PERFECTOIDS AND GALOIS COHOMOLOGY: A PEDAGOGICAL INTRODUCTION TO p-ADIC HODGE THEORY

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ABSTRACT. Since Tate proposed the famous Hodge-Tate decomposition conjecture in the 1960s, p-adic Hodge theory has undergone profound and continuous development over the subsequent sixty years, with new ideas and tools constantly emerging. Among these, the theory of perfectoid rings is one of the most striking breakthroughs and has gradually become a foundational language for understanding modern p-adic geometry.

This lecture notes starts from a historical perspective to explain the role and status of perfectoids in p-adic geometry, and uses this as a main thread to introduce the basic framework and core ideas of p-adic Hodge theory. We will present the deep and beautiful techniques of p-adic geometry to graduate students and advanced undergraduates in a friendly and detailed manner.

More specifically, we begin with how Tate used local class field theory to compute Galois cohomology in the discretely valued case, then introduce the notion of perfectoids and prove several key results, including the tilting correspondence, cohomological descent in the arc topology, and the almost purity theorem. Using these tools, we compute the cohomology of the fundamental group of smooth algebraic varieties, which has been a central topic of p-adic Hodge theory over the past sixty years. Finally, we discuss the extension of these methods to general (non-discrete) valuation rings and look ahead to the future development of p-adic Hodge theory.

CONTENTS

1. A GLANCE AT p-ADIC HODGE THEORY

1.a. **Hodge decomposition.** One of the most fundamental theorem in complex geometry concerning about the singular cohomology of complex manifolds is the following so-called *Hodge decomposition.*

Theorem 1.1. *Let X be a projective smooth variety over $\mathbb{C}$. Then, there is a canonical decomposition*

$$H^n_{\mathrm{sing}}(X,\mathbb{C}) \cong \bigoplus_{i+j=n} H^j(X,\Omega^i_{X/\mathbb{C}}). \tag{1.1.1}$$

The standard proof used essentially techniques in analysis: the n-th de Rham cohomology classes are represented uniquely by n-th harmonic forms ([Voi02, 5.23]), and the latter can be decomposed into direct sums of (i,j)-type harmonic forms ([Voi02, 6.10]), the space of which is canonically isomorphic to $H^j(X,\Omega^i_{X/\mathbb{C}})$ ([Voi02, 6.18]).

Let's take a view from the p-adic geometry.

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1.b. $\mathbb{C}$ **and** $\mathbb{C}_p$**.** If we complete the field of rational numbers $\mathbb{Q}$ with respect to the archimedean norm, we obtain the field of real number $\mathbb{R}$; if we complete $\mathbb{Q}$ with respect to a non-archimedean norm, we obtain the field of p-adic numbers $\mathbb{Q}_p$. Recall that $\mathbb{R}$ and $\mathbb{Q}_p$ are the only two types of completions that $\mathbb{Q}$ has by a theorem of Ostrowski.

Recall that the non-archimedean norm on $\mathbb{Q}_p$ corresponds to the discrete valuation ring

$$\mathbb{Z}_p = \lim_{n\to\infty} \mathbb{Z}/p^n\mathbb{Z} \tag{1.1.2}$$

where the valuation map is

$$\begin{aligned} v_p : \mathbb{Z}_p &\longrightarrow \mathbb{N}\cup\{\infty\}, \\ p^n u &\longmapsto n,\ \forall n\in\mathbb{N} \text{ and } u\in\mathbb{Z}_p^\times, \\ 0 &\longmapsto \infty. \end{aligned} \tag{1.1.3}$$

The discrete valuation field $\mathbb{Q}_p$ is the fraction field of $\mathbb{Z}_p$ given by inverting p: $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. We refer to [Bou06a] for basic theory on valuation rings.

Taking an algebraic closure of $\mathbb{R}$, we obtain the field of complex numbers $\mathbb{C}$ which has degree 2 over $\mathbb{R}$; taking an algebraic closure of $\mathbb{Q}_p$, we obtain an infinite Galois extension $\overline{\mathbb{Q}}_p$. Notice that $\overline{\mathbb{Q}}_p$ is still a valuation field (but not discrete) with respect to the valuation ring $\overline{\mathbb{Z}}_p$, where the latter is the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$ (see [Bou06a, VI, §8.6, Proposition 6] and [Sta25, 04GH]). The extended valuation map is

$$\begin{aligned} v_p : \overline{\mathbb{Z}}_p &\longrightarrow \mathbb{Q}_{\geq 0}\cup\{\infty\}, \\ x &\longmapsto v_p(\mathrm{N}_{\mathbb{Q}_p(x)/\mathbb{Q}_p}(x))/[\mathbb{Q}_p(x):\mathbb{Q}_p]. \end{aligned} \tag{1.1.4}$$

But $\overline{\mathbb{Q}}_p$ is not complete with respect to its valuation (i.e., $\overline{\mathbb{Z}}_p$ is not p-adically complete, $\overline{\mathbb{Z}}_p \neq \widehat{\overline{\mathbb{Z}}}_p = \lim_{n\to\infty} \overline{\mathbb{Z}}_p/p^n\overline{\mathbb{Z}}_p$ as $1+p^{1+1/p}+p^{2+1/p^2}+\cdots\in\widehat{\overline{\mathbb{Z}}}_p$ is transcendental over $\mathbb{Q}_p$). We put $\mathbb{C}_p = \widehat{\overline{\mathbb{Z}}}_p[1/p]$, which is a complete algebraically closed valuation field by Krasner's lemma.

Notice that $\mathbb{C}$ and $\mathbb{C}_p$ have the same cardinalities as that of $\mathbb{R}$. Hence, they have the same transcendental degree $\mathfrak{a}$ over $\mathbb{Q}$ and thus they are both algebraic closures of the purely transcendental extension $\mathbb{Q}(T_i|i\in\mathfrak{a})$ of $\mathbb{Q}$ (see [Sta25, 030D, 09GV]). In conclusion, there is a field isomorphism

$$\mathbb{C}\cong\mathbb{C}_p. \tag{1.1.5}$$

Although these two fields are isomorphic, the way they are defined actually endows them totally different topology. It is clear that the Euclidean topology on $\mathbb{C}$ is connected, while the non-archimedean topology on $\mathbb{C}_p$ is totally disconnected.

1.c. **Hodge-Tate decomposition.** The same thing happens to a projective smooth variety X over $\mathbb{C}_p$. When fixing a field isomorphism $\mathbb{C}\cong\mathbb{C}_p$, we have an isomorphism of schemes $X_{\mathbb{C}}\cong X$. However, the Euclidean topology on $X_{\mathbb{C}}$ as a manifold is totally different from the étale or Zariski topology on X as a scheme.

But a surprising fact that these two different topology actually give the same cohomological invariants (which thus reflects the geometric nature of X) as long as we fix $\mathbb{C}\cong\mathbb{C}_p$:

$$H^n_{\mathrm{sing}}(X_{\mathbb{C}},\mathbb{C})\cong H^n_{\text{ét}}(X,\mathbb{C}_p), \tag{1.1.6}$$

where the latter is defined as $\mathbb{C}_p\otimes_{\mathbb{Z}_p}(\lim_{r\to\infty} H^n_{\text{ét}}(X,\mathbb{Z}/p^r\mathbb{Z}))$. This is Artin's comparison theorem, see [SGA 4_{III}, XI.4.4].

Therefore, the terms involved in the Hodge decomposition (1.1.1) actually come from algebraic geometry and Theorem 1.1 implies that

$$H^n_{\text{ét}}(X,\mathbb{C}_p)\cong\bigoplus_{i+j=n} H^j(X,\Omega^i_{X/\mathbb{C}}). \tag{1.1.7}$$

A priori, this isomorphism depends on the arbitrary choice of the field isomorphism $\mathbb{C}\cong\mathbb{C}_p$. But both sides are algebraic, we naturally ask

Question 1.2. *Is there a purely algebraic proof or a canonical construction of* (1.1.7)*? If so, how is the valuation ring structure* $\mathbb{Q}_p\supseteq\mathbb{Z}_p$ *involved here?*

This question is the central theme of p-adic Hodge theory. It started by Tate [Tat67], where he explained what does the "canonical construction" should mean and solve the question for abelian varieties over a finite extension K of $\mathbb{Q}_p$ with good reductions. Although it looks like a very special case, his strategy is generalized greatly by Faltings [Fal88] to solve the question for proper smooth varieties over K. Thus, the canonical decomposition (1.1.7) is also called the *Hodge-Tate decomposition.* While Tate's proof specializes only to abelian varieties, Faltings invented a bunch of new techniques to realize Tate's strategy over general smooth varieties, including *almost purity theorem* and *Galois cohomology computation*. Nowadays, Faltings' techniques have been developed and subsumed within *perfectoid theory* after Scholze [Sch12, Sch13], which we are going to explain to graduate and undergraduate students in a friendly and detailed manner in this lecture series.

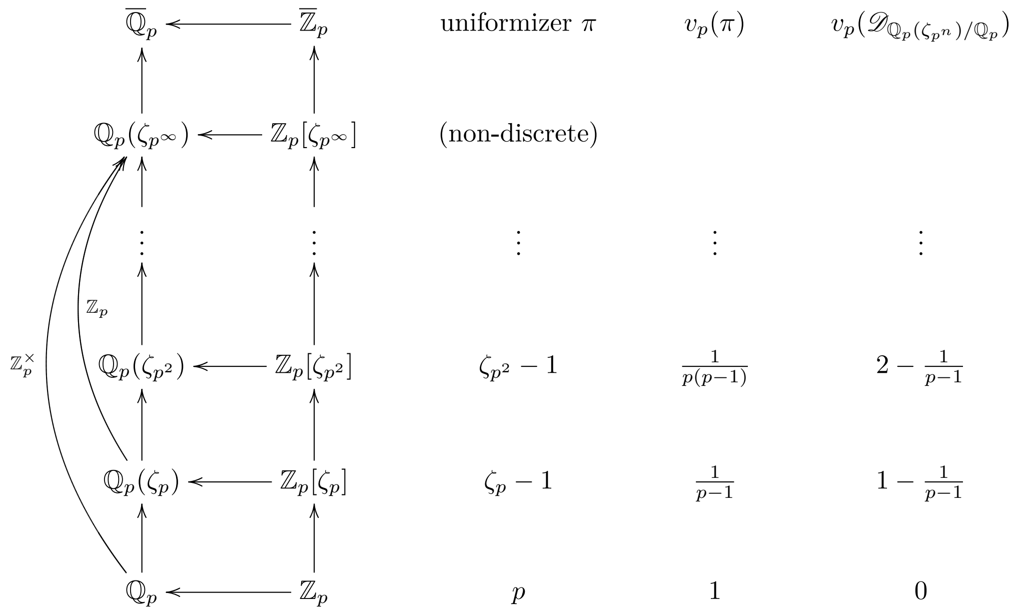
It would be too technical and difficult if we start directly with these deep techniques. Instead, we begin with Tate's groundbreaking work [Tat67] to trace the origins of these modern techniques.

Question 1.2 essentially requires a good understanding of the p-adic cohomology. The key making p-adic cohomology distinguished is the valuation ring structure $\mathbb{Q}_p \supseteq \mathbb{Z}_p$ and the ramification above it. Before we try to understand ramification above X following Faltings, let's simply understand ramification above the single point $\mathbb{Q}_p$ following Tate.

1.d. Ramification of $\overline{\mathbb{Q}}_p$ over $\mathbb{Q}_p$.

Example 1.3. Consider a compatible system of primitive p^n-th roots of unity $(\zeta_{p^n})_{n\in\mathbb{N}}$, i.e., $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ and $\zeta_p \neq \zeta_1 = 1$. Using basics in algebraic number theory, we can prove that $\mathbb{Q}_p(\zeta_{p^n})$ is a totally ramified extension of $\mathbb{Q}_p$ with valuation ring $\mathbb{Z}_p[\zeta_{p^n}]$ (see [Ser79] or [He25, 5.4]). This explicit expression of valuation rings (or integral closures) enables us to compute every invariant about the ramification behavior. For instance, $\zeta_{p^n} - 1$ is a uniformizer of $\mathbb{Q}_p(\zeta_{p^n})$ with valuation $v_p(\zeta_{p^n} - 1) = \frac{1}{p^{n-1}(p-1)}$, and the valuation of the different ideal $\mathscr{D}_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}$ is $n - \frac{1}{p-1}$ for $n \geq 1$.

(1.3.1)



But how to understand ramification above $\mathbb{Q}_p(\zeta_{p^\infty})$? Tate answers this question by the following theorem.

Theorem 1.4 ([Tat67, §3])**.** *Let K be a complete discrete valuation field extension of $\mathbb{Q}_p$, K_∞ a totally ramified $\mathbb{Z}_p$-extension of K. Let K_n be the subfield of K_∞ corresponding to the closed subgroup $p^n\mathbb{Z}_p$ of $\mathrm{Gal}(K_\infty/K) = \mathbb{Z}_p$ for any $n \in \mathbb{N}$.*

(1) *(Regular ramification) There is a constant c and a bounded sequence $(a_n)_{n\in\mathbb{N}}$ of integers such that for any $n \in \mathbb{N}$, the valuation of the different ideal*

(1.4.1) $$v_p(\mathscr{D}_{K_n/K}) = n + c + p^{-n}a_n.$$

(2) *(Almost unramification) For any finite field extension L of K, if we denote by L_n the composite of L with K_n for any $n \in \mathbb{N}\cup\{\infty\}$. Then,*

(1.4.2) $$v_p(\mathscr{D}_{L_n/K_n}) \to 0 \text{ when } n \to \infty.$$

In other words, $\mathscr{D}_{L_\infty/K_\infty}$ ([He25, 4.1.2]) *is equal to $\mathfrak{m}_{L_\infty}$ or $\mathcal{O}_{L_\infty}$ (we call L_∞ is* almost unramified *over K_∞).*

Remark 1.5. (1) Tate's proof to these results essentially used higher ramification groups and local class field theory.

(2) Tate used these results to compute the p-adic cohomology for $\mathbb{Q}_p$, i.e., $H^n_{\text{ét}}(\mathrm{Spec}(\mathbb{Q}_p), \mathbb{C}_p)$, see [Tat67, §3.3].

(3) Faltings adopted the same strategy to understand the ramification above a smooth variety X. Roughly speaking, for a small smooth algebra R over $\mathbb{C}_p$, he constructed a "regularly ramified" tower $R \to R_\infty$ such that there is no more ramification beyond R_∞ in the almost sense. In fact, this R_∞ is "pre-perfectoid" and we will show the almost purity theorem for perfectoid rings and Galois cohomology computation for this specific tower $R \to R_\infty$.

2. Definition of Perfectoids

2.a. **Review of deformation theory.** We refer to [Ill71] and [Ill72] for a systematic development of deformation theory and suggest to read Grothendieck's definitions of smoothness [EGA IV_4, §17] or Illusie's expository notes [Ill96, §1,2] at first before jumping into the most general theory.

Recall that a *thickening* of affine schemes is a closed immersion $\mathrm{Spec}(R_0) \to \mathrm{Spec}(R)$ such that $R_0 = R/I$ with $I^2 = 0$. For example, each closed immersion in $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}/p^3\mathbb{Z}) \to \cdots$ is a thickening.

Question 2.1. *Given a flat R_0-algebra A_0, is there a flat R-algebra A with $A_0 = A \otimes_R R_0$?*

(2.1.1)
$$\begin{array}{ccc} \mathrm{Spec}(A) & \leftarrow\cdots & \mathrm{Spec}(A_0) \\ \vdots & \llcorner & \downarrow \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R_0) \end{array}$$

Example 2.2. Consider the baby case $A_0 = R_0[T]$. Then, there is an obvious lifting $A = R[T]$.

(2.2.1)
$$\begin{array}{ccc} R[T] & \longrightarrow & R_0[T] \\ \uparrow & & \uparrow \\ R & \longrightarrow & R_0 \end{array}$$

In fact, any flat lifting of $R_0[T]$ is isomorphic to $R[T]$: let A' be a flat R-algebra with $A'/IA' = R_0[T]$. Then, we consider the R-algebra homomorphism $R[T] \to A'$ sending T to $T' \in A'$ a lifting of $T \in R_0[T]$. It is an isomorphism by the exact sequence $0 \to IA' \to A' \to A'/IA' \to 0$ and the identity $IA' = I \otimes_R A' = I \otimes_{R_0} A'/IA'$. Moreover, the automorphism group of the flat lifting $R[T]$ is isomorphic to $IA = I \otimes_{R_0} A_0$, where each $a \in IA$ corresponds to the automorphism sending T to $T + a$.

In general, there is a standard simplicial resolution of A_0 by free algebras over R_0 ([Ill71, I.1.5.5.6], see also [Sta25, 08N8])

(2.2.2)
$$\cdots \Rrightarrow P_1 = R_0[R_0[A_0]] \rightrightarrows P_0 = R_0[A_0] \longrightarrow A_0.$$

The *cotangent complex* of A_0 over R_0 is the associated complex of A_0-modules ([Ill71, II.1.2.3], see also [Sta25, 08PL])

(2.2.3)
$$\mathbb{L}_{A_0/R_0} = (\cdots \to \Omega^1_{P_1/R_0} \otimes_{P_1} A_0 \to \Omega^1_{P_0/R_0} \otimes_{P_0} A_0).$$

Theorem 2.3 ([Ill71, III.2.1.2.3])**.** *For the lifting problem* 2.1*, we have:*

(1) *There is an element $\omega \in \mathrm{Ext}^2_{A_0}(\mathbb{L}_{A_0/R_0}, A_0 \otimes_{R_0} I)$, which vanishes if and only if there exists a flat lifting A.*
(2) *When $\omega = 0$, the set of isomorphism classes of all the flat liftings A is a torsor under $\mathrm{Ext}^1_{A_0}(\mathbb{L}_{A_0/R_0}, A_0 \otimes_{R_0} I)$.*
(3) *The automorphism group of a flat lifting A is canonical isomorphic to $\mathrm{Ext}^0_{A_0}(\mathbb{L}_{A_0/R_0}, A_0 \otimes_{R_0} I)$.*

In particular, when $A_0 = R_0[T]$, we see that $\mathbb{L}_{A_0/R_0} = \Omega^1_{A_0/R_0} \cong A_0$ is a free A_0-module of rank 1. Thus, we can deduce 2.2 from 2.3.

Question 2.4. *Given a morphism of flat R_0-algebras $f_0 : A_0 \to A'_0$ together with fixed flat R-algebras A and A' with $A_0 = A \otimes_R R_0$ and $A'_0 = A' \otimes_R R_0$, is there a morphism $f : A \to A'$ with $f_0 = f \otimes_R R_0$?*

(2.4.1)
$$\begin{array}{ccc} \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(A'_0) \\ \Big\downarrow f & \lrcorner & \Big\downarrow f_0 \\ \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(A_0) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R_0) \end{array}$$

Theorem 2.5 ([Ill71, III.2.2.2])**.** *For the lifting problem* 2.4*, we have:*

(1) *There is an element $\omega \in \mathrm{Ext}^1_{A_0}(\mathbb{L}_{A_0/R_0}, A'_0 \otimes_{R_0} I)$, which vanishes if and only if there exists a lifting f.*
(2) *When $\omega = 0$, the set of isomorphism classes of all the liftings f is a torsor under $\mathrm{Ext}^0_{A_0}(\mathbb{L}_{A_0/R_0}, A'_0 \otimes_{R_0} I)$.*

2.b. **Universal p-deformation: Witt rings.** We fix a *perfect* $\mathbb{F}_p$-algebra R in this subsection, i.e., the Frobenius map $\mathrm{Frob} : R \to R$ sending x to x^p is an isomorphism.

Lemma 2.6 ([GR03, 6.5.13.(i)])**.** *The cotangent complex $\mathbb{L}_{R/\mathbb{F}_p} = 0$ in the derived category of R-modules.*

Proof. The Frobenius induces an endomorphism of the standard resolution

(2.6.1)
$$\begin{array}{ccccccc} \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow 0 \\ & \Big\downarrow \mathrm{Frob} & & \Big\downarrow \mathrm{Frob} & & \Big\downarrow \mathrm{Frob} & \\ \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow 0. \end{array}$$

Since $\mathrm{Frob} : R \xrightarrow{\sim} R$ is an isomorphism, by the functoriality of cotangent complexes ([Ill71, II.1.2.3.2]), we see that $\mathrm{Frob} : \mathbb{L}_{R/\mathbb{F}_p} \to \mathbb{L}_{R/\mathbb{F}_p}$ is an isomorphism of complexes (this morphism does not coincides with (2.6.2) in the level of complexes). On the other hand, it coincides with the following morphism in the derived category of R-modules ([Ill71, II.1.2.6.2])

(2.6.2)
$$\begin{array}{ccccc} \cdots \longrightarrow & \Omega^1_{P_1/\mathbb{F}_p} \otimes_{P_1} R & \longrightarrow & \Omega^1_{P_0/\mathbb{F}_p} \otimes_{P_1} R \\ & \Big\downarrow \mathrm{Frob} & & \Big\downarrow \mathrm{Frob} \\ \cdots \longrightarrow & \Omega^1_{P_1/\mathbb{F}_p} \otimes_{P_1} R & \longrightarrow & \Omega^1_{P_0/\mathbb{F}_p} \otimes_{P_1} R. \end{array}$$

Since $\mathrm{Frob}(\mathrm{d}x) = \mathrm{d}x^p = px^{p-1}\mathrm{d}x = 0$ for any $\mathrm{d}x \in \Omega^1_{P_n/\mathbb{F}_p}$ and $n \in \mathbb{N}$. We see that the isomorphism $\mathrm{Frob} : \mathbb{L}_{R/\mathbb{F}_p} \to \mathbb{L}_{R/\mathbb{F}_p}$ is the zero map in the derived category of R-module and thus $\mathbb{L}_{R/\mathbb{F}_p} = 0$. ☐

Proposition 2.7. *There exists a p-adically complete and flat $\mathbb{Z}_p$-algebra W with $W/pW = R$. Moreover, it is unique up to a unique isomorphism.*

Proof. By deformation theory (2.3 and 2.6), there is a unique flat $\mathbb{Z}/p^2\mathbb{Z}$-algebra R_2 with $R_2/pR_2 = R$. Consider the derived tensor product of $\mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})}$ with the exact sequence of R_2-modules $0 \to pR_2 \to R_2 \to R \to 0$, we obtain a distinguished triangle (where we used the fact that $R \otimes^{\mathrm{L}}_{R_2} \mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})} = \mathbb{L}_{R/\mathbb{F}_p}$ by [Ill71, II.2.2.1])

(2.7.1)
$$pR_2 \otimes^{\mathrm{L}}_R \mathbb{L}_{R/\mathbb{F}_p} \longrightarrow \mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})} \longrightarrow \mathbb{L}_{R/\mathbb{F}_p} \longrightarrow$$

which implies that $\mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})} = 0$ by 2.6. Repeating this argument, we obtain unique (up to a unique isomorphism) flat liftings

(2.7.2)
$$\begin{array}{ccccccc} \cdots \twoheadrightarrow & R_3 & \twoheadrightarrow & R_2 & \twoheadrightarrow & R_1 = R \\ & \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\ \cdots \twoheadrightarrow & \mathbb{Z}/p^3\mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p^2\mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \end{array}$$

with $\mathbb{L}_{R_n/(\mathbb{Z}/p^n\mathbb{Z})} = 0$ in the derived category.

Then, we take $W = \lim_{n\to\infty} R_n$. As $R_{n+1}/p^n R_{n+1} = R_n$ by construction, we have $W/p^nW = R_n$ for any $n \geq 1$ ([Sta25, 09B8]) and thus W is p-adically complete.

Consider the injection $\mathbb{Z}/p^{n-1}\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^n\mathbb{Z}$. Tensoring with the flat $\mathbb{Z}/p^n\mathbb{Z}$-module R_n, we obtain an injection $R_{n-1} \xrightarrow{\cdot p} R_n$. Taking inverse limit over $n \in \mathbb{N}$, we get an injection $W \xrightarrow{\cdot p} W$, in other words, W is p-torsion free (hence flat over $\mathbb{Z}_p$).

The uniqueness of W follows from that of the diagram (2.7.2). □

Definition 2.8. We denote by $W(R)$ the unique p-adically complete flat $\mathbb{Z}_p$-algebra with $W(R)/pW(R) = R$. We call it the *Witt ring* of the perfect $\mathbb{F}_p$-algebra R.

Remark 2.9. By deformation theory (2.5 and 2.6), any morphism of perfect $\mathbb{F}_p$-algebras $R \to R'$ lifts uniquely to a ring homomorphism $W(R) \to W(R')$. In other words, there is an equivalence of categories

(2.9.1)
$$\{p\text{-complete flat } \mathbb{Z}_p\text{-algebras } A \text{ with } A/pA \text{ perfect}\} \cong \{\text{perfect } \mathbb{F}_p\text{-algebras } R\}$$
$$A \mapsto A/pA$$
$$W(R) \leftarrow\!\shortmid R.$$

Lemma 2.10. *There is a unique multiplicative section* $[\ ] : R \to W(R)$ *of the canonical surjection* $W(R) \to R$.

Proof. For any $x \in R$ and $n \in \mathbb{N}$, we take a lifting $y_n \in W(R)/p^nW(R)$ of $x^{1/p^n} \in R$. Since $(y_n + pz)^{p^n} \equiv y_n^{p^n} \mod p^nW(R)$ for any $z \in W(R)$, we see that $y_n^{p^n} \in W(R)/p^nW(R)$ is a lifting of $x \in R$ independent of the choice of y_n. We take $[x] = (\dots, y_2^{p^2}, y_1^p, y_0) \in W(R) = \lim_{n\to\infty} W(R)/p^nW(R)$. It is clear that $[\ ] : R \to W(R)$ is a well-defined multiplicative section of $W(R) \to R$. This verifies the existence.

For the uniqueness, let $[\ ]' : R \to W(R)$ be another multiplicative section. For any $x \in R$, we have $[x]' = [x] + py$ for some $y \in W(R)$. Taking p^n-th power, we get $[x^{p^n}]' \equiv [x^{p^n}] \mod p^nW(R)$. Since R is perfect, any element of R is of the form x^{p^n} for some $x \in R$. Thus, $[x]' \equiv [x] \mod p^nW(R)$ for any $x \in R$ and $n \in \mathbb{N}$. Taking inverse limit over $n \in \mathbb{N}$, we get $[x]' = [x]$ in $W(R)$. □

Proposition 2.11 (Teichmüler expansion)**.** *For any* $x \in W(R)$*, there is a unique sequence* $x_0, x_1, x_2, \dots$ *in* R *such that*

(2.11.1)
$$x = [x_0] + p[x_1] + p^2[x_2] + \cdots.$$

In particular, $x \in W(R)^\times$ *if and only if* $x_0 \in R^\times$*.*

Proof. For any $x \in W(R)$, let x_0 be its image in R. Then, $x = [x_0] + px'$ for a unique $x' \in W(R)$ by the flatness of $W(R)$ over $\mathbb{Z}_p$. Inductively repeating the construction, we obtain the existence and uniqueness of the sequence $x_0, x_1, x_2, \dots$.

If $x \in W(R)^\times$, then its image $x_0 \in R$ is also a unit. The converse is also true since $W(R)$ is p-adically complete. □

Remark 2.12.
(1) (Frobenius) By 2.9, there is a unique ring isomorphism $F : W(R) \to W(R)$ lifting the Frobenius on R. In particular, for any $x = [x_0] + p[x_1] + p^2[x_2] + \cdots \in W(R)$, we have $F(x) = [x_0^p] + p[x_1^p] + p^2[x_2^p] + \cdots$.
(2) (Verschiebung) There is a canonical additive map $V = pF^{-1} : W(R) \to W(R)$ sending $x = [x_0] + p[x_1] + p^2[x_2] + \cdots \in W(R)$ to $V(x) = p[x_0^{1/p}] + p^2[x_1^{1/p}] + p^3[x_2^{1/p}] + \cdots \in W(R)$.
(3) (Witt vectors) There is a canonical bijection

(2.12.1)
$$W(R) \xrightarrow{\sim} \prod_{n=0}^{\infty} R$$
$$\sum_{n=0}^{\infty} p^n[x_n] \longmapsto (a_n^{p^n})_{n\in\mathbb{N}}.$$

The latter is the usual presentation of the elements in Witt rings, see [Bou06b, IX.§1] or [Ser79, II.§6].
(4) (Addition and multiplication formulas in Teichmüler expansions) For any $x, y \in W(R)$, we put $x = [x_0]+p[x_1]+p^2[x_2]+\cdots$ and $y = [y_0]+p[y_1]+p^2[y_2]+\cdots$. We hope to write explicitly the Teichmüler expansions of $x + y$ and xy in terms of $x_0, x_1, \dots, y_0, y_1, \dots$. Unwinding the

construction 2.10 of Teichmüler liftings, we can compute out by hand that

$$(x+y)_0 = x_0 + y_0, \tag{2.12.2}$$

$$(x+y)_1 = x_1 + y_1 + \frac{(x_0^{1/p})^p + (y_0^{1/p})^p - (x_0^{1/p} + y_0^{1/p})^p}{p} = x_1 + y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_0^{\frac{i}{p}} y_0^{\frac{p-i}{p}}, \tag{2.12.3}$$

$$(xy)_0 = x_0 y_0, \tag{2.12.4}$$

$$(xy)_1 = x_0 y_1 + x_1 y_0, \tag{2.12.5}$$

$$(xy)_2 = x_0 y_2 + x_2 y_0 + x_1 y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} (x_1 y_0)^{\frac{i}{p}} (x_0 y_1)^{\frac{p-i}{p}}. \tag{2.12.6}$$

In general, after passing to the form of Witt vectors in (3), then the explicit addition and multiplication formulas are inductively computed out in [Bou06b, IX.§1.3, (12) and (13)] or [Ser79, II.§6, Theorem 6], which can be translated back into the following theorem.

Theorem 2.13 ([Bou06b, IX.§1.3, (a) and (b)])**.** *For any $x, y \in W(R)$, we put $x = [x_0] + p[x_1] + p^2[x_2] + \cdots$ and $y = [y_0] + p[y_1] + p^2[y_2] + \cdots$.*

(1) *There is a homogeneous polynomial $S_n \in \mathbb{Z}[X_0^{1/p^n}, X_1^{1/p^{n-1}}, \dots, X_n, Y_0^{1/p^n}, Y_1^{1/p^{n-1}}, \dots, Y_n]$ of degree 1 for any $n \in \mathbb{N}$ such that for any $x, y \in W(R)$,*

$$x + y = [S_0(x,y)] + p[S_1(x,y)] + p^2[S_2(x,y)] + \cdots \in W(R), \tag{2.13.1}$$

where $S_n(x,y) \in R$ is the value of the polynomial S_n at $X_i = x_i$ and $Y_i = y_i$ for any $0 \le i \le n$.

(2) *There is a homogeneous polynomial $P_n \in \mathbb{Z}[X_0^{1/p^n}, X_1^{1/p^{n-1}}, \dots, X_n, Y_0^{1/p^n}, Y_1^{1/p^{n-1}}, \dots, Y_n]$ of degree 2 that is homogeneous of degree 1 with respect to the variables $(X_0^{1/p^n}, X_1^{1/p^{n-1}}, \dots, X_n)$ and also homogeneous of degree 1 with respect to the variables $(Y_0^{1/p^n}, Y_1^{1/p^{n-1}}, \dots, Y_n)$ for any $n \in \mathbb{N}$ such that for any $x, y \in W(R)$,*

$$xy = [P_0(x,y)] + p[P_1(x,y)] + p^2[P_2(x,y)] + \cdots \in W(R), \tag{2.13.2}$$

where $P_n(x,y) \in R$ is the value of the polynomial P_n at $X_i = x_i$ and $Y_i = y_i$ for any $0 \le i \le n$.

2.c. **Universal ξ-deformation: perfect prisms.** Since $W(R)$ is the "universal p-deformation" of a perfect $\mathbb{F}_p$-algebra R, in order to define the "mixed-characteristic analogue of perfect algebras", we would like to realize $W(R)$ as a "universal ξ-deformation". We firstly need to define what ξ is.

Definition 2.14. A *perfect prism* is a pair (A, I) consisting of a ring A and an ideal I of A such that

(1) A is a p-complete flat $\mathbb{Z}_p$-algebra with $R = A/pA$ perfect (i.e., $A = W(R)$).
(2) $I = (\xi)$ for some $\xi = [\xi_0] + p[\xi_1] + p^2[\xi_2] + \cdots \in W(R)$ such that R is ξ_0-complete and $\xi_1 \in R^\times$ (we call such an element of $W(R)$ *distinguished*).

Remark 2.15. Since we want to realize $A = W(R)$ as a "universal ξ-deformation", it is natural to require that it is ξ-complete and ξ-torsion free. We will see that they are guaranteed by the second condition 2.14.(2) in the following lemmas.

Lemma 2.16. *Let R be a perfect $\mathbb{F}_p$-algebra, $d \in R$. Then, any element of R that is killed by a power of d is also killed by a p-power root of d, i.e., $R[d^\infty] = R[d^{1/p^\infty}]$. In particular, R is d-torsion-bounded.*

Proof. If $dx = 0$, then $(dx)^{1/p^n} = 0$ by perfectness, i.e., $d^{1/p^n} x^{1/p^n} = 0$. Hence, $d^{1/p^n} x = d^{1/p^n} x^{1/p^n} \cdot x^{1 - 1/p^n} = 0$. □

Lemma 2.17 (completeness)**.** *Any perfect prism $(A, (\xi))$ is (p, ξ)-complete.*

Proof. Firstly, we take induction on $n \ge 1$ to see that $W(R)/p^n$ is ξ-complete (where $R = A/pA$). By $\mathbb{Z}_p$-flatness of $W(R)$, there is an exact sequence $0 \to W(R)/p^{n-1} \xrightarrow{\cdot p} W(R)/p^n \to W(R)/p = R \to 0$. Since R is ξ-torsion bounded by 2.16, taking ξ-completion still produces an exact sequence $0 \to (W(R)/p^{n-1})^\wedge \to (W(R)/p^n)^\wedge \to \widehat{R} = R \to 0$ ([He25, 8.8]), where R is ξ-complete by definition 2.14.(2). By induction, we see that $(W(R)/p^n)^\wedge = W(R)/p^n$.

Then, as $W(R)$ is p-adically complete by definition, we have $W(R) = \lim_{n\to\infty} W(R)/p^n = \lim_{n\to\infty} \lim_{m\to\infty} W(R)/(p^n, \xi^m) = \lim_{r\to\infty} W(R)/(p^r, \xi^r) = \lim_{r\to\infty} W(R)/(p,\xi)^r$, where the last equality follows from $(p^{2r}, \xi^{2r}) \subseteq (p,\xi)^{2r} \subseteq (p^r, \xi^r)$. In other words, $W(R)$ is (p,ξ)-complete. □

Lemma 2.18. *Let A be a p-complete $\mathbb{Z}_p$-flat algebra with A/pA perfect, ξ a distinguished element of A, $x \in A$. Then, $\xi \cdot x$ is distinguished if and only if $x \in A^\times$.*

Proof. We write $A = W(R)$ and $\xi = [\xi_0] + p\xi'$, $x = [x_0] + px' \in W(R)$.

If $x \in W(R)^\times$, i.e., $x_0 \in R^\times$ by 2.11. Then, $\xi \cdot x = [\xi_0 x_0] + p(\xi'[x_0] + [\xi_0]x' + p\xi' x')$. On the one hand, R is $\xi_0 x_0$-complete as it is ξ_0-complete. On the other hand, since $\xi'[x_0] + [\xi_0]x' + p\xi'x' \equiv \xi' x_0 \mod (p, \xi)$ is a unit, it is also a unit in $W(R)$ by (p, ξ)-completeness (2.17). Therefore, $\xi \cdot x$ is distinguished.

Conversely, if $\xi \cdot x$ is distinguished, then $\xi'[x_0] + [\xi_0]x' + p\xi'x'$ is a unit in $W(R)$. Modulo (p, ξ), we see from the previous discussion that $\xi' x_0$ is a unit in $R/\xi_0 R$. This implies that $x_0 \in R^\times$ (as R is ξ_0-complete) and thus $x \in W(R)^\times$. □

Lemma 2.19 (nonzero divisor)**.** *Let (A, I) be a perfect prism. Then, any generator ξ of I is a distinguished nonzero divisor of A.*

Proof. By definition, there exists a distinguished generator $\xi = [\xi_0] + p\xi'$ of I, i.e., $\xi' \in W(R)^\times$ by 2.11. Then, any generator of I is still distinguished by 2.18.

To see any generator of I is a nonzero divisor, consider $x = [x_0] + p[x_1] + p^2[x_2] + \cdots \in W(R)$ and suppose that $\xi \cdot x = 0$. Then, we have $([\xi_0] + p\xi')x = 0$. For any positive odd number n, we have $([\xi_0]^n + p^n \xi'^n)x = 0$ and thus $p^n x \in [\xi_0^n]W(R)$. By the uniqueness of the Teichmüler expansion, we see that each $x_i \in \xi_0^n R$ for any odd number n. But since R is ξ_0-complete, we must have $x_i = 0$, i.e., $x = 0$. □

Now we start to technically realize A as a "universal ξ-deformation" of $A/\xi A$.

Lemma 2.20. *Let R be a ring, I an ideal of R such that R/I is of characteristic p and that R is I-adically complete. Then, the canonical map*

$$\varprojlim_{\mathrm{Frob}} R \longrightarrow \varprojlim_{\mathrm{Frob}} R/IR \tag{2.20.1}$$

is a bijection, where $\varprojlim_{\mathrm{Frob}} R := \lim(\cdots \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R)$ as a multiplicative monoid.

Proof. For $(\cdots, x_2, x_1, x_0) \in \varprojlim_{\mathrm{Frob}} R/IR$, we take liftings $\cdots, y_2, y_1, y_0$ of these coordinates in R. Notice that for any $n, m \in \mathbb{N}$ and $z \in I$, $(y_{n+m} + z)^{p^n} \equiv y_{n+m}^{p^n} \mod I^{n+1}R$ as $p \in I$. Thus, $y_{n+m}^{p^n} \in R/I^{n+1}R$ does not depend on the choice of y_{n+m}. Then, we see that $\lim_{n\to\infty} y_{n+m}^{p^n}$ is a well-defined element in $R = \lim_{n\to\infty} R/I^{n+1}R$. We put

$$y = (\cdots, \lim_{n\to\infty} y_{n+2}^{p^n}, \lim_{n\to\infty} y_{n+1}^{p^n}, \lim_{n\to\infty} y_n^{p^n}) \in \varprojlim_{\mathrm{Frob}} R. \tag{2.20.2}$$

It is clearly that y is well-defined and the assignment $x \mapsto y$ gives an inverse to the canonical map (2.20.1). □

Proposition 2.21. *The following functor from the category of of perfect prisms to the category of rings*

$$\begin{aligned} \{\textit{perfect prisms}\} &\longrightarrow \{\textit{rings}\} \\ (A, I) &\longmapsto A/I, \end{aligned} \tag{2.21.1}$$

is fully faithful.

Proof. Let S be a ring lying in the essential image of (2.21.1). We take a perfect prism (A, I) with $A/I \cong S$. Then, we have

$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S \cong A/I & \twoheadrightarrow & S/p \cong R/\xi_0. \end{array} \tag{2.21.2}$$

Since R is a ξ_0-complete perfect $\mathbb{F}_p$-algebra, $R \xleftarrow{\sim} \varprojlim_{\mathrm{Frob}} R \xrightarrow{\sim} \varprojlim_{\mathrm{Frob}} R/\xi_0 R \cong S^\flat := \varprojlim_{\mathrm{Frob}} S/pS$ by 2.20. In particular, the canonical map $S^\flat \to S/pS$ is surjective. By deformation theory (2.5 and 2.6), the canonical surjection $S^\flat \to S/pS$ lifts uniquely to a morphism $\theta : W(S^\flat) \to S$ (which remains surjective by déviassage). By deformation theory again, we see that the isomorphism $A/I \cong S$ lifts uniquely to an isomorphism $A \cong W(S^\flat)$. All in all, the functor from the essential image of (2.21.1) to the category of perfect prisms sending S to $(W(S^\flat), \ker(\theta))$ is well-defined and forms a quasi-inverse to (2.21.1). □

Definition 2.22. A *perfectoid ring* is a ring S such that $S \cong A/I$ for some perfect prism (A, I).

Remark 2.23. Note that possibly many perfectoid rings S could have the same perfect $\mathbb{F}_p$-algebra $S^\flat$, since the choice of an distinguished principal ideal (ξ) on $W(S^\flat)$ could be many (even if we fix $\xi_0 \in R = S^\flat$, it seems that different choices of $\xi_1 \in R^\times$ could lead to different ideals $I = (\xi) \subseteq W(R)$). But I don't have an explicit example in hand.

However, this issue does not exists when we work over a fixed perfect prism, see the tilting correspondence in the following.

2.d. **Tilting correspondence of perfectoids.** Our definition for perfectoids immediately implies the tilting correspondence as long as we have the following rigidity lemma:

Lemma 2.24 (rigidity). *Let $(A, I) \to (B, J)$ be a morphism of perfect prisms. Then, $J = IB$.*

Proof. We only need to show that for generator ξ of J, if $\xi \cdot x$ is distinguished then $x \in B^\times$. This is proved in 2.18. □

Theorem 2.25. *Given a perfect prism (A, I), we put*

(2.25.1)
$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S = A/\xi & \twoheadrightarrow & S/p = R/\xi_0. \end{array}$$

Then, the base change induces equivalences of categories

(2.25.2)
$$\begin{array}{ccc} \{\textit{perfect } (A,I)\textit{-prisms } (A', IA')\} & \xrightarrow[\beta]{\sim} & \{\xi_0\textit{-complete perfect } R\textit{-algebras } R'\} \\ \alpha \downarrow \wr & & \gamma \downarrow \wr \\ \{\textit{perfectoid } S\textit{-algebras } S'\} & \xrightarrow[\delta]{\sim} & \{\textit{relatively perfect } (S/p) = (R/\xi_0)\textit{-algebras } T \textit{ with } T = T^\flat/\xi_0^\flat T^\flat\} \end{array}$$

where $\xi_0^\flat = (\cdots, \xi_0^{1/p^2}, \xi_0^{1/p}, \xi_0) \in (R/\xi_0 R)^\flat$.

Proof. By the rigidity lemma 2.24, the category of perfect (A, I)-prisms (A', IA') is the category of perfect prisms (B, J) with a morphism $(A, I) \to (B, J)$. Hence, it is equivalent to the category of perfectoid rings S' with a morphism $S = A/I \to S'$ by 2.21, i.e., α is an equivalence.

Unwinding the definition 2.14, the category of perfect (A, I)-prisms (A', IA') is the category of p-complete $\mathbb{Z}_p$-algebras A' with A'/pA' perfect ξ_0-complete and a morphism $A \to A'$. Hence, it is equivalent to the category of ξ_0-complete perfect $\mathbb{F}_p$-algebras R' with a morphism $R = A/pA \to R'$ by 2.9, i.e., β is an equivalence.

Recall that $R \xleftarrow{\sim} R^\flat \xrightarrow{\sim} (R/\xi_0 R)^\flat$ identifying ξ_0 with $\xi_0^\flat$ by 2.20 and that the Frobenius induces an isomorphism $R/\xi_0^{1/p} \xrightarrow{\sim} R/\xi_0$. The same holds true for any ξ_0-complete perfect R-algebra R'. In particular, γ is a well-defined functor. To see that it is an equivalence, we only need to show that for any relatively perfect (R/ξ_0)-algebra T with $T = T^\flat/\xi_0^\flat T^\flat$, $T^\flat$ is a ξ_0-complete perfect R-algebra. As $R = (R/\xi_0)^\flat$, we see $T^\flat$ is naturally a perfect R-algebra. Moreover, $T^\flat = \lim(\cdots \xrightarrow{\mathrm{Frob}} T^\flat/\xi_0 T^\flat \xrightarrow{\mathrm{Frob}} T^\flat/\xi_0 T^\flat \xrightarrow{\mathrm{Frob}} T^\flat/\xi_0 T^\flat) = \lim(\cdots \to T^\flat/\xi_0^{p^2} T^\flat \to T^\flat/\xi_0^p T^\flat \to T^\flat/\xi_0 T^\flat)$, where we applied the identification $\mathrm{Frob}^n : T^\flat/\xi_0 T^\flat \xrightarrow{\sim} T^\flat/\xi_0^{p^n} T^\flat$. This shows that $T^\flat$ is ξ_0-complete. Hence, γ is an equivalence.

The proof of 2.21 shows that δ is a well-defined functor making the diagram (2.25.2) commutative. Hence, δ is an equivalence. □

Remark 2.26. We couldn't simply apply deformation theory in the setting of 2.25 because the a relatively perfect $(S/p) = (R/\xi_0)$-algebra T may not be flat. To resolve this issue, one may consider instead relatively perfect *animated* $(S/p) = (R/\xi_0)$-algebra T, i.e., animated algebra T such that the relative Frobenius $T \otimes^{\mathrm{L}}_{R/\xi_0, \mathrm{Frob}} R/\xi_0 \to T$ is an isomorphism, and then apply deformation theory for animated algebras, see [Bha25, 3.2.6]. In another way, one can impose flatness assumptions in order to use the classical deformation theory as follows.

Theorem 2.27. *Given a perfect prism (A, I), we put*

(2.27.1)
$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S = A/\xi & \twoheadrightarrow & S/p = R/\xi_0. \end{array}$$

Then, the base change induces equivalences of categories

(2.27.2)
$$\begin{array}{ccc} \{(p,\xi)\text{-completely flat perfect } (A,I)\text{-prisms } (A',IA')\} & \xrightarrow[\beta]{\sim} & \{\xi_0\text{-completely flat } \xi_0\text{-complete perfect } R\text{-algebras } R'\} \\ \alpha \downarrow \wr & & \gamma \downarrow \wr \\ \{p\text{-completely flat perfectoid } S\text{-algebras } S'\} & \xrightarrow[\delta]{\sim} & \{\text{flat relatively perfect } (S/p)=(R/\xi_0)\text{-algebras } T\} \end{array}$$

where "I-completely flat" means "flat after modulo I^n for any $n \in \mathbb{N}$" here.

Proof. Let T be a flat relatively perfect $(S/p) = (R/\xi_0)$-algebra. By deformation over $R \to R/\xi_0$ and $\mathbb{L}_{T/(R/\xi_0)} = 0$ ([GR03, 6.5.13.(i)]), there exists a unique ξ_0-completely flat ξ_0-complete R-algebra R' with $R'/\xi_0 R' = T$ (see the proof of 2.7 and 2.9). To see that γ is an equivalence, it remains to check that R' is perfect. As T is relatively perfect, we have $R'/\xi_0 R' \otimes_{R/\xi_0 R,\mathrm{Frob}^{p^n}} R/\xi_0 R = R'/\xi_0 R'$ for any $n \in \mathbb{N}$. Since $\mathrm{Frob}^{p^n} : R/\xi_0 R \to R/\xi_0 R$ factors as $R/\xi_0 R \xrightarrow{\sim} R/\xi_0^{p^n} R \to R/\xi_0 R$, the uniqueness of the liftings implies that the Frobenius induces isomorphism $\mathrm{Frob}^{p^n} : R'/\xi_0 R' \xrightarrow{\sim} R'/\xi_0^{p^n} R'$. Thus, $R' = \lim_{n\to\infty} R'/\xi_0^{p^n} R' = \varprojlim_{\mathrm{Frob}} R'/\xi_0 R' = R'^\flat$ is perfect.

Similarly, by deformation over $S \to S/p$ and $\mathbb{L}_{T/(S/p)} = 0$, there exists a unique p-completely flat p-complete S-algebra S' with $S'/pS' = T$. To see that δ is an equivalence, it remains to check that S' is perfectoid. It suffices to check that S' lies in the essential image of α. As the diagram (2.27.2) commutes, we only need to prove that β is an equivalence.

We claim that a perfect (A,I)-prism (A',IA') is (p,ξ)-completely flat if and only if $A'/(p,\xi)A'$ is flat over $A/(p,\xi)A$. Since A' is p-torsion-free, we have $A' \otimes^{\mathrm{L}}_A A/pA = A'/pA'$. Thus, $A' \otimes^{\mathrm{L}}_A A/(p,\xi) = A'/pA' \otimes^{\mathrm{L}}_{A/pA} A/(p,\xi) = R' \otimes^{\mathrm{L}}_R R/\xi_0$, where $R' = A'/pA'$ is a perfect $\mathbb{F}_p$-algebra. In particular, $\mathrm{Tor}^A_1(A', A/(p,\xi)) = \mathrm{Tor}^R_1(R', R/\xi_0) = 0$ by 2.28. Then, the claim follows directly from [Sta25, 051C].

The claim implies that the category of (p,ξ)-completely flat perfect (A,I)-prisms (A',IA') is equivalent to the category of p-complete $\mathbb{Z}_p$-algebras A' with A'/pA' perfect ξ_0-complete ξ_0-completely flat and a morphism $A \to A'$. Hence, it is equivalent to the category of ξ_0-completely flat ξ_0-complete perfect $\mathbb{F}_p$-algebras R' with a morphism $R = A/pA \to R'$ by 2.9, i.e., β is an equivalence. ☐

Lemma 2.28. *Let $R \to R'$ be a morphism of perfect $\mathbb{F}_p$-algebras, $d \in R$. Consider the following conditions:*

(1) *$R/dR \to R'/dR'$ is flat.*
(2) *$R/d^nR \to R'/d^nR'$ is flat for any $n \in \mathbb{N}$.*
(3) *$R[d] \otimes_R R' \to R'[d]$ is an isomorphism.*
(4) *$R[d] \otimes_R R' \to R'[d]$ is surjective.*
(5) *$\mathrm{Tor}^R_1(R', R/d) = 0$.*
(6) *$R/dR \otimes^{\mathrm{L}}_R R' \to R'/dR'$ is an isomorphism.*

Then, we have (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6).

Proof. (1) ⇒ (2): As R and R' are perfect, the Frobenius induces isomorphism between $R/dR \to R'/dR'$ with $R/d^{p^n}R \to R'/d^{p^n}R'$. Thus, the latter is also flat.

(2) ⇒ (3): Note that $R[d] \cap dR = 0$ by 2.16. Thus, the sequence of R/d^2R-modules $0 \to R[d] \to R/dR \xrightarrow{\cdot d} R/d^2R$ is exact. Tensoring with R'/d^2R', the flatness implies that $R'[d] = R[d] \otimes_R R'$.

(3) ⇒ (4): This is clear.

(4) ⇒ (5): Consider the exact sequence $0 \to R[d] \to R \xrightarrow{\cdot d} R \to R/dR \to 0$. Then, $R/dR \otimes^{\mathrm{L}}_R R'$ is represented by the total complex of $R[d] \otimes^{\mathrm{L}}_R R' \to R' \xrightarrow{\cdot d} R'$. In particular, $\mathrm{Tor}^R_1(R', R/d) = H_1(R/dR \otimes^{\mathrm{L}}_R R') = \mathrm{Coker}(R[d] \otimes_R R' \to R'[d]) = 0$.

(5) ⇒ (6): As R is perfect, $R[d] = R[d^{1/p^\infty}]$ by 2.16 so that $R/R[d]$ is a perfect $\mathbb{F}_p$-algebra. Recall that $R/R[d] \otimes^{\mathrm{L}}_R R' = R/R[d] \otimes_R R'$ by [BS17, 11.6]. We deduce from the exact sequence $0 \to R[d] \to R \to R/R[d] \to 0$ that $R[d] \otimes^{\mathrm{L}}_R R' = R[d] \otimes_R R'$. In particular, $R/dR \otimes^{\mathrm{L}}_R R'$ is concentrated in degree $[-1,0]$ by previous discussion. Thus, condition (5) implies that $R/dR \otimes^{\mathrm{L}}_R R' = R/dR \otimes_R R' = R'/dR'$. ☐

2.e. **Properties of perfectoids.** We fix a perfectoid ring S in this section. Recall that it is associated with a commutative diagram

$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S = A/\xi & \twoheadrightarrow & S/p = R/\xi_0. \end{array} \tag{2.28.1}$$

where $R = S^\flat$ by 2.21 and its proof.

Definition 2.29. A *strict uniformizer* ϖ of a perfectoid ring S is the image of $[\xi_0] \in W(R)$.

Lemma 2.30 (p^∞-roots)**.** *Any strict uniformizer $\varpi \in pS^\times$ and admits a compatible p-power roots $(\varpi^{1/p^n})_{n\in\mathbb{N}}$ in S.*

Proof. We write $\xi = [x_0] + p\xi' \in W(R)$ with $\xi' \in W(R)^\times$. Modulo ξ, we see that $\varpi = -p\xi'$ in S so that $\varpi \in pS^\times$. On the other hand, $([\xi_0^{1/p^n}])_{n\in\mathbb{N}}$ is a compatible p-power roots of ϖ. □

Lemma 2.31 (Frobenius isomorphism)**.** *The Frobenius induces an isomorphism $S/\varpi^{1/p}S \xrightarrow{\sim} S/\varpi S$.*

Proof. Since R is perfect, the Frobenius on $R/\xi_0 R$ is surjective with kernel generated by $\xi_0^{1/p}$. Since $R/\xi_0 = S/p$ identifying $\xi_0^{1/p}$ with $\varpi^{1/p}$ via the commutative diagram (2.28.1), the conclusion follows immediately. □

Lemma 2.32 (almost torsion-free)**.** $S[\varpi^\infty] = S[\varpi^{1/p^\infty}] = R[\xi_0^{1/p^\infty}] = R[\xi_0^\infty]$. *In particular, S is p-torsion bounded.*

Proof. Since p, ξ are both nonzero divisors on A by 2.19, there are canonical isomorphisms

$$(A/\xi)[p] \xleftarrow{\sim} \frac{\{(x,y) \in A^2 \mid \xi x = py\}}{\{(pz, \xi z) \mid z \in A\}} \xrightarrow{\sim} (A/p)[\xi] \tag{2.32.1}$$
$$y \longleftarrow\!\shortmid (x,y) \longmapsto x.$$

Thus, we have $S[\varpi] = R[\xi_0]$ as $(S/p) = (R/\xi_0)$-modules. Then, for any $n \geq 1$, we have $S[\varpi^{1/p^n}] = (S[\varpi])[\varpi^{1/p^n}] = (R[\xi_0])[\xi_0^{1/p^n}] = R[\xi_0^{1/p^n}]$.

Since R is perfect, we have $R[\xi_0^{1/p^\infty}] = R[\xi_0]$ by 2.16. The above discussion implies that $S[\varpi^{1/p^\infty}] = S[\varpi]$. This implies furthermore that $S[\varpi^{1/p^\infty}] = S[\varpi^\infty]$. □

Lemma 2.33 (completeness)**.** *S is p-complete.*

Proof. As ξ is a nonzero divisor on A (2.19), there is an exact sequence $0 \to A \xrightarrow{\cdot\xi} A \to S \to 0$. Since S is p-torsion bounded by 2.32, taking p-completion still produces an exact sequence $0 \to A \to A \to \widehat{S} \to 0$ ([He25, 8.8]), where $\widehat{A} = A$ by 2.17. Hence, we get $\widehat{S} = S$, i.e., S is p-complete. □

Proposition 2.34 (perfectoidness criterion)**.** *A p-torsion-free ring S is perfectoid if and only if the following conditions hold:*

(1) *S is p-complete.*
(2) *There exists $\pi \in S$ such that $\pi^p \in pS^\times$.*
(3) *The Frobenius induces an isomorphism $S/\pi S \xrightarrow{\sim} S/pS$.*

Proof. These conditions are necessary by 2.33, 2.30 and 2.31. To see they are also sufficient, consider $S^\flat = \varprojlim_{\mathrm{Frob}} S/pS$. The surjectivity part of condition (3) implies that the canonical projection $S^\flat \to S/pS$, $(\cdots, x_2, x_1, x_0) \mapsto x_0$ is surjective. Hence, we can take $\xi_0 = (\cdots, \pi_2, \pi_1 = \pi, \pi_0 = \pi^p) \in S^\flat$. Since S is p-torsion-free, the injectivity part of condition (3) implies that the kernel of $S^\flat \to S/pS$ is generated by ξ_0. By deformation theory and a dévissage argument (2.5 and 2.6, see also 2.9), the exact sequence $S^\flat \xrightarrow{\cdot\xi_0} S^\flat \to S/pS \to 0$ lifts uniquely to an exact sequence $0 \to W(S^\flat) \to W(S^\flat) \to S \to 0$, where we used the fact that S is p-complete and p-torsion-free. Let ξ be the image of 1 under the map $W(S^\flat) \to W(S^\flat)$ and denote the surjection $W(S^\flat) \to S$ by θ.

To see that S is perfectoid, it remains to show that ξ is distinguished. By condition (2) we write $\pi_0 = \pi^p = pu$. As $\theta([\xi_0^{1/p}]) \equiv \pi_1 \mod pS$, we have $\theta([\xi_0]) \equiv \pi_1^p \equiv pu \mod p^2 S$. This shows that $\theta([\xi_0]) = pv$ for some $v \in S^\times$ as S is p-complete and p-torsion-free. Let $w \in W(S^\flat)^\times$ be a lifting of v. Then, $[\xi_0] - pw \in \ker(\theta) = (\xi)$ and is distinguished by construction. This implies that ξ is also distinguished by a similar argument of 2.18. □

2.f. **Examples of perfectoids.**

Lemma 2.35 (adding p^∞-roots)**.** *Let S be a perfectoid ring. Then, the p-adic completion $S\langle X^{1/p^\infty}\rangle$ of $S[X^{1/p^\infty}]$ is also perfectoid.*

Proof. Since $(S/pS)[X^{1/p^\infty}]$ is a flat relative perfect (S/pS)-algebra with the unique flat lifting $S\langle X^{1/p^\infty}\rangle$. Thus, the proof of 2.27 shows that $S\langle X^{1/p^\infty}\rangle$ is perfectoid. ☐

Lemma 2.36 (perfect=perfectoid over $\mathbb{F}_p$)**.** *Let S be an $\mathbb{F}_p$-algebra. Then, S is perfectoid if and only if S is perfect.*

Proof. If S is perfect, then $S = W(S)/pW(S)$ with perfect prism $(W(S),(p))$ (i.e., p is distinguished).

If S is perfectoid, then $\mathbb{F}_p \to S$ is a morphism of perfectoid rings, which corresponds to a morphism of perfect prisms $(\mathbb{Z}_p,(p)) \to (W(S^\flat), I)$ by 2.21. Then, $I = pW(S^\flat)$ by the rigidity lemma 2.24. Hence, $S = W(S^\flat)/I = S^\flat$ is perfect. ☐

Lemma 2.37 (perfectoid valuation ring)**.** *Let V be a p-complete valuation ring extension of $\mathbb{Z}_p$ that is not absolutely unramified. Then, V is perfectoid if and only if the Frobenius is surjective on V/pV. In particular, if the fraction field of V is algebraically closed, then V is perfectoid.*

Proof. Firstly, we claim that there exists $\pi \in V$ with $\pi^p \in pV^\times$. As V is not absolutely unramified, we can write $p = \pi_1\pi_2$ for some elements $\pi_1,\pi_2 \in \mathfrak{m}_V$. Then, the surjectivity of the Frobenius on V/pV implies that $\pi_i = x_i^p + py_i$ for some $x_i, y_i \in V$ (where $i = 1,2$). Notice that $x_i^p = \pi_i - py_i \in \pi_i V^\times$ by construction. We get $x_1^p x_2^p \in pV^\times$.

Since V is integrally closed in $V[1/p]$, the Frobenius induces an injection $V/\pi V \to V/pV$ (see [He24, 5.21]). Thus, the conclusion follows from the perfectoidness criterion 2.34. ☐

Lemma 2.38 (torsion-free quotient)**.** *Let S be a perfectoid ring. Then, its maximal p-torsion-free quotient $\overline{S} = S/S[p^\infty]$ is also perfectoid.*

Proof. Let ϖ be a strict uniformizer of S. Then, $S[p^\infty] = S[\varpi^\infty] = S[\varpi^{1/p^\infty}]$ by 2.32. In particular, $S[p^\infty] \cap \varpi^{1/p^n}S = 0$ for any $n \in \mathbb{N}$. The exact sequence $0 \to S[p^\infty] \to S \to \overline{S} \to 0$ induces exact sequences $0 \to S[p^\infty] \to S/\varpi^{1/p^n}S \to \overline{S}/\varpi^{1/p^n}\overline{S} \to 0$. In particular, the Frobenius induces an isomorphism $\overline{S}/\varpi^{1/p}\overline{S} \xrightarrow{\sim} \overline{S}/\varpi\overline{S}$ by 2.31. Hence, the conclusion follows from the perfectoidness criterion 2.34. ☐

Lemma 2.39 (integral closure)**.** *Let S be a perfectoid ring. Then, its integral closure S^+ in $S[1/p]$ is also perfectoid.*

Proof. After 2.38, we may assume that S is p-torsion-free. The injectivity of the Frobenius $S/\varpi^{1/p}S \to S/\varpi S$ implies that S is p-integrally closed, i.e., for any $x \in S[1/p]$ with $x^p \in S$ we have $x \in S$ (see [He24, 5.21]). Then, $S \to S^+$ is an almost isomorphism, i.e., $\varpi^{1/p^\infty}S^+ \subseteq S$ (see [He24, 5.25]).

We claim that the Frobenius induces an isomorphism $S^+/\varpi^{1/p}S^+ \xrightarrow{\sim} S^+/\varpi S^+$. It is injective as S^+ is p-integrally closed. For any $z \in S^+$, the previous discussion allows us to write $\varpi^{1/p}z = x^p + \varpi y$ for some $x, y \in S$. As $z = (x/\varpi^{1/p^2})^p + \varpi^{1-1/p}y$, we see that $x' = x/\varpi^{1/p^2} \in S^+$. We continue to write $z = x'^p + \varpi^{1-1/p}y = x'^p + \varpi^{1-1/p}(y'^p + \varpi^{1-1/p}z')$ for some $y', z' \in S^+$. Thus, $z = (x' + \varpi^{1/p-1/p^2}y')^p + \varpi z''$ for some $z'' \in S^+$. This shows the surjectivity of the Frobenius on $S^+/\varpi S^+$.

In conclusion, S^+ is perfectoid by the perfectoidness criterion 2.34. ☐

Lemma 2.40 (tensor product)**.** *Let $S_2 \leftarrow S_1 \to S_3$ be morphisms of perfectoid rings. Then, the p-completed tensor product $S_2\widehat{\otimes}_{S_1}S_3$ is perfectoid.*

Proof. Let $\xi = [\xi_0] + p\xi'$ be a distinguished generator of $\mathrm{Ker}(W(S_1^\flat) \to S_1)$ (and thus also a distinguished generator of for S_2 and S_3 by the rigidity lemma 2.24). The given morphisms $S_2 \leftarrow S_1 \to S_3$ induce morphisms of perfect $\mathbb{F}_p$-algebras $S_2^\flat \leftarrow S_1^\flat \to S_3^\flat$. It is clear that the ξ_0-completed tensor product $S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat$ is still perfect.

Firstly, we claim that $W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p^n = W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)/p^n$ for any $n \in \mathbb{N}$. This holds for $n = 1$. In general, it follows from taking induction and the following exact sequences

(2.40.1)

$$\begin{array}{ccccccccc}
 & & W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p^n & \longrightarrow & W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)/p^n & \longrightarrow & S_2^\flat \otimes_{S_1^\flat} S_3^\flat & \longrightarrow & 0.
\end{array}$$

Therefore, we have $W(S_2^\flat)\widehat{\otimes}^p_{W(S_1^\flat)}W(S_3^\flat) = W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)$, where the completion is p-adic, and thus there is an exact sequence $0 \to W(S_2^\flat)\widehat{\otimes}^p_{W(S_1^\flat)}W(S_3^\flat) \xrightarrow{\cdot p} W(S_2^\flat)\widehat{\otimes}^p_{W(S_1^\flat)}W(S_3^\flat) \to S_2^\flat \otimes_{S_1^\flat} S_3^\flat \to 0$. Since $S_2^\flat \otimes_{S_1^\flat} S_3^\flat$ is ξ-torsion-bounded by 2.16, taking ξ-completion still produces an exact sequence $0 \to W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat) \xrightarrow{\cdot p} W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat) \to S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat \to 0$.

To show that $W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p^n = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/p^n$ for any $n \in \mathbb{N}$. We still take induction on n. The case for $n = 1$ is proved above. In general, it follows from the following exact sequences (2.40.2)

$$\begin{array}{ccccccccc}
 & & W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p^n & \longrightarrow & W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
0 & \longrightarrow & W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/p^n & \longrightarrow & S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat & \longrightarrow & 0.
\end{array}$$

Therefore, we have $W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat) = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)$. In particular, we have $S_2 \otimes_{S_1} S_3/p^n = W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/(p^n,\xi) = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/(p^n,\xi)$. Taking inverse limit on $n \in \mathbb{N}$, we see that $S_2\widehat{\otimes}_{S_1}S_3 = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/\xi$ is perfectoid. $\square$

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