C}^\times$-invariant coherent ideal sheaf on $X \times \mathbb{C}$.

Lemma 5.6. $\exists\, m_0 \gg 1$, s.t. the sheaf $\mathcal{O}((m+m_0)p_1^*L)\otimes \mathfrak{a}_m$ is generated by global section on $X\times\mathbb{C}$, for each $m\geq 1$.

(c.f. Demailly-Ein-Lazarsfeld;

A subadditivity property of multiplier ideals.

Michigan Math. 2000.)

Lemma 5.7. Set $\varphi_m := (m+m_0)^{-1}\varphi_{\mathfrak{a}_m}$. Then.

(i) $\varphi_m \in \mathcal{H}^{NA}$

(ii) $u_{NA} \leq \varphi_m \leq \frac{m}{m+m_0} u_{NA} + \frac{1}{m}(A_X+1)$ on $X_{\mathbb{Q}}^{div}$

(iii). $L_\theta^{NA}(u_{NA}) = \lim_{m\to\infty} L_\theta^{NA}(\varphi_m)$

Pf. (i) OK. $\mathfrak{a}_m \rightsquigarrow (\mathcal{X}_m, \mathcal{L}_m)$ semiample

$$\mathcal{X}_m = Bl_{\mathfrak{a}_m}(X\times\mathbb{C}) \xrightarrow{\mu_m} X\times\mathbb{C}$$

$$\mathcal{L}_m = \mu_m^*(L) - \frac{1}{m+m_0}E_m.$$

(ii). $w(\mathcal{J}(mU)) \leq m\,W(U) \leq w(\mathcal{J}(mU)) + \underset{\substack{\parallel \\ A_{X\times\mathbb{C}}}}{A(w)}$

$$\underset{w=\sigma(v)}{\overset{\text{by def}}{\Longrightarrow}} (m+m_0)\varphi_m(v) \geq m\,u_{NA}(v) \geq (m+m_0)\varphi_m(v) - A_X(v) - 1.$$

$$\begin{array}{c} \shortparallel \\ -\sigma(v)(u) \end{array}$$

which implies (ii). since $\varphi_m \leq 0$. ($\varphi_m(v_{triv}) = 0$)

(iii). By (ii). $L_\theta^{NA}(\varphi_m) = \inf_{X_{\mathbb{Q}}^{div}} \{A_\theta + \varphi_m\} \geq \inf_{X_{\mathbb{Q}}^{div}} \{A_\theta + u_{NA}\}$

$$= L_\theta^{NA}(u_{NA})$$

Pick $\varepsilon > 0$, and $v \in X_{\mathbb{Q}}^{div}$. s.t. $A_\theta(v) + u_{NA}(v) \leq L_\theta^{NA}(u_{NA}) + \varepsilon$

$$L_\theta^{NA}(\varphi_m) \geq L_\theta^{NA}(u_{NA}) \geq \underline{A_\theta}(v) + u_{NA}(v) - \varepsilon$$

$$\geq \underline{L_\theta^{NA}(\varphi_m)} + u_{NA}(v) - \underline{\varphi_m(v)} - \varepsilon$$

which implies (iii'). since $\varphi_m(v) \to u_{NA}(v)$. by (ii). $\square$

Lemma 5.8. $\forall m$. $E^{NA}(\varphi_m) \geq \lim_{t \to \infty} t^{-1} E(u_t)$

Pf. By lemma 4.6. $\exists$ psh ray $u^m : \mathbb{R}_{\geq 0} \to \mathcal{E}^1$ with alg. sing.

s.t. $u^m_{NA} = \varphi_m$, $\quad E(u^m_t) = t E^{NA}(\varphi_m) + O(1)$

since $u_{NA} \leq \varphi_m$. $\exists C > 0$. s.t. $u_t \leq u^m_t + C$. for $t \geq 1$

By monotonicity of E, $\quad E(u_t) \leq E(u^m_t) + O(1) = t E^{NA}(\varphi_m) + O(1)$

$\square$

cont. to prove Thm 3.1

Arguing by contradiction.

suppose $\theta \geq 0$, $\Delta_\theta^{NA} \geq \varepsilon J^{NA}$ on $\mathcal{H}^{NA}$

Assume D_θ is not coercive,

Step 1. construct a destabilizing geodesic ray U for D_θ

by Cor 2.17, $D_\theta(U_t) \leq D_\theta(U_0)$

$$D_\theta'^{\infty}(U) = \lim_{t\to\infty} \frac{D_\theta(U_t)}{t} \leq 0.$$

without loss of generality, we can assume $U_t \leq O(1)$ as $t \to +\infty$ with $E(U_t) = ct$, $c<0$.

Step 2. Multiplier Approximation.

$$\to (\mathcal{X}_m, \mathcal{L}_m) \in \mathcal{H}^{NA}.$$
$$\quad \parallel$$
$$\quad \varphi_m.$$

Step 3.

$$D^{NA}(\varphi_m) \geq \varepsilon J^{NA}(\varphi_m)$$
$$\parallel$$
$$-E^{NA}(\varphi_m) + L_\theta^{NA}(\varphi_m)$$

$$\Longleftrightarrow L_\theta^{NA}(\varphi_m) \geq (1-\varepsilon) E^{NA}(\varphi_m)$$

$$\begin{array}{ccc} & \downarrow & \vee \\ & L_0^{NA}(U^{NA}) & (1-\varepsilon)\,E^{\infty}(U) \\ & \| & \| \\ & L_0'^{\infty}(U) & (1-\varepsilon)c \\ & \wedge & \\ D'^{\infty}(U)\le 0 & E'^{\infty}(U) & \\ \| & \| & \\ -E'^{\infty}(U)+L_0'^{\infty}(U) \ \Rightarrow & c & \Rightarrow\ c \ge (1-\varepsilon)c \end{array}$$

contradiction.

$\square$.

• The Berkovich analytification

X^{NA} = topological space whose points can be understood as semivaluation on X.

i.e. valuation $v : \mathbb{C}(Y)^* \longrightarrow \mathbb{R}$. trivial on $\mathbb{C}$

where Y is the subvar. of X

In particular, $X^{\text{div}}_{\mathbb{Q}} \subseteq X^{NA}$

The topology of X^{NA} is generated by functions of the form $\hat{f} : v \longmapsto v(f)$, with f a regular function on some Zariski open set $U \subseteq X$.

One shows that X^{NA} is cpt, Hausdorff. and $X^{div}_{\mathbb{Q}} \subseteq X^{NA}$ is dense. $p_1 : X \times \mathbb{C} \to X \rightsquigarrow$

$$\begin{tikzcd} (X\times\mathbb{C})^{NA} \arrow[r, "p_1"] & X^{NA} \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

(c.f. Boucksom-Jonsson:
Singular semipositive metrics on line bundles on var. over trivially valued fields
arXiv: 1801.08229 v1).

$\forall$ T.C. $(\mathcal{X}, \mathcal{L})$. $\varphi_{(\mathcal{X},\mathcal{L})} : X^{NA} \to \mathbb{R}$.

$\mathbb{C}^*$-inv. ideal on $X \times \mathbb{C} \rightsquigarrow \varphi_{\mathfrak{a}} : X^{NA} \to [-\infty, +\infty)$ by

$$\varphi_{\mathfrak{a}}(v) = -\sigma(v)(\mathfrak{a}).$$

$\varphi_{\mathfrak{a}}$ is continuous, and finite valued

$\Longleftrightarrow$ $\mathfrak{a}$ is cosupported on $X \times \{0\}$.

<u>Def</u>: An <u>L-psh function</u> is a functional $\varphi : X^{NA} \to \mathbb{R}, \not\equiv \infty$ that can be written as the limit of a decreasing sequence in $\mathcal{H}^{NA}$

<u>Thm 6.2</u>. $\forall$ psh ray $U : \mathbb{R}_{\geq 0} \to$ PSH of linear growth, the

function $U_{NA}: X^{div}_{\mathbb{Q}} \to \mathbb{R}$. admits a unique extension to a function in PSH^{NA}

$E^{NA}: \mathcal{H}^{NA} \to \mathbb{R}$, admits a unique extension to a monotone, u.s.c. functional

$$E^{NA}: PSH^{NA} \longrightarrow [-\infty, +\infty)$$

$$\varphi \longmapsto E^{NA}(\varphi) = \inf \left\{ E^{NA}(\psi) \,\middle|\, \begin{array}{l} \psi \in \mathcal{H}^{NA} \\ \psi \geq \varphi \end{array} \right\}$$

finite energy: if $E^{NA}(\varphi) > -\infty$

$\mathcal{E}^{1,NA}$ = the space of finite energy L-psh function

· Maximal geodesic ray:

Thm 6.4 $\forall$ psh ray $U: \mathbb{R}_{\geq 0} \to \mathcal{E}^1$ of linear growth, the associated L-psh function $U_{NA} \in \mathcal{E}^{1,NA}$.

$$E^{NA}(U_{NA}) \geq E'^{\infty}(U) > -\infty$$

Def: A psh geodesic ray U is maximal if any psh ray of linear growth $V: \mathbb{R}_{>0} \to \mathcal{E}^1$ with $\lim_{t \to 0} V_t \leq U_0$ and $V_{NA} \leq U_{NA}$, then $V \leq U$.

Thm 6.6: $\forall u \in \mathcal{E}^1$, $\varphi \in \mathcal{E}^{1,NA}$, there exists a unique maximal geodesic ray U. s.t. $U_0 = u$. and $U_{NA} = \varphi$.

Cor 6.7 U is maximal iff $E^{NA}(U_{NA}) = \lim_{t\to\infty} \frac{E(U_t)}{t}$

Thm (Chi Li).

M^{NA} is coercive on $\mathcal{E}^{1,NA}$, then M is coercive on $\mathcal{E}^1$

Key lemma: $M'^{\infty}(U) < +\infty$, $\Rightarrow$ U is maximal.