Reference

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- [F15] Farrell. "Bundles with extra geometric or dynamic structure", 2015
- [S77] Sullivan. "Hyperbolic geometry and homeomorphisms" 1977
- [O01] Okun, "Nonzero degree tangential maps between dual symmetric spaces", 2001.
- [FJ89] Farrell, Jones, "negatively curved mfd with exotic smooth structures", 1989
- [KM63] Kervaire, Milnor. "Groups of homotopy spheres, I". 1963
- [L85] Levine, "Lectures on grps of homotopy spheres", 1985.
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- [C70] Cerf, "La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie", 1970
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Q: $\exists$ element of ∞ order in $\pi_k T^{<0}(M)$?

$$\Downarrow \ k \geq 2$$

$$\pi_k T^{<}(M) \otimes \mathbb{Q} \neq 0 \ ?$$

$$D^n \triangleq \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$$

$$\mathrm{Diff}(D^n, \partial) \triangleq \{ f : D^n \xrightarrow[\text{diff.}]{\cong} D^n \mid f|_{\partial D^n} = id \}$$

with C^∞- top.

$$\Omega \mathrm{Diff}(D^{n-1}, \partial) \triangleq \{ [0,1] \xrightarrow[\text{cont. map}]{\beta} \mathrm{Diff}(D^n, \partial) \mid \beta(0) = \beta(1) \}$$ with compact-open top.

based loop space of $\mathrm{Diff}(D^{n-1}, \partial)$

called a (cont.) loop

called a smooth loop.

$\Omega_s \mathrm{Diff}(D^{n-1}, \partial)$

$$\triangleq \left\{ \beta : [0,1] \longrightarrow \mathrm{Diff}(D^{n-1}, \partial) \mid \beta(0) = \beta(1). \ \begin{matrix} D^{n-1} \times I \longrightarrow D^{n-1} \times I \\ (x, t) \longmapsto (\beta_t(x), t) \end{matrix} \text{ smooth} \right\}$$

The inclusion

$i : \Omega_s \mathrm{Diff}(D^{n-1}, \partial) \longrightarrow \Omega \mathrm{Diff}(D^{n-1}, \partial)$ is a homotopy equivalence.

we will assume (when necessary)

- loops are smooth
- loops are constant near 0 and 1

Define the Gromoll's map

$$\alpha_n : \Omega \mathrm{Diff}(D^{n-1}, \partial) \longrightarrow \mathrm{Diff}(D^n, \partial)$$

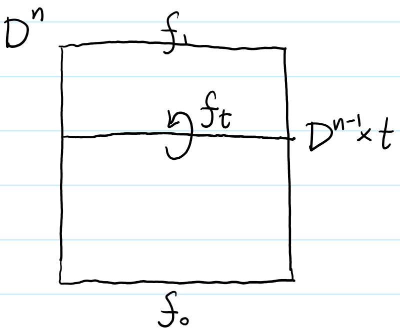
as follows:

$\forall f \in \Omega \mathrm{Diff}(D^{n-1}, \partial)$

$\rightsquigarrow$ a cont. family of self-diffeom.

$f_t \in \mathrm{Diff}(D^{n-1}, \partial)$, $t \in [0,1]$

with $f_0 = f_1 = id$



Identify D^n with $D^{n-1} \times [0,1]$

Define

$$\begin{aligned} \alpha_n(f) : D^n &\longrightarrow D^n \\ \| \quad & \qquad \| \\ D^{n-1} \times [0,1] & \qquad D^{n-1} \times [0,1] \\ (x, t) &\longmapsto (f_t(x), t) \end{aligned}$$

$\Rightarrow \alpha_n(f)\big|_{\partial D^{n-1} \times [0,1] \cup D^{n-1} \times \{0,1\}} = id$

$\Rightarrow \alpha_n(f) \in \mathrm{Diff}(D^n, \partial)$.

$\alpha_n : \Omega \mathrm{Diff}(D^{n-1}, \partial) \longrightarrow \mathrm{Diff}(D^n, \partial)$

$\rightsquigarrow \alpha_{n*} : \pi_{i+1} \mathrm{Diff}(D^{n-1}, \partial) \longrightarrow \pi_i \mathrm{Diff}(D^n, \partial)$

Th ([BFJ18]) Let $i, n \in \mathbb{Z}_{>0}$, $i \geq n+1$. Assume

$\pi_{i+1} \mathrm{Diff}(D^{n-1}, \partial) \otimes \mathbb{Q} \xrightarrow{\alpha_{n*}} \pi_i \mathrm{Diff}(D^n, \partial) \otimes \mathbb{Q} \xrightarrow{\alpha_{n+1}*} \pi_{i-1} \mathrm{Diff}(D^{n+1}, \partial) \otimes \mathbb{Q}$ is nontrivial $(*)$

Then for every closed hyp. mfd N of dim n.
$\exists$ finite sheeted cover M of N s.t.

$$F_* : \pi_{i+1} T^{<0}(M) \otimes \mathbb{Q} \longrightarrow \pi_{i+1} T(M) \otimes \mathbb{Q}$$

is nontrivial. In particular,

$$\pi_{i+1} T^{<0}(M) \otimes \mathbb{Q} \neq 0$$

The existence of i, n satifying $(*)$ is guaranteed by Weiss' discovery of nontrivial rational Pontrjagin classes $\in H^{4k+4m}(BTop(2m); \mathbb{Q})$ for $k > 0$

$$Top(2m) = \{ f : \mathbb{R}^{2m} \xrightarrow{\approx} \mathbb{R}^{2m} \}.$$

Th ([BFJ18]) $\exists C > 0$ s.t. the following holds:
for every closed real hyp. mfd of M^n of

dim $n \geq C$, $\exists$ a finite sheeted cover $\hat{M}$ of M s.t.

$$F_*: \pi_i T^{<0}(\hat{M}) \otimes \mathbb{Q} \longrightarrow \pi_i T(\hat{M}) \otimes \mathbb{Q}$$

is nontrivial for some $i \in [n+1, 2n]$.

$$\Rightarrow \pi_i T^{<0}(\hat{M}) \otimes \mathbb{Q} \neq 0.$$

Now sketch pf for following (A special case for Th3.3)

Th Let M^6 be a closed hyp. mfd of dim 6. Then $\exists$ a finite sheeted cover N of M s.t.

$$F_*: \pi_1 T^{<0}(N) \longrightarrow \pi_1 T(N)$$

is nontrivial. ($\Rightarrow \pi_1 T^{<0}(N) \neq 0$

$\Rightarrow T^{<0}(N)$ is non contractible).

Rk: The condition of "existence of exotic sphere of dim $n+1$" is dropped here because $\exists$ exotic 7-sphere.

Sketch pf (cf. [F15])

Fact 1: Given a closed hyp. mfd M, $\exists$ a finite sheeted cover $\hat{M}$ of M, which is stably parallelizable, i.e.

$$T\hat{M} \oplus \varepsilon^1 \text{ is a trivial bundle}$$

tangent bundle of $\hat{M}$

trivial line bundle

(cf. [S77] [O01])

Rk: any cover of a stably parallelizable mfd is still stably parallelizable.

Fact 2: Given $R \in \mathbb{R}_{>0}$ and a closed hyp. mfd M, $\exists$ finite sheeted cover $\hat{M}$ s.t. any cover of $\hat{M}$ has a geodesic ball of radius R.
(cf. [FJ89, p.901])

By Fact 1, 2, taking large enough finite sheeted cover of the real hyp. mfd M^6, we obtain a real hyp. mfd (N^6, g) s.t.

① N contains a geodesic ball of radius $4r$ (where r is large, how large will become apparent as we proceed)

② N is stably parallelizable

Strategy: construct $f \in \mathrm{Diff}_0(N)$ s.t.

(1) f is not isotopic to id, i.e.
$\nexists$ $F: I \longrightarrow \mathrm{Diff}(N)$ cont. s.t.
$F(0) = f$, $F(1) = id$

(2) $\exists$ path α in $\mathrm{Met}^{<0}(N)$
s.t. $\alpha(0) = f \cdot g$, $\alpha(1) = g$

Claim: If $\exists f \in \mathrm{Diff}_0(N)$ s.t. (1) (2) hold,
$\Rightarrow$ $F_*: \pi_1 T^{<0}(N) \longrightarrow \pi_1 T(N)$ nontrivial

pf: Consider fiber bundle

$$\underbrace{\mathrm{Diff}_0(N)}\circlearrowright \mathcal{R}^{<0}(N) \longrightarrow T^{<0}(N) \ni [g]$$
$$h \longmapsto h \cdot g$$

$\sigma(\mathrm{Diff}_0(N))$ is the fiber over $[g]$.

$\rightsquigarrow$ $\pi_1 T^{<0}(N) \xrightarrow{\delta_1} \pi_0 \mathrm{Diff}_0(N) \xrightarrow{\sigma_*} \pi_0 \mathcal{R}^{<0}(N)$ exact

$$
\begin{tikzcd}
\arrow[dr, "F_*"'] & \circlearrowleft & \arrow[u, "\cong"'] \\
& & \pi_1 T(N)
\end{tikzcd}
$$

(Reason:

$$
\begin{tikzcd}
\mathrm{Diff}_0(N) \arrow[r, hook, "\sigma"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & R^{<0}(N) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & T^{<0}(N) \arrow[d, "F"] \\
\mathrm{Diff}_0(N) \arrow[r] & R(N) \arrow[r] & T(N)
\end{tikzcd}
$$

$$
\Longrightarrow
\begin{tikzcd}
\pi_1 R^{<0}(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \pi_1 T^{<0}(N) \arrow[r, "\delta_1"] \arrow[d, "F_*"] \arrow[dr, phantom, "\circlearrowleft"] & \pi_0 \mathrm{Diff}_0(N) \arrow[d, equal] & \\
\pi_1 R(N) \arrow[r] \arrow[d, equal] & \pi_1 T(N) \arrow[r, "\cong"] & \pi_0 \mathrm{Diff}_0(N) \arrow[r] & \pi_0 R(N) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}
$$
).

$$\Longrightarrow \ker \sigma_* = \operatorname{im} F_*$$

$$\ker\sigma_* = \{[h] \in \pi_0 \mathrm{Diff}_0(N) \mid \sigma_*([h]) = [g] \in \pi_0 R^{<0}(N)\}, \quad \sigma_*([h]) = [h\cdot g]$$

(1) $\Longrightarrow [f] \neq [\mathrm{id}] \in \pi_0 \mathrm{Diff}_0(N) = \pi_1 T(N)$

(2) $\Longrightarrow [f\cdot g] = [g] \in \pi_0 R^{<0}(N) \Longrightarrow [f] \in \ker\sigma_* = \operatorname{im} F_*$

$$\Longrightarrow F_* : \pi_1 T^{<0}(N) \longrightarrow \pi_1 T(N) = \pi_0 \mathrm{Diff}_0(N)$$

$$\exists\, \bullet \longmapsto [f] \neq [\mathrm{id}]$$

is nontrivial.

$\square$

The construction of f needs more knowledge on exotic spheres.

$$\Theta_n \triangleq \{[\Sigma^n] \mid \Sigma^n \text{ closed orientable smooth mfd of dim } n,\ \Sigma^n \approx S^n\}$$

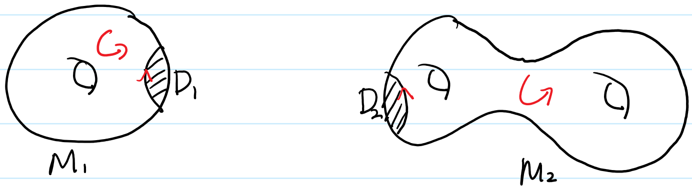
$[\Sigma^n]$: orientation-preserving diffeomorphism classes of Σ^n

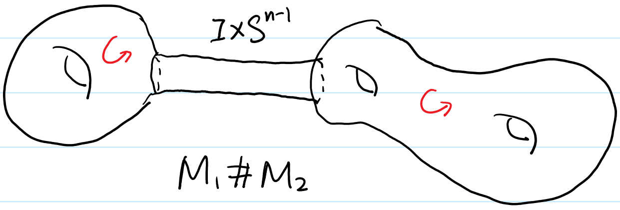
Given two connected n-mfds M_1, M_2, remove a small open n-ball D_i^n from each

$\rightsquigarrow M_i \setminus \mathring{D}_i^n$, $i=1,2$

Take $f: \partial(M_2\setminus \mathring{D}_2) \xrightarrow{\cong} \partial(M_1 \setminus \mathring{D}_1)$

$M_1\setminus\mathring{D}_1 \cup_f M_2\setminus\mathring{D}_2$ is the <u>connected sum</u> of M_1 and M_2, denoted by $M_1 \# M_2$.





$(\Theta_n, \#)$ <u>forms an abelian grp for $n\neq 4$</u>, is called grp of homotopy spheres / grp of exotic spheres

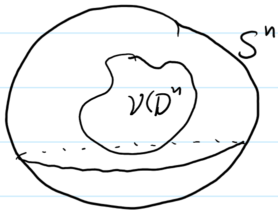
<u>Rk</u>: • Θ_n is finite.

•

n	≤ 3	4	5	6	7	8	9	10	11	12	13
$\#\Theta_n$	1	?	1	1	28	2	8	6	992	1	3

(cf. [KM63][L85])

Given embedding $\nu: D^n \longrightarrow S^n$.



Consider $\pi_0 \mathrm{Diff}(D^n,\partial) \xrightarrow{\phi} \Theta_{n+1}$

$[h] \longmapsto [D^{n+1}\cup_{\hat{h}} D^{n+1}]$

$h\in \mathrm{Diff}(D^n,\partial) \underset{\text{by id on } S^n\setminus\nu(D^n)}{\overset{\text{extend}}{\rightsquigarrow}} \hat{h}\in \mathrm{Diff}(S^n)$

$\rightsquigarrow D^{n+1}\cup_{\hat{h}} D^{n+1} \approx S^{n+1}$.

Φ is $\cong$:

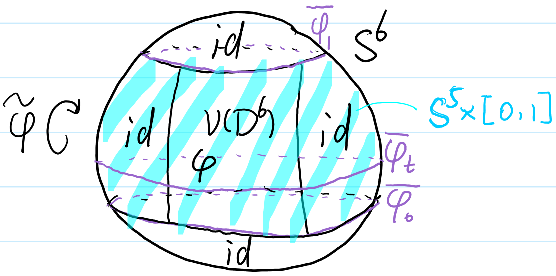
Φ surjective: cf. [S61] Smale's h-cob. th

Φ injective: cf. [C70]

Now let $n=6$, $\Theta_7 \neq 0$.

Consider $\pi_0 \mathrm{Diff}(D^6, \partial) \xrightarrow[\Phi]{\cong} \Theta_7$
$[\varphi] \longmapsto [\Sigma^7] \neq 0$

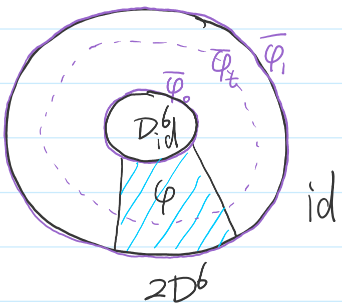
(Assume $\nu: D^6 \hookrightarrow S^5 \times [0,1] \hookrightarrow S^6$)



$\varphi \in \mathrm{Diff}(D^6, \partial) \overset{\text{extend}}{\rightsquigarrow} \overline{\varphi} \in \mathrm{Diff}(S^5 \times [0,1])$

$\overline{\varphi}_t : S^5 \xrightarrow{\cong} S^5$, $t \in [0,1]$

f is constructed as picture below:



$\subseteq N^6 \circlearrowleft f \in \mathrm{Diff}(N)$

Claim: $f \in \mathrm{Diff}_0(N)$.

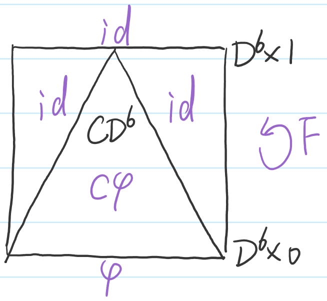
Pf: $\mathrm{Diff}(D^6, \partial) \xrightarrow[\text{by id}]{\text{extend}} \mathrm{Diff}(N)$ for some embedding $D^6 \hookrightarrow N$
$\varphi \longmapsto f$

To show $f \sim id : N \to N$, it suffices to show

$\overset{\text{extend by id.}}{\varphi \sim id : D^6 \to D^6 \text{ rel } \partial D^6}$

given as follows:

$CD^6 \subset D^6 \times [0,1]$



Define $F : D^6 \times [0,1] \circlearrowleft$

$$F(x) = \begin{cases} C\varphi(x), & x \in CD^6 \\ x, & x \in D^6 \times [0,1] \setminus CD^6 \end{cases}$$

$\Rightarrow F$ cont. $\quad F(\cdot, 0) = \varphi$, $F(\cdot, 1) = id$

$\square$

Claim: f satisfy (1), i.e.

f is not isotopic to id

pf: $\boxed{\text{Suppose } f \text{ is isotopic to id.}}$

$\Rightarrow M_f \triangleq N \times [0,2] / (x,0) \sim (f(x),2)$

mapping torus of f

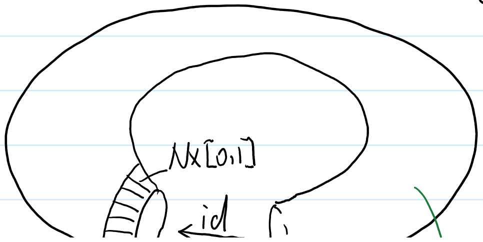
$\Vert S$ ①

$M_{id} \triangleq N \times [0,2] / (x,0) \sim (x,2) = N \times S^1$

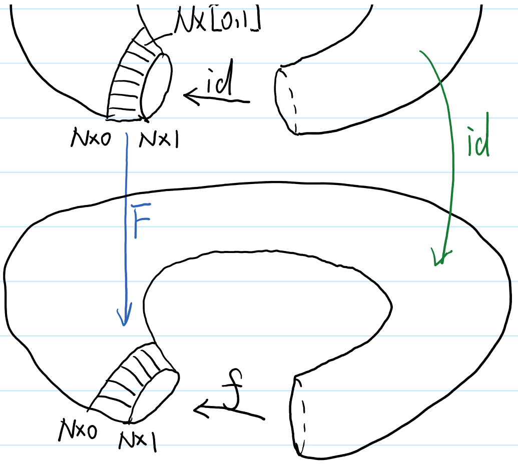
(Reason for ①: Since f is isotopic to id

$\Rightarrow \exists\, F \in \mathrm{Diff}(N \times I)$ s.t. $F(x,0) = (x,0)$

$F(x,1) = (f(x),1)$



$= M_{id}$

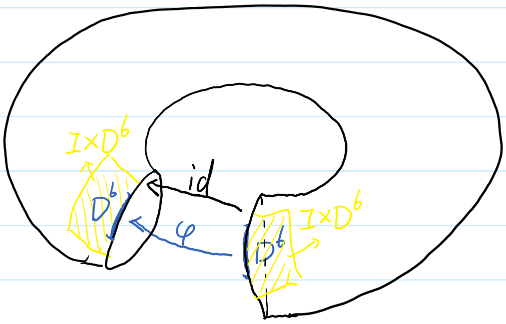


$= M_f$

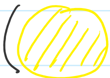
The picture gives diffeomorphism : $M_{id} \cong M_f$

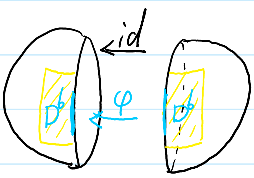
On the other hand, we show

② $M_f \cong N \times S^1 \# \Sigma$

$M_f =$ 

$M_f \setminus$ $= N \times S^1 \setminus \mathring{D}^7$

Note $\partial($  $) = S^6$ and $\cup_{S^6} D^7$

$=$ 

$= D^7 \cup_{\tilde{\varphi}} D^7 = \Sigma^7$

$\Rightarrow$ $= \Sigma^7 \setminus \mathring{D}^7$

$\Rightarrow M_f = N \times S^1 \setminus \mathring{D}^7 \cup_{id} \Sigma \setminus \mathring{D}^7 = N \times S^1 \# \Sigma$

$\Rightarrow \quad N \times S^1 \# \Sigma \not\cong N \times$

$$\Rightarrow \quad \begin{array}{ccc} N\times S^1 \# \Sigma & \not\cong & N\times S^1 \\ \text{②}\,\|\,S & & \|\,S\,\text{①} \\ M_f & & M_f \end{array}$$

The proof is difficult. (using N is parallelizable).

$\Rightarrow$ contradiction! claim follows $\square$

Rk: $\exists$ smooth closed mfd M^n s.t.
for any exotic sphere Σ^n,
$$M^n \# \Sigma \cong M^n.$$
eg. when $M = \mathbb{H}P^2$,
(cf. [KS07])