**Refinement**:

$\exists ! \, \ell \in [1, N] \cap \mathbb{N}$, denoted by $\dim(X)$ essential dim. or rectifiable dim.

s.t.

$$m(X \setminus \mathcal{R}_\ell) = 0$$

Pf. Use RLE.

$\mathrm{RCD}(N-1, N)$, $N = \frac{3}{2}$

$([0, \pi], d_{Eucl}, \sin^{N-1} t \, dt)$

$[N]$ = the integer part of $N$

**Rem.**

- $\dim_H X \in \{0\} \cup \{1\} \cup [2, [N]]$ (Haus. dim.)
- If $N \in \mathbb{N}$, then $\dim_H X \in \{0\} \cup \{1\} \cup [2, N-1] \cup \{N\}$
- $\exists$ example s.t. $2 = \dim(X) < \dim_H(X) \notin \mathbb{N}$ (Pan-Wei)

**Rem.**

It is unknown whether

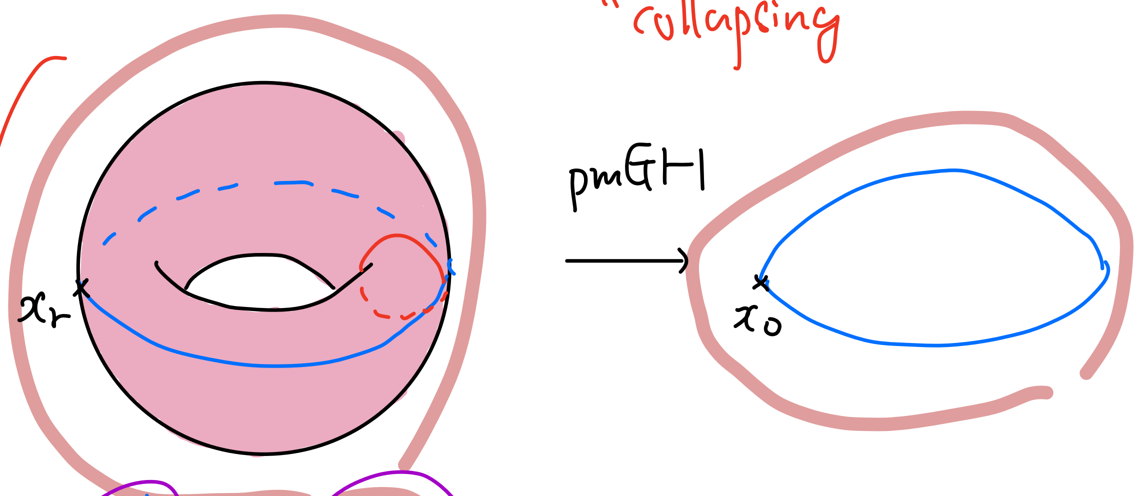
$$\mathcal{R}_n = \emptyset \text{ for } \forall n \neq \dim(X)$$

Rem.

As checked in step 2 in the pf. of Goal

dim. is lower semi cont. w.r.t pmGH conv.

"collapsing"




$$\left( \underset{2\pi}{S^1(1)} \times \underset{2\pi r}{S^1(r)},\ x_r,\ \frac{\mathcal{H}^2}{4\pi r} \right) \qquad \left( S^1(1),\ x_0,\ \frac{\mathcal{H}^1}{2\pi} \right).$$

$R_k \ni x$

$x \in R_k$

$\dim X \geq \dim \underbrace{T_x X}_{\cong \mathbb{R}^k} = k$

$\boxed{\text{Q. if } k > \dim(X) \text{, then } R_k = \emptyset \text{ ? Yes!}}$

## Rectifiability

Denoting $\ell := \dim(X)$, for ${}^\forall \varepsilon > 0$

${}^\exists A_i \subset X$ : Borel (c. $\mathbb{R}^\ell$)

${}^\exists \Phi_i : A_i \hookrightarrow \mathbb{R}^\ell$ (top. emb.)

s.t.

Ⓐ $\Phi_i$ : $(1 \pm \varepsilon)$-bi-Lip.

$\left( C(1-\varepsilon) d(x,y) \le |\Phi_i(x) - \Phi_i(y)|_{\mathbb{R}^\ell} \le C(1+\varepsilon) d(x,y) \right)$

Ⓑ $m\left(X \setminus \bigcup_{i=1}^{\infty} A_i\right) = 0$

Ⓒ $m\llcorner_{A_i} \ll \mathcal{H}^\ell \llcorner_{A_i} \ll m\llcorner_{A_i}$

metric rect.

metric meas. rect.

$(\Phi_i)_\# m\llcorner_{A_i}$

Pf.

- Ⓐ & Ⓑ : Similar to Goal.

- Ⓒ : Cheeger's differentiability thm.

  &

  Converse of Rademacher's thm.

---

$\forall f : \mathbb{R}^n \to \mathbb{R}$ : Lip $\Rightarrow$ $f$ is diff'ble for $\mathcal{L}^n$-a.e.

$\forall \varphi : \mathbb{R}^n \to \mathbb{R}$ : Lip is diff'ble for $g\mathcal{L}^n$

$g \in L^1_{loc}$

If $\forall \varphi$ : Lip. on $\mathbb{R}^n$ is diff. for $m$-a.e $x \in \mathbb{R}^n$,

then $m = g\mathcal{L}^n$

# §6 Special class of RCD

**Non-collapsed**

An $RCD(K,N)$ sp $(X, d, m)$ is said to be non-collapsed ( $ncRCD(K,N)$ )

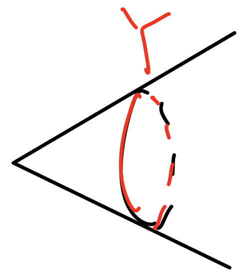
if $\quad m = \mathcal{H}^N$

## Nicer properties

$(X, d, \mathcal{H}^N)$ : ncRCD$(K,N)$

non-collapsed

Then.




Ⓐ $\dim(X) = N$

Ⓑ $\forall$ tang. cone is a cone over an RCD$(N-2, N-1)$ sp

Ⓒ $\displaystyle\lim_{r \downarrow 0} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} \le 1$ for $\forall x \in X$

Ⓓ $\mathcal{R} = \mathcal{R}_N$ & ($x \in \mathcal{R}_N \Leftrightarrow$ the eq. in Ⓒ holds)

Ⓔ $\Delta f = \mathrm{tr}\,\mathrm{Hess}\, f$

Ⓕ $\forall x \in \mathcal{R} = \mathcal{R}_N$, $\forall \varepsilon < 1$, $\exists r > 0$

$\exists \Phi : B_r(x) \to \mathbb{R}^N$ : harm. s.t.

sharp

$(1-\varepsilon)\, d(y,z)^{1+\varepsilon} \le |\Phi(y) - \Phi(z)| \le (1+\varepsilon)\, d(y,z)$

$B_{(1-\varepsilon)r}(0_N) \subset \mathrm{Image}\,\Phi \subset B_{(1+\varepsilon)r}(0_N)$

Pf. of only (B) & (F)

- (B) : Applying a metric cone rigidity.
- (F) : Applying Reifenberg flatness around $x$ (i.e. $\forall y \in B_r(x)$ is "almost" regular) with characterization of harmonic functions on metric cones via Transformation thm.

Rem. (Sharpness of Ⓕ)

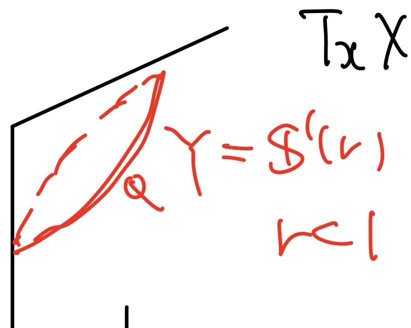
- $\exists (X, d, \mathcal{H}^2)$ : ncRCD$(0,2)$ s.t.

  $X \setminus \mathcal{R}$ : dense.

- $\forall x \in \mathcal{R} = \mathcal{R}_2$ & Assume

  $\Phi : B_r(x) \to \mathbb{R}^2$ : harm. as above

  is $C_i$-Lip.




- $\forall y \in B_r(x) \setminus \mathcal{R}$

- $\underbrace{\tilde{\Phi}}_{C_i\text{-Lip}} : \underbrace{T_y X}_{\cong C(Y)} \to \mathbb{R}^2$ : blow-up harm. map

  of $\Phi$ at $y$

Then

- $\tilde{\Phi}$ : linear, thus $g_{C(Y)} \overset{\text{splitting thm}}{=} \tilde{\Phi}^* g_{\mathbb{R}^2} \oplus {}^{\exists} h$

But ______ $\Rightarrow h = 0 \Rightarrow C(Y) \cong \mathbb{R}^2$

$\Rightarrow$ contradiction.

because of ______

Weakly noncollapsed

If an $RCD(K,N)$ sp $(X,d,m)$ satisfies $\dim X = N$, then $m = cH^N$ for $\exists c > 0$

"weakly nc" "nc upto normalization"

Pf.

- $\Phi_\epsilon : X \longrightarrow L^2(X,m)$

$$x \longmapsto (y \longmapsto p(x,y,t))$$

heat kernel

- $g_\epsilon := \Phi_\epsilon^* g_{L^2}$

Use $g_\epsilon$ as "smoothing" of $(X,d,m)$.

Openness of nc.

The moduli of nc RCD(K, N) spaces is open w.r.t. pmGH.

$N - \delta = \dim X - \delta \le \dim Y \le N$

pf. This is a direct conseq. of

- weakly nc $\Rightarrow$ nc. $\mathfrak{m} = c\mathcal{H}^N$
- the lower semicontinuity of dim. w.r.t pmGH

- $(X, d, \mathcal{H}^N)$ : nc RCD(K, N)
- $(Y, \tilde{d}, \tilde{\mathfrak{m}})$ : RCD(K, N)

$d_{pmGH}((X, d, \mathcal{H}^N, x), (Y, \tilde{d}, \tilde{\mathfrak{m}}, y)) < \varepsilon$

## Topological stability ($n \in \mathbb{N}$)

${}^\forall (M^n, g)$ : closed

${}^\forall K$, ${}^\forall \varepsilon > 0$ ${}^\exists \delta > 0$ s.t.

if $d_{GH}((X, d), (M^n, d^g)) < \delta$

for sum $(X, d, m)$ : $RCD(K, n)$,

then ${}^\exists F : X \to M^n$ : homeo.

s.t.

$(1-\varepsilon) d(x, y)^{1+\varepsilon} \leq d^g(F(x), F(y))$

$\leq (1+\varepsilon) d(x, y)$

for ${}^\forall x, {}^\forall y \in X$.

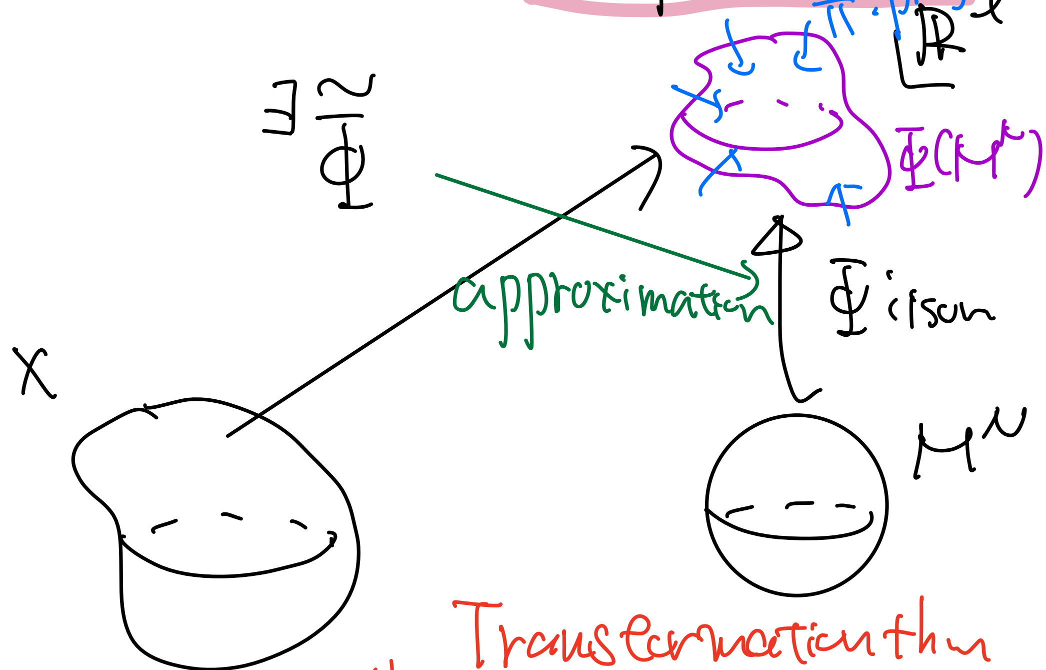
Pf.

- $\dim X = n \Rightarrow m = cH^n$

$\Rightarrow$ Reifenberg flatness

$\Rightarrow$ $X$: top. $n$-mfd.

- $\exists \tilde{\Phi}$




- $F := \Phi^{-1} \circ \pi \circ \tilde{\Phi}$

Transformation thm & results in §4 etc.

$\Rightarrow$

$\Rightarrow$ $F$: homeo.

invariance of domain with

# Open problems

① Establish regularity results on $\underline{m}$

e.g. "$m = e^{-f} \mathcal{H}^{\ell}$, $\ell = \dim(X)$"

on Reg. set.  metric measure rect.

② Establish "local RCD theory".

$$\left(M_i^n, g_i, \frac{\mathrm{vol}^{g_i}}{\mathrm{vol}^{g_i} B_1(x_i)}, x_i\right) \xrightarrow{\mathrm{pmGH}} (X, d, m, x)$$

· $\forall R > 0$  $\mathrm{Ric}^{g_i}|_{B_R(x_i)} \geq K_R$  ($K_R \to -\infty$, $R \uparrow \infty$)

$e_i \to f$ in $C^2$  $h_t^i e_i \to h_t f$

③ Establish "Smoothing" via geometric flow.

(e.g. "$g_t^i$")

④ Determine the topology around sing. pt in nc.RCD(K, 3) sp.

⑤ $(X, d, \mathcal{H}^n)$ : ncRCD(K, N)
& cpt

$\Rightarrow t^{\frac{N+2}{2}} \sum_i e^{-\lambda_i t} \int_X \left((\Delta f_i)^2 - |\mathrm{Hess}_{f_i}|^2\right)$

$\lambda_i$ is enough

$\cdots$

~~$\mathrm{Ric} \geq K$~~

$\leq C$ for $^\forall t$ : small

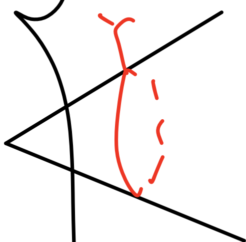
$\Delta f_i + \lambda_i f_i = 0$ & $\|f_i\|_{L^2} = 1$

$\Leftarrow$ RCD ver. of a conj of Yau
"asym. of $\lambda_i$ $\Rightarrow$ $C^1$ - Scal. bd"

⑥ $g \in H^{1,2}_{loc}$ ? ($g \in W^{1,2}_{loc}$)
Riem. met. of $(X, d, n)$

⑦ For

- tang. cones $C(Y)$ at $p$ of a ncRCD$(K,N)$ sp $X$.



or

- tang. cones $C(Y)$ at $\infty$ of a ncRCD$(0,N)$ sp $X$ with Euclidean vol. growth,

$$(0,\infty) \setminus \bigcup_{Y} \mathrm{Spec}\{-\Delta_Y\} \neq \emptyset ?$$

$Y \in$ moduli of such $Y$ is cpt

$\rightarrow$ $\exists$ nontrivial harm. fct with polynomial growth on $X$ (RCD'ver of a conj. of Yau)

⑧ Establish ~~spectral~~ conv.

conv. of heat flow associated with

for

- Hodge Lap. acting on 1-form.
- Connection Lap acting on tensors of any type.

w.r.t. nc-mGH conv. of nc RCD(K,N) sps.

$\Rightarrow$ $\exists$ Generalizations of §4. $\int |\nabla \omega|^2$

$\int |d\omega|^2 + |\delta\omega|^2$

(9) When

$$\mathrm{Hess}_{f_i} \xrightarrow{L^2_{\mathrm{str}}} \mathrm{Hess}_f \quad ?$$

for $f_i : X_i \to \mathbb{R}$

$$f : X \to \mathbb{R}$$

w.r.t $(X_i, d_i, m_i, x_i) \xrightarrow{\mathrm{pmGH}} (X, d, m, x)$

---

$(X, d, m)$ : $RCD(K, N)$

$$K \geq 0$$

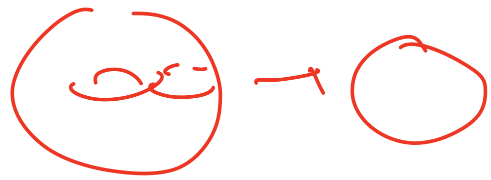
$$b_1(X) \leq n$$

the standard def. of

$\dim\{\mathrm{harm.}\ 1\text{-forms}\} \geq$ ? Bochner-Weit.

$(X_i, d_i, \mathcal{H}^N, x_i)$ : $RCD(K,N)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \wedge$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ nc

$\Rightarrow \exists\, i(j)\quad \exists\, (X, d, x)$ : proper



s.t.

$(X_{i(j)}, d_{i(j)}, \mathcal{H}^N, x_{i(j)})$ $\mathcal{H}^2 \longrightarrow 0$

$$\xrightarrow{\text{pmGH}} (X, d, \mathcal{H}^N, x)$$

$$\mathcal{H}^N(B_r(y_{i(j)})) \longrightarrow \mathcal{H}^N(B_r(y))$$

for $\forall r > 0$, $\forall y_{i(j)} \longrightarrow y$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \in \quad\quad\quad\quad \in$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad X_{i(j)} \quad\quad\quad X$

Moreover

$$\lim_{j\to\infty} \mathcal{H}^n(B_1(x_{i(j)})) > 0,$$

$$\shortparallel$$

$$\mathcal{H}^n(B_1(y))$$

then $(X, d, \mathcal{H}^n)$ : ncRCD$(K, n)$

$(M^n, d^g)$ : closed

$f \in C^\infty(M^n)$ $\quad f \not\equiv$ const

$\Rightarrow$ $\forall N > n$ $\quad \exists K \in \mathbb{R}$

$(M^n, d^g, e^{-f}\mathrm{vol}^g)$

: RCD$(K, N)$.

$\underline{a}$, $\exists\,(X_i, d_i, c_i\mathcal{H}^N)$

$: RCD(\hat{K}, N)$ s.t.

$$(X_i, d_i, c_i\mathcal{H}^N) \xrightarrow{mGH} (H^n, d^g, e^{-f}\,vol^g)$$