Lecture No 24 May 31, 2022 (Tue)

2. Semilinear SPDEs of parabolic type

We discuss semilinear SPDEs of higher order in general form, and are especially concerned with the regularity of solutions.

- Consider the SPDEs for $u = u(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$:

$$\partial_t u = Au + B\{b(x,u)\} + C\{c(x,u)\dot{W}(t,x)\}. \qquad (1)$$

- $A = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C_b^\infty(\mathbb{R}^d)$, $m \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, where

$$|\alpha| = \sum_i \alpha_i, \quad D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}.$$

The coefficients satisfy uniform ellipticity condition:

$$\inf_{x,\sigma\in\mathbb{R}^d, |\sigma|=1} (-1)^{m+1} \sum_{|\alpha|=2m} a_\alpha(x)\sigma^\alpha > 0,$$

where $\sigma^\alpha = \sigma_1^{\alpha_1}\cdots\sigma_d^{\alpha_d}$ for $\sigma = (\sigma_1, \ldots, \sigma_d)$.

- $b(x,u)$, $c(x,u)$ are nonlinear functionals of u.
  The case $c = c(x)$ is called additive noise, while in general $c = c(x,u)$ is called multiplicative noise.
- $B = \sum_{|\alpha| \le n} b_\alpha(x) D^\alpha$ with $b_\alpha \in C_b^\infty(\mathbb{R}^d)$, $n \in \mathbb{Z}$.
- $C = \sum_{|\alpha| \le \ell} c_\alpha(x) D^\alpha$ with $c_\alpha \in C_b^\infty(\mathbb{R}^d)$, $\ell \in \mathbb{Z}$.
- The integers n and ℓ may be negative, then they are regarded as integral operators. Here we assume $n, \ell \ge 0$.
- $\dot{W}(t,x)$ is the space-time Gaussian white noise.
- TDGL equation (2) of non-conservative type:
  $$m = 1, n = 0, \ell = 0.$$
  TDGL equation (3) of conservative type:
  $$m = 2, n = 2, \ell = 1.$$

Recall

$$\partial_t u = \frac{1}{2}\Delta u - \frac{1}{2}V'(u) + \dot{W}(t,x), \tag{2}$$

$$\partial_t u = -\frac{1}{2}\Delta^2 u + \frac{1}{2}\Delta\{V'(u)\} + \nabla \cdot \dot{\mathbb{W}}(t,x). \tag{3}$$

2.1. Concepts of Solutions

- We take the weighted L^2-spaces

$$L_r^2 = L^2(\mathbb{R}^d, e^{-2r\chi(x)}dx), \quad r > 0,$$

as the state spaces for solutions of (1), where $\chi \in C^\infty(\mathbb{R}^d)$ such that $\chi(x) = |x|$ for $|x| \geq 1$.

[Definition 1] $u(t, x)$ is called a solution of (1) with initial value u_0 in the sense of generalized functions, if it satisfies

$$\begin{aligned} \langle u(t), \varphi \rangle =& \langle u_0, \varphi \rangle + \int_0^t \{\langle u(s), A^* \varphi \rangle + \langle b(\cdot, u(s)), B^* \varphi \rangle\} ds \\ &+ \int_0^t \int_{\mathbb{R}^d} C^* \varphi(x) c(x, u(s)) W(dsdx), \end{aligned} \tag{4}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, where $\langle u, \varphi \rangle = \int_{\mathbb{R}^d} u(x)\varphi(x)dx$. □

- Another way to give a mathematical meaning to (1) is based on Duhamel's principle:

[Definition 2] $u(t,x)$ is called a mild solution of (1), if it satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)B\{b(\cdot, u(s))\}ds + \int_0^t T(t-s)C\{c(\cdot, u(s))dW_s\},$$

where $T(t) = e^{tA}$ is a semigroup generated by the operator A in L^2_r. The last term is defined as a stochastic integral w.r.t. white noise process for non-random operator as its integrand. □

- In typical cases, two notions of solutions are equivalent (as we saw on $[0,1]$ with boundary conditions).
- If $b(x,u)$ and $c(x,u)$ are Lipschitz continuous on L_r, under the condition on m, n, ℓ stated below, the (mild) solution exists uniquely. This is shown by standard successive approximation.

2.2. Regularity of solutions

[Proposition 1] Assume $2m > 2\ell + d$ and in addition, for simplicity, $n < 2\ell + \frac{d}{2}$. Then, for the solution $u(t,x)$ of (1), we have that

$$u(t,x) \in C^{\alpha-,\beta-}((0,\infty) \times \mathbb{R}^d), \quad \text{a.s.}, \tag{5}$$

with

$$\alpha = \frac{2m - 2\ell - d}{4m} \quad \text{and} \quad \beta = \frac{2m - 2\ell - d}{2}.$$

□

- In particular, for the TDGL equation (2) of non-conservative type,

$$u(t,x) \in C^{\frac{2-d}{4}-,\frac{2-d}{2}-}((0,\infty)\times\mathbb{R}^d), \quad \text{a.s.}$$

- For the TDGL equation (3) of conservative type,

$$u(t,x) \in C^{\frac{2-d}{8}-,\frac{2-d}{2}-}((0,\infty)\times\mathbb{R}^d), \quad \text{a.s.}$$

- In these cases, the solutions live in the usual function spaces, only when $d = 1$.

Proof of Proposition 1:

- $q(t,x,y)$: fundamental solution of $\partial_t - A$. Then,

$$\left|\partial_t^j D_x^\alpha D_y^\beta q(t,x,y)\right| \le t^{-\frac{|\alpha|+|\beta|}{2m}-j}\bar{q}(t,x,y),$$

for $t \in (0,\, T], x, y \in \mathbb{R}^d$, where

$$\bar{q}(t,x,y) = K_1 t^{-\frac{d}{2m}} \exp\left\{-K_2 \left(\frac{|x-y|^{2m}}{t}\right)^{\frac{1}{2m-1}}\right\},$$

see Eidel'man "Parabolic Systems" 1969, F Osaka J. Math. 1991.

- We consider the mild solution and set

$$u(t,x) = u_1(t,x) + u_2(t,x) + u_3(t,x), \tag{6}$$

$$u_1(t,x) = \int_{\mathbb{R}^d} q(t,x,y)u_0(y)dy,$$

$$u_2(t,x) = \int_0^t \int_{\mathbb{R}^d} B_y^* q(t-s,x,y)b(y,u(s))dsdy,$$

$$u_3(t,x) = \int_0^t \int_{\mathbb{R}^d} C_y^* q(t-s,x,y)c(y,u(s))W(dsdy).$$

- Then, for the term involving the stochastic integrals, Burkholder's inequality and Itô isometry prove for $p \geq 1$

$$E\left[\left|u_3(t,x) - u_3(t',x')\right|^{2p}\right] \leq C\left\{\left|t-t'\right|^{p\frac{2m-2l-d}{2m}} + \left|x-x'\right|^{p(2m-2l-d-\delta)\wedge 2}\right\}, \quad t,t' \in (0,T], x,x' \in \mathbb{R}^d,\ \delta > 0,$$

  as long as both exponents are positive.
- Applying Kolmogorov-Čentsov's theorem, we obtain (5) for $u_3(t,x)$. Other terms u_1 and u_2 have better regularity at least if $n < 2\ell + \frac{d}{2}$.

2.3. Invariant measures, reversible measures (infinite-dimensional case)

- F, Nagoya Math J. 1983: polygonal approximations
- F, Osaka J. Math. 1991: reversible $\Leftrightarrow$ Gibbs measure, by co-cycle property (reduce to finite volume domain) and reducing the problem to that for OU process.

Part B: Stochastic motion by mean curvature

3.1. Background

  3.1.1. Motion by mean curvature (MMC without noise)

  3.1.2. Its derivation under sharp interface limit (SIL)

  3.1.3. Stochastic MMC (SMMC)

3.2. A quick survey of known results

  3.2.1. Motion by mean curvature

  3.2.2. Stochastic MMC

3.3. Some further progress

  3.3.1. SMMC with a direction-dependent smooth noise (DFY)

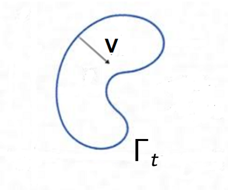
  3.3.2. Volume preserving MMC with noise (FY)

3.1. Background

3.1.1. MMC (without noise)

- MMC is a time evolution of $(d-1)$-dimensional hypersurface Γ_t in $\mathbb{R}^d$ defined by

$$V = \kappa,$$



where V is an inward normal velocity, κ is the mean curvature $\times(d-1)$.

- MMC is geometrically interesting object. From physical view point, it appears as an evolutional law of phase separating interfaces.

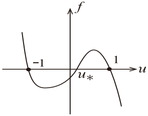
3.1.2. Its derivation under sharp interface limit (SIL)

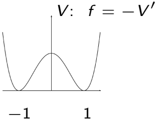
- Allen-Cahn equation (Reaction-diffusion equation):

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2} f(u), \quad t > 0,\ x \in D \subset \mathbb{R}^d,$$

  with Neumann condition at ∂D.

- Here $\varepsilon > 0$ is a small parameter, f is bistable with stable points ± 1 and unstable point $u_* \in (-1, 1)$:





- We always assume no flux condition (=balance condition):

$$A(f) := \int_{-1}^{1} f(u)\, du \ \big(= V(-1) - V(1) \big) = 0,$$

  where V is the corresponding potential s.t. $f = -V'$.

- A traveling wave solution $m = m(y), y \in \mathbb{R}$ with speed $c = c(f) \in \mathbb{R}$ is determined by

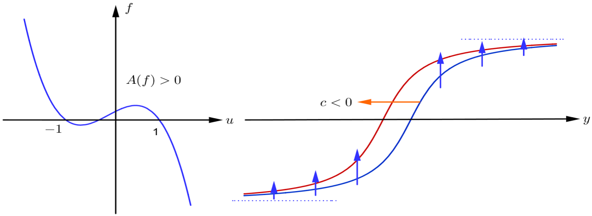
$$\begin{cases} m'' + cm' + f(m) = 0, \quad y \in \mathbb{R}, \\ m(\pm\infty) = \pm 1. \end{cases}$$

- Namely, $v(t, y) = m(y - ct)$ is a solution of

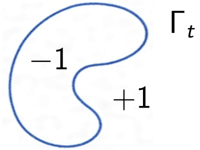
$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + f(v), \quad t > 0, \ y \in \mathbb{R}. \tag{7}$$

- We normalize m as $m(0) = 0$.

- $A(f)$ and $-c(f)$ has the same signs, in particular, $A(f) = 0 \iff c(f) = 0$ (no flux condition).



- For the solution $u = u^\varepsilon$ of Allen-Cahn equation, we expect $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = +1$ or -1. The problem is to find the evolutional law of the interface Γ_t separating two phases ± 1 under proper time scale.



- In fact, if $A(f) = 0$, the proper time scale is $O(1)$, i.e.,

$$u^\varepsilon(t,x) \underset{\varepsilon\downarrow 0}{\longrightarrow} \chi_{\Gamma_t}(x) := \begin{cases} +1, & \text{outside of } \Gamma_t, \\ -1, & \text{inside of } \Gamma_t, \end{cases}$$

  and Γ_t moves according to the MMC (except the case of $d = 1$).

Heuristic derivation of MMC

- Assume the transition $+1 \leftrightarrow -1$ near the interface Γ_t happens as a scaling of m (as in the Drumhead model):

$$u^\varepsilon(t,x) \sim m\Big(\frac{d(x,\Gamma_t)}{\varepsilon}\Big),$$
$$d(x,\Gamma_t) = \text{signed distance between } x \text{ and } \Gamma_t.$$

  Then,

$$\begin{aligned} 0 &= \frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) \qquad \text{(by Allen-Cahn eq)} \\ &\sim \frac{1}{\varepsilon} m'\left(\frac{d}{\varepsilon}\right) \frac{\partial d}{\partial t} - \left\{ \frac{1}{\varepsilon} m'\left(\frac{d}{\varepsilon}\right) \Delta d + \frac{1}{\varepsilon^2} m''\left(\frac{d}{\varepsilon}\right) |\nabla d|^2 \right\} - \frac{1}{\varepsilon^2} f(m) \\ &= \frac{1}{\varepsilon} m'\left(\frac{d}{\varepsilon}\right) \left\{ \frac{\partial d}{\partial t} - \Delta d \right\}. \end{aligned}$$

- The last line follows from the equation $m'' + f(m) = 0$ and $|\nabla d| = 1$ near Γ_t.
- Since $V = \frac{\partial d}{\partial t}$ and $\kappa = \Delta d$ on Γ_t, we obtain MMC.

3.1.3. Stochastic MMC

- Recall Time-dependent Ginzburg-Landau (TDGL) equation of non-conservative type:

$$\partial_t u = -\frac{1}{2}\frac{\delta H}{\delta u(x)}(u) + \dot{W}(t,x),\ x \in \mathbb{R}^d.$$

- Here $\dot{W}(t,x)$ is a space-time Gaussian white noise with covariance structure formally given by

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y),$$

and

$$H(u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}|\nabla u(x)|^2 + V(u(x)) \right\} dx.$$

- Since the functional derivative is given by

$$\frac{\delta H}{\delta u(x)} = -\Delta u - f(u(x)), \quad f = -V',$$

TDGL eq (of non-conservative type) has the form:

$$\partial_t u = \frac{1}{2}\Delta u + \frac{1}{2}f(u) + \dot{W}(t,x). \tag{8}$$

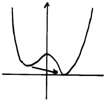
- Stochastic PDE (8) is ill-posed, when $d \geq 2$.
- $d = 2, 3$: Regularity structure by Hairer, Paracontrolled calculus by Gubinelli-Imkeller-Perkowski.

Heuristic derivation of Stochastic MMC: $V = \kappa + \bar{c}\dot{W}_t$

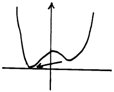
- Consider TDGL equation dropping $\frac{1}{2}$ and with suitably scaled in ε:

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}\left\{f(u) + \varepsilon \dot{W}_t\right\}, \quad \dot{W}_t = \dot{W}(t,x).$$

- In other words, the potential V is randomly perturbed to $V(u) - (\varepsilon \dot{W}_t)u$ and this yields a small traveling wave toward the minimizer of the perturbed potential.



- This gives a fluctuation $\bar{c}\dot{W}_t$ in the limit.



- More precisely, for $a \in \mathbb{R}$ (with small $|a|$), define $m = m(y;a), c = c(a)$ by

$$\begin{cases} m'' + cm' + \{f(m) + a\} = 0, \quad y \in \mathbb{R}, \\ \qquad\qquad m(\pm\infty) = m_\pm^*, \end{cases} \tag{9}$$

where $m_\pm^* \equiv m_\pm^*(a) = \pm 1 + O(a)\,(a \to 0)$ are solutions of $f(m_\pm^*) + a = 0$.

- The solution $u = u^\varepsilon$ is expected to behave as

$$u^\varepsilon(t,x) \sim m(d(x,\Gamma_t)/\varepsilon; \varepsilon \dot{W}_t),$$
$$d(x,\Gamma_t) = \text{signed distance between } x \text{ and } \Gamma_t.$$

- Then, we have

$$\begin{aligned} 0 &= \frac{\partial u^\varepsilon}{\partial t} - \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) - \frac{1}{\varepsilon}\dot{W}_t \\ &\sim \frac{1}{\varepsilon} m'\left(\frac{d}{\varepsilon}\right)\frac{\partial d}{\partial t} - \left\{\frac{1}{\varepsilon} m'\left(\frac{d}{\varepsilon}\right)\Delta d + \frac{1}{\varepsilon^2} m''\left(\frac{d}{\varepsilon}\right)|\nabla d|^2\right\} \\ &\qquad\qquad - \frac{1}{\varepsilon^2}\left\{f(m) + \varepsilon \dot{W}_t\right\} \\ &\sim \frac{1}{\varepsilon} m'\left(\frac{d}{\varepsilon}\right)\left\{\frac{\partial d}{\partial t} - \Delta d + \bar{c}\dot{W}_t\right\}. \end{aligned}$$

- The last line follows from (9), $|\nabla d| = 1$ near Γ_t and $c(a) = c(0) + c'(0)a + O(a^2) = -\bar{c}a + O(a^2)$.
- (9) was used to cancel the terms of order $O(1/\varepsilon^2)$.

- Thus the condition to cancel the terms of the order $O(1/\varepsilon)$ becomes

$$\frac{\partial d}{\partial t} = \Delta d + \bar{c}\dot{W}_t,$$

- Since $V = \frac{\partial d}{\partial t}$ and $\kappa = \Delta d$ on Γ_t, we obtain the limit equation

$$V = \kappa + \bar{c}\dot{W}_t.$$

- $\bar{c}$ is called the inverse surface tension.
- We didn't clearly state the dependence of $\dot{W}_t$ on x, but heuristically one can deal with the noise $\dot{W}_t = \dot{W}(t,x)$ and even $\dot{W}(t,x,\mathbf{n})$, space and direction-dependent noise.