ON THE VIRTUAL RASMUSSEN INVARIANT

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Abstract. We produce chain-level generators of the virtual Lee complex $Kh'(V)$ and use them to convert the computable bounds on the Rasmussen invariant of classical knots due to Kawamura and Lobb into bounds on the virtual Rasmussen invariant as defined by Dye, Kaestner, and Kauffman. We also exhibit a class of diagrams for which the bounds are tight. In addition, we use the chain-level generators to show that the virtual Rasmussen invariant is additive with respect to connect sum.

Keywords: virtual knot theory, Khovanov homology, Rasmussen invariant, knot concordance

Mathematical subject classification 2010: 57M25, 57M27

1. Introduction

1.1. **The slice genus.** The *slice genus* of a knot $K \subset S^3 \subset \partial B^4$, denoted $g^*(K)$, is the least genus of all smoothly embedded surfaces in B^4 which bound K. If $g^*(K) = 0$ we say that K is *slice*. Rasmussen [16] used Lee's degeneration of Khovanov homology [12] to produce a knot invariant, specifically a homomorphism on the smooth concordance group of knots

$$s : \mathcal{C} \to 2\mathbb{Z}$$

which yields a lower bound on the slice genus i.e.

$$|s(K)| \leq 2g^*(K).$$

In principle it is tough to compute $s(K)$, however, as it is equivalent to the maximal filtration grading of all elements homologous to a certain generator of the Lee homology of K.

Kawamura and Lobb [13] independently defined diagram-dependent upper bounds on $s(K)$, denoted $U(D)$ (for D a diagram of K), which are easily computable by hand, along with an error term, $\Delta(D)$, the vanishing of which implies that $s(K) = U(D)$, in fact. More precisely,

$$U(D) - 2\Delta(D) \leq s(K) \leq U(D).$$

The bounds $U(D)$ are henceforth referred to as the *strong slice-Bennequin bounds.*

1.2. **Virtual knot theory.** Kauffman initated the study of virtual links [10] which are

- 4-valent planar graphs with three possible decorations of the vertices - the over- and under-crossings of classical knot theory and a *virtual crossing*,  - up to certain equivalences. Such equivalence classes of graphs are known as *virtual links.*

or equivalently

- Embeddings of disjoint unions of S^1 into thickened genus g surfaces up to self-diffeomorphism and handle stabilisation of the surface.

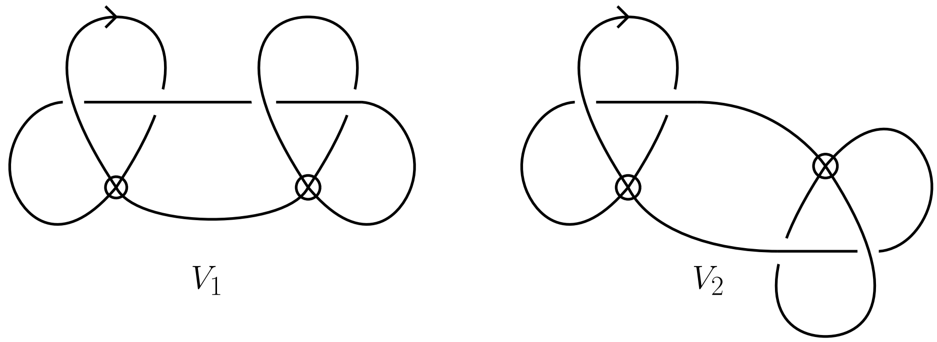


FIGURE 1. Diagrams of two inequivalent virtual knots which are connect sums of the same initial pair of virtual knots. They are distinguished by the virtual Alexander polynomial [17].

In the latter interpretation a classical link is as an embedding of $\bigsqcup S^1$ into a thickened S^2.

There is a third interpretation of virtual links which we shall make use of below [4, 8]. Given a virtual link diagram we can algorithmically produce a closed curve on a surface with boundary which deformation retracts onto the curve. Such a curve and surface pair is known as an *abstract link diagram*. An *abstract link* is an equivalence class of abstract link diagrams up to *stable equivalence* as defined in Definition 2.2.

1.3. **Virtual cobordism and the slice genus.** In direct analogue to those of the classical case we make the following definitions (see [6] and [11]). Two virtual knot diagrams V_1 and V_2 are *cobordant* if one can be obtained from the other by a finite sequence of births and deaths of circles, oriented saddles, and virtual Reidemeister moves. Such a sequence describes a compact, oriented surface, C, such that $\partial C = V_1 \sqcup V_2$. If $g(C) = 0$ we say that V_1 and V_2 are *concordant*. If $V_2 = U$, the unknot, and V_1 is concordant to U we say that V_1 is *slice*. In general, we define the *slice genus* of a virtual knot V, denoted g^*, as

$$g^*(V) = \min\{g(C) \mid C \text{ a compact, oriented, connected surface such that } \partial C = V\}$$

(here we have simply capped off the unknot in ∂C with a disc). Behind the scenes, the cobordism surface C is embedded in a 4-manifold of the form $M \times [0,1]$ where M is a compact, oriented 3-manifold with $\partial M = \Sigma_k \sqcup \Sigma_l$, where Σ_i denotes a surface of genus i. The 3-manifold M is described in the standard way in terms of codimension 1 submanifolds and critical points: starting from $\partial M = \Sigma_k$, codimension 1 submanifolds are Σ_k until we pass a critical point, after which they are $\Sigma_{k\pm 1}$. Critical points of M correspond to handle stabilisation. A finite number of handle stabilisations are needed to reach Σ_l.

As detailed below, the Rasmussen invariant of classical knots can be extended to an invariant of virtual knots. Thus, treating a classical knot K as a virtual knot without virtual crossings, this yields a lower bound on the genus of surfaces $C \hookrightarrow \left(M \setminus B^3\right) \times [0,1]$ where $\partial C = K \hookrightarrow S^2 \times [0,1]$ and M is a closed, oriented 3-manifold. It is unknown whether it is possible to lower the slice genus of a classical knot by treating it as a virtual knot, but in light of the above observation it is clear that the Rasmussen invariant provides an interval within which the virtual slice genus must lie. Precisely, for a classical knot K

$$\frac{1}{2}s(K) \leq g^*_{\mathrm{v}}(K) \leq g^*_{\mathrm{c}}(K)$$

for $g^*_{\mathrm{v}}(K)$, $g^*_{\mathrm{c}}(K)$ the virtual and classical slice genera, respectively.

Unlike in the classical case of embeddings $S^1 \hookrightarrow S^3$ it is not immediately apparent whether the set of concordance classes of virtual knots forms a group under the connect sum. In fact, connect sum is not a well-defined operation on virtual knots: one can connect sum two virtual knots together in more than one inequivalent way. An example is given in Fig. 1. The connect sum operation is well-defined, however, on the set of *based* virtual knot diagrams:

Definition 1.1. *A* based *(virtual knot) diagram is an oriented (virtual knot) diagram with a distinguished point on one of its arcs. This distinguished point is known as the* basepoint*.*

Definition 1.2. *Given two based diagrams V_1^* and V_2^* the* connect sum $V_1^* \# V_2^*$ *is defined as the diagram formed by connect summing V_1^* and V_2^* together in the standard way at the site dictated by the basepoints. The basepoints are removed on $V_1^* \# V_2^*$. By an abuse of notation we denote by $V_1^* \# V_2^*$ both the diagram and the virtual equivalence class it represents.*

In this manner the diagrams given in Fig. 1 are seen to be the connect sums of two distinct pairs of based virtual diagrams which differ only in placement of the distinguished point. It is not known whether pairs of virtual knots such as these are concordant. In general, the answer to the following question is not known:

> if V_1^*, V_3^* are any two based diagrams of a virtual knot (that is, they are based versions of (possibly different) diagrams of the same virtual knot), and V_2^*, V_4^* are any two based diagrams of another, are $V_1^* \# V_2^*$ and $V_3^* \# V_4^*$ concordant?

That they were would be sufficient to define a virtual concordance group. Until this is settled, however, it is possible that the virtual Rasmussen invariant is not a homomorphism. We show below, however, that it is at least additive.

1.4. **Structure.** Whilst Manturov [14] first defined Khovanov homology for virtual links we shall work with the reformulation of his theory due to Dye, Kaestner, and Kauffman [6]. In [6] it is shown that virtual Khovanov homology is amenable to the construction of Lee and a virtual Rasmussen invariant is defined in essentially identical fashion to that of the classical case. Our plan is to recreate the strong slice-Bennequin bounds in this setting.

In Section 2.1 we review important definitions of [6], before in Section 2.2 producing chain-level generators of $Kh'(V)$. In Section 3 we then show using these generators that the virtual Rasmussen invariant is additive. Finally, in Section 4, we convert the strong slice-Bennequin bounds to the virtual setting. We assume familiarity with virtual Khovanov homology and the virtual Rasmussen invariant.

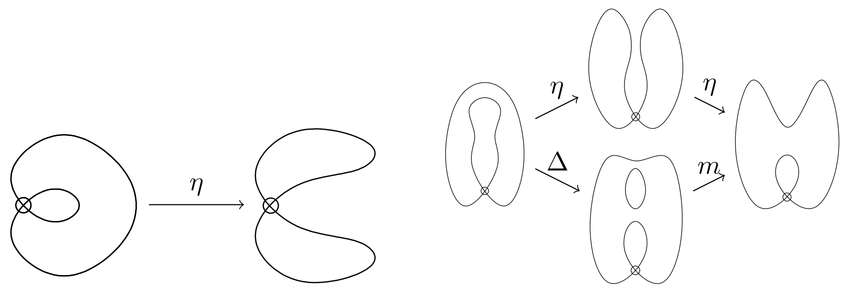
2. Chain-level canonical generators

In [6] generators of the Lee homology of a virtual knot are produced at a diagram level but there is no clear way to push them to chains. In this section we describe a procedure to do just that.

2.1. **Review of** [6]**.**

We review the definitions of the virtual Khovanov complex and the virtual Rasmussen invariant.

2.1.1. *The virtual Khovanov complex.* The fundamental obstruction to transferring Khovanov homology to the virtual setting is the existence of the *single-cycle smoothing* depicted in Fig. 2(A) (otherwise known as a *one-to-one bifurcation*). In order to preserve the quantum grading the map associated to this smoothing, denoted η, must be identically zero. The prototypical face is depicted in Fig. 2(B). Notice that the differential along the top and



(A) The single-cycle smoothing. (B) The problem face.

FIGURE 2

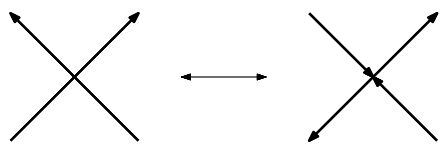


FIGURE 3. The source-sink decoration.

right-hand edges is $\eta \circ \eta = 0$, but along the left-hand and bottom edges it is $m \circ \Delta \neq 0$ so that $d^2 \neq 0$.

Remark: Tubbenhauer [18] has constructed a virtual Khovanov homology theory in the manner of Bar-Natan [2] using non-orientable cobordisms, but there are compatibility issues with the theory presented in [6].

Let $\mathcal{A} = \mathcal{R}[X]/(X^2 - t)$ for $\mathcal{R}$ a commutative ring and $t \in \mathcal{R}$. In order to detect the problem face a symmetry present in $\mathcal{A}$ (which corresponds to the two possible orientations of S^1) is exploited using the following automorphism:

Definition 2.1. *The* barring operator *is the map*

$$\overline{} : \mathcal{A} \to \mathcal{A}, \;\; X \mapsto -X. \tag{1}$$

Applying the barring operator is referred to as conjugation.

Note that if $\mathcal{R} = \mathbb{R}$ and $t = -1$ then $\mathcal{A} = \mathbb{C}$ and the barring operator is just standard complex conjugation. How the barring operator is applied within the Khovanov complex is determined using an extra decoration on link diagrams, the source-sink decoration as depicted in Fig. 3. A new diagram is formed by replacing the classical crossings with the source-sink decoration, which induces an orientation on the incident arcs of a crossing. Arcs of the diagram on which the induced orientations due to separate crossings disagree are marked by a *cut locus*. We refer the reader to [6].

2.1.2. *Lee's degeneration and the virtual Rasmussen invariant.* There is a degeneration of Khovanov homology due to Lee [12]. There is such a degeneration of virtual Khovanov homology also. Dye, Kaestner, and Kauffman use the methods of Bar-Natan and Morrison [3] to show this. Specifically, they employ the Karoubi envelope of a category and the interpretation of virtual links as abstract links [4, 8].

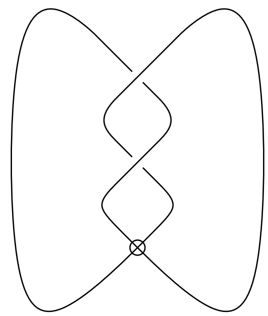


FIGURE 4. A two-crossing virtual knot diagram.

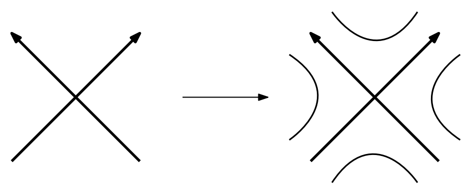


FIGURE 5. Component of the surface of an abstract link diagram about a classical crossing.

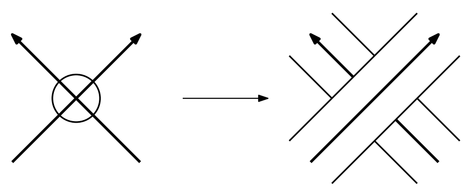


FIGURE 6. Component of the surface of an abstract link diagram about a virtual crossing.

As such diagrams are used extensively below, we describe the process given in [8] to obtain a (representative of an) abstract link from a (representative of a) virtual link (examples are given in Section 2.2). Let V be a diagram of a virtual link, as in Fig. 4, then

i) About the classical crossings place a disc as shown in Fig. 5.
ii) About the virtual crossings place two discs as shown in Fig. 6.
iii) Join up these discs with collars about the arcs of the diagram.

The result is a knot diagram on a surface which deformation retracts onto the underlying curve of the knot diagram. We will denote abstract link diagrams by (F, D) for D a knot diagram and F a compact, oriented surface (which deformation retracts on to the underlying curve of D). We treat such diagrams up to stable equivalence, defined below.

Definition 2.2. *(Definition 3.2 of* [4]*) Let (F_1, D_1) and (F_2, D_2) be abstract link diagrams. We say that (F_1, D_1) and (F_2, D_2) are equivalent, denoted $(F_1, D_1) \leftrightsquigarrow (F_2, D_2)$, if there exists a closed, connected, oriented surface F_3 and orientation-preserving embeddings $f_1 : F_1 \to F_3$, $f_2 : F_2 \to F_3$ such that $f_1(D_1)$ and $f_2(D_2)$ are related by Reidemeister moves on F_3. We say that two abstract link diagrams (F, D) and (F', D') are* stably equivalent *if*

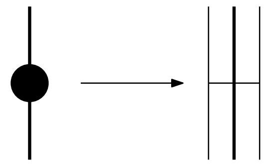


Figure 7. Cross cuts on an abstract link diagram inherited from cut loci.

there is a chain of equivalences

$$(F, D) = (F_0, D_0) \leftrightsquigarrow (F_1, D_1) \leftrightsquigarrow \cdots \leftrightsquigarrow (F_n, D_n) = (F', D')$$

for $n \in \mathbb{N}$.

Stable equivalence classes of abstract link diagrams are in bijective correspondence to equivalence classes of virtual link diagrams [8].

Definition 2.3. *A* smoothing *of an abstract link diagram* (F, D) *is a diagram formed by smoothing the crossings of* D *into either their* 0- *or* 1*-resolution on* F*. The result is a collection of disjoint copies of* S^1 *on the surface* F*. A copy of* S^1 *is called a* cycle.

The diagram-level canonical generators of the Lee complex given in [6] are smoothings of abstract link diagrams with extra information added. This extra information is needed to keep track of the source-sink structure of the virtual knot. The information is in the form of *cross cuts* which are added in the following way: before beginning the procedure described above mark the virtual knot diagram with cut loci as inherited from the source-sink orientation and preserve them on the abstract link diagram. Replace each cut locus with a cross cut which bisects the surface as shown in Fig. 7. Henceforth by *abstract link diagram* we mean an *abstract link diagram with cross cuts*.

Using the source-sink decoration we add yet more information to abstract link diagrams in the form of a *checkerboard colouring*:

Definition 2.4. *From an abstract link diagram* (F, D) *form its associated* checkerboard coloured *abstract link diagram from the surface and curve pair* $(F, S(D))$ *(where* $S(D)$ *denotes the source-sink diagram formed by replacing each crossing by the source-sink decoration) by colouring the surface* F *using the recipe given in Fig. 8 and Fig. 9. Notice that Fig. 8 allows us to induce a checkerboard colouring of smoothings of abstract link diagrams by simply joining the shaded or unshaded areas produced by smoothing the crossing.*

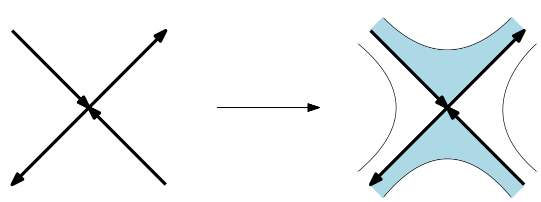


Figure 8. Checkerboard colouring at a crossing.

From checkerboard coloured smoothings of abstract link diagrams we are able to produce the tool used by Dye, Kaestner, and Kauffman to prove theorems analogous to those in [3]. Henceforth we set $\mathcal{R} = \mathbb{Q}$ and $t = -1$.

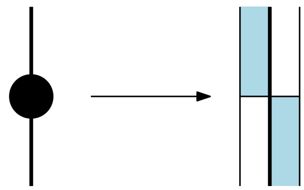


FIGURE 9. Checkerboard colouring at a cut locus.

Definition 2.5. *Let $\{r, g\}$ be the basis for $\mathcal{A}$ where*

$$\text{“red”} = r = \frac{1+X}{2}$$
$$\text{“green”} = g = \frac{1-X}{2}.$$

On the level of diagrams, arcs of a smoothing are coloured red or green to denote which generator they are labelled with.

Lemma 4.1 of [6] lists the properties of r and g. The most important for our purposes is that r and g are conjugates with respect to the barring operator. That is

$$\overline{r} = g \text{ and } \overline{g} = r.$$

Definition 2.6. *(Analogue of Definition* 1.1 *of* [3]*) An* alternately coloured smoothing of an abstract link diagram *is a smoothing for which the arcs have been coloured either red or green such that the arcs passing through each small neighbourhood of a smoothing are coloured different colours. At a cut locus the colouring of an arc switches.*

Using alternately coloured smoothings the following theorems are stated and proved:

Theorem 2.7. *(Theorem 4.2 of* [6]*) Within the Karoubi envelope the Lee complex of a virtual link V is homotopy equivalent to a complex with one generator for each alternately coloured smoothing of K on an abstract link diagram with cross cuts and with vanishing differentials.*

Theorem 2.8. *(Theorem 4.3 of* [6]*) A virtual link V with c-components has exactly 2^c alternately coloured smoothings on an abstract link diagram with cross cuts. These smoothings are in bijective correspondence with the 2^c orientations of V.*

In Section 2.2 we describe the bijective correspondence of Theorem 2.8, but we conclude this section by stating the definition of the virtual Rasmussen invariant and its properties.

Definition 2.9. *Let V be a virtual knot diagram, $CKh'(V)$ and $Kh'(V)$ the associated Lee complex and Lee homology, respectively. Let s be the grading on $Kh'(V)$ induced by j on $CKh'(V)$. Define*

$$s_{min}(V) = \min\{s(x) | x \in Kh'(V),\ x \neq 0\}$$
$$s_{max}(V) = \max\{s(x) | x \in Kh'(V),\ x \neq 0\}.$$

The virtual Rasmussen invariant of V is

$$s(V) = \frac{1}{2}\left(s_{max} + s_{min}\right).$$

Proposition 2.10. *(Parts of Proposition* 6.5 *and Theorem* 5.6 *of* [6]*) The virtual Rasmussen invariant satisfies the following*

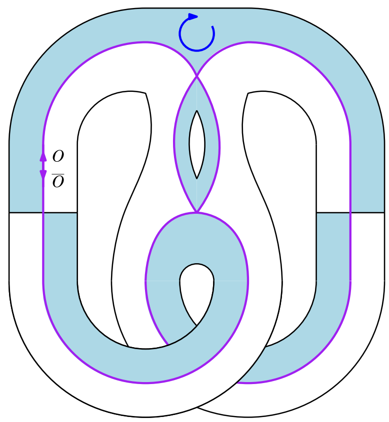


FIGURE 10. A checkboard coloured abstract link diagram corresponding to the virtual knot diagram of Fig. 4, with orientations o and $\overline{o}$.

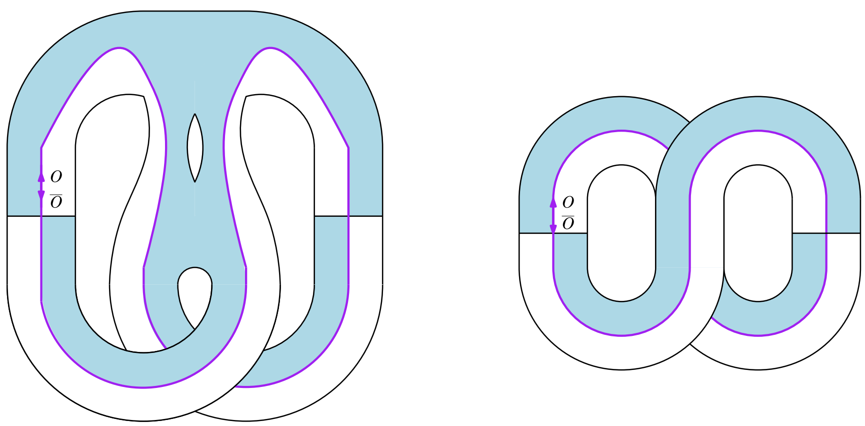
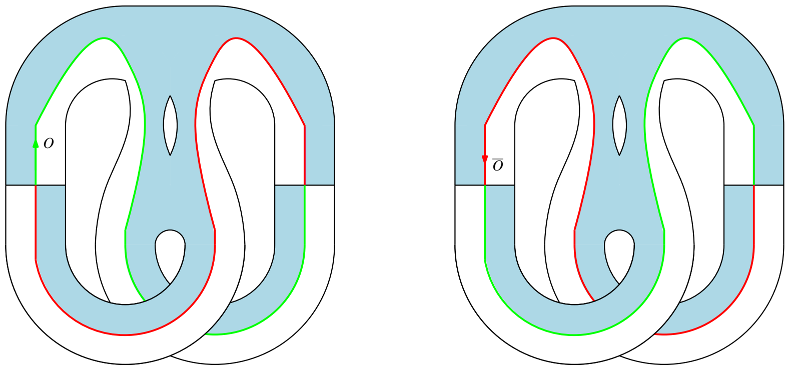


FIGURE 11. Two representatives of the stable equivalence class of smoothings of the checkboard coloured abstract link diagram depicted in Fig. 10, with orientations o and $\overline{o}$.

(1) $s(V) = s_{max} - 1 = s_{min} + 1$.
(2) $s(\overline{V}) = -s(V)$, *for* $\overline{V}$ *the mirror image of* V*: the diagram formed by switching all positive classical crossings to negative classical crossings and vice versa.*
(3) $|s(V)| \leq 2g^*(V)$, *where* $g^*(V)$ *denotes the slice genus of* V*.*

Notice that the virtual Rasmussen invariant lacks the out-of-the-box additivity of its classical counterpart (another consequence of there being multiple ways to connect sum two virtual knots together). In Section 3 we show, however, that the virtual s invariant is indeed additive.

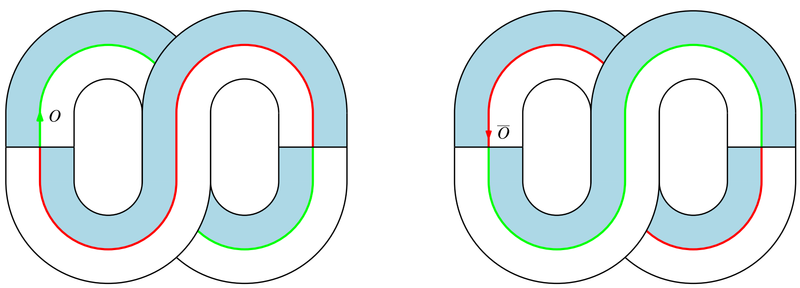
2.2. **Chain-level generators.** In [6] canonical generators are produced at a diagrammatic level i.e. they are alternately coloured smoothings of (checkerboard-coloured) abstract link diagrams. These generators are sufficient to prove Theorems 2.7 and 2.8. Below, we give a method to produce the corresponding chain-level generators of $Kh'(V)$. Before doing so,



(A) The alternately coloured smoothing associated to orientation o.

(B) The alternately coloured smoothing associated to orientation $\overline{o}$.

FIGURE 12. The alternately coloured smoothings on abstract link diagrams corresponding to the two possible orientations of the virtual knot diagram.



(A) A smoothing stably equivalent to that of Fig. 12(A).

(B) A smoothing stably equivalent to that of Fig. 12(B).

FIGURE 13. Alternately coloured smoothings stably equivalent to those of Fig. 12.

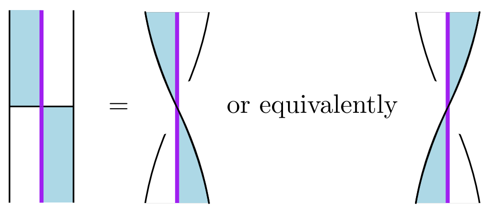
however, it is instructive to recall the bijection of Theorem 2.8 between orientations of a virtual link and alternately coloured smoothings of the associated abstract link diagram as given in [6]. We use Fig. 4 as an example.

(i) Given a virtual link diagram V construct the checkerboard coloured abstract link diagram as in Definition 2.4. Note that for a virtual knot the checkerboard colouring is independent of the orientation, a consequence of the invariance of the source-sink decoration under 180° rotations. See Fig. 10.
(ii) For a given orientation o of V form the corresponding oriented smoothing on the checkerboard coloured abstract link diagram as in Definition 2.3. See Fig. 11.
(iii) Place a clockwise orientation on the shaded regions of the oriented smoothing, which in turn induces a new orientation on the arcs of the smoothing. On each arc compare this orientation to that induced by o. If these two orientations agree colour the arc red, if they disagree colour the arc green (as in Definition 2.5). See Fig. 12.

At this stage we have produced alternately coloured smoothings on abstract link diagrams as in Definition 2.6. We need a way of reading off from these diagrams elements of $C^0Kh'(V)$ (as the oriented resolution is always at height 0), which will be the chain-level canonical generators of $Kh'(V)$. We are unable to do so at this point as the cycles of the alternately coloured smoothings possess more than one colour. We now describe a process by which single coloured smoothings can be produced, and hence chain-level generators of $Kh'(V)$.

Firstly, we utilise the stable equivalence relation given in Definition 2.2 to work with alternately coloured smoothings of abstract link diagrams for which the surface deformation retracts onto the curve of the smoothing, for example the abstract link diagrams given in Fig. 13. We can always do this as the curve of the smoothing is simply a disjoint union of copies of S^1. Note that the resulting smoothing (of a checkerboard coloured abstract link diagram) may not be connected.

Next, we interpret the cross cuts as half-twists with the parity of the twist ignored. That is,



Replacing cross cuts with appropriate half-twists we are able to view the surface of the smoothing (of a checkerboard coloured abstract link diagram) as a two-sided surface such that the curve of the smoothing appears on both sides. That cross cuts always come in pairs ensures that the surface has two sides. Importantly, on each side of the surface the curve of the smoothing is coloured exactly one colour. This is because passing a cross cut causes the arc to change to change colour (c.f. Definition 2.6), and to pass a cut locus is to pass onto the other side of the surface. (From this one can see that passing a cut locus, or equivalently moving on to the other side of the surface, is replicated in $\mathcal{A}$ by applying the barring operator.)

In summary, we view alternately coloured smoothings (of checkerboard coloured abstract link diagrams) such as those in Fig. 13 as two sided surfaces such that the curve of the smoothing is coloured exactly one colour on each side. At this point it is clear that in order to read off generators of CKh_0 from such alternately coloured smoothings we must make a choice of side (or sides, if the surface of the smoothing is disconnected) of the surface to read. Further, we must also ensure that this choice is the same for both the alternately coloured smoothings associated to o and $\overline{o}$. We must have this as they are both coloured versions of the same smoothing of an abstract link diagram (the oriented smoothing) c.f. the left hand smoothing of Fig. 11 with Fig. 12. In effect we are making the choice on this uncoloured smoothing, which the alternately coloured smoothings then inherit.

With all this in mind, let us make a choice: given a virtual knot diagram V with orientations o and $\overline{o}$, let A denote the oriented smoothing of the checkerboard coloured abstract link diagram associated to V. On A cancel an arbitrary pair of adjacent cross cuts against one another so that the strand they bound is removed. An example is given in Fig. 14. This cancellation of cross cuts is simply 'flipping' the segment of the surface they bound so that the other side of the surface is shown. Continue cancelling available arbitrary pairs of cross cuts until all have been removed. In our interpretation, that the smoothing has no cross cuts means that we are looking at exactly one side of surface. Now return to part (iii) of

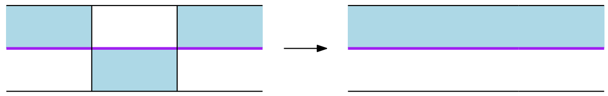


FIGURE 14. Removing a strand by cancelling cross-cuts.

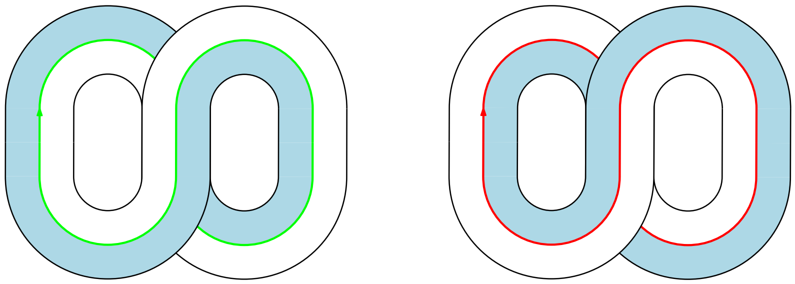


FIGURE 15. The possible ways to cancel the alternately coloured smoothing corresponding to orientation o of V.

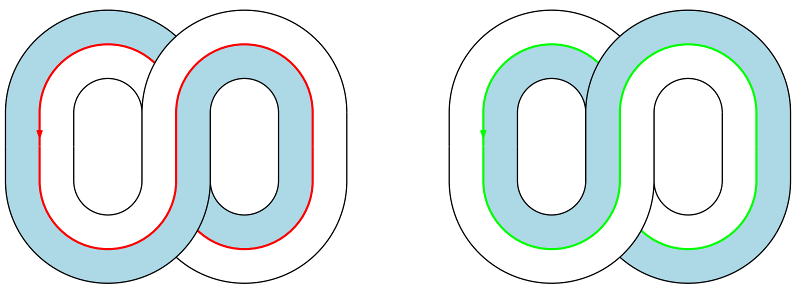


FIGURE 16. The possible ways to cancel the alternately coloured smoothing corresponding to orientation $\overline{o}$ of V.

the process given on page 9, and colour the cycles of the oriented smoothings associated to o and $\overline{o}$ as dictated there. Denote by A_o and $A_{\overline{o}}$ the resulting alternately coloured abstract link diagrams associated to o and $\overline{o}$, respectively. That the cycles of A_o and $A_{\overline{o}}$ are coloured with opposite colours follows from the fact that their orientations are opposite but the checkerboard colouring of A_o and $A_{\overline{o}}$ is the same.

Examples of such single coloured smoothings are given in Fig. 15 and Fig. 16. In this case a choice of top and bottom is equivalent to picking either the two smoothings on the left of the Figures, or the two on the right.

After all that we are left with smoothings of abstract link diagrams the cycles of which are coloured with exactly one colour, either red or green. We form the canonical generators of $Kh'(V)$, denoted $\mathfrak{s}_o$ for o an orientation of V, by taking the appropriate tensor product of r and g as dictated by the colours of the cycles. In this way we obtain two distinct algebraic generators.

We conclude by remarking that the s invariant is independent of this choice of which side of the surface to read. Making another choice results in an application of the barring operator to one or more tensor factors of $\mathfrak{s}_o$ and $s_{\overline{o}}$, because if a cycle is coloured green on one side of the surface it must be coloured red on the other. But conjugation does not interact with the filtration, that is

$$j(\overline{r}) = j(g) \text{ and } j(\overline{g}) = j(r).$$

To conclude this section we prove a Lemma analogous to Lemma 3.5 of Rasmussen [16] which we will use in both the following sections.

Lemma 2.11. *(Lemma 3.5 of Rasmussen [16]) Let n be the number of components of V. There is a direct sum decomposition $Kh'(V) \cong Kh'_o(V) \oplus Kh'_e(V)$, where $Kh'_o(V)$ is generated by all states with q-grading conguent to $2+n \mod 4$, and $Kh'_e(V)$ is generated by all states with q-grading congruent to $n \mod 4$. If o is an orientation on V, then $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ is contained in one of the two summands, and $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ is contained in the other.*

Proof. The first statement follows exactly as in the classical case. Regarding the second statement, following [16] let $\iota : CKh'(V) \to CKh'(V)$ be the map which acts by the identity on $CKh'_e(V)$ and multiplication by -1 on $CKh'_o(V)$. We claim that $\iota(\mathfrak{s}_o) = \pm\mathfrak{s}_{\overline{o}}$. To show this we define a new grading on $\mathcal{A}$ with respect to which X has grading 2 and 1 has grading 4. We have that $\overline{X} = -X$ and $\overline{1} = 1$ so that $\overline{r} = g$ and $\overline{g} = r$, and the map

$$\overline{}^{\otimes n} : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$$

(which applies the barring operator to all tensor factors) acts as the identity on elements with new grading congruent to $0 \mod 4$ and multiplication by -1 on elements with new grading congruent to $2 \mod 4$. The new grading differs from the q-grading by an overall shift so that

$$\iota(\mathfrak{s}_o) = \pm\overline{\mathfrak{s}_o}^{\otimes n} = \pm\mathfrak{s}_{\overline{o}}$$

as in the classical case. □

A direct corollary of Lemma 2.11 is that $\mathfrak{s}_o$ is not of top filtered degree, that is:

$$s(\mathfrak{s}_o) = s(\mathfrak{s}_{\overline{o}}) = s_{min}(V). \tag{2}$$

3. Additivity of the virtual Rasmussen invariant

We can use the chain-level generators of $Kh'(V)$ to show that the virtual Rasmussen invariant is additive with respect to connect sum, confirming that the virtual invariant behaves in the same way as its classical counterpart in this respect.

Theorem 3.1. *For $V_1^* \# V_2^*$ the connect sum of based virtual knot diagrams V_1^* and V_2^* (c.f. Definition 1.1 and Definition 1.2)*

$$s(V_1^* \# V_2^*) = s(V_1) + s(V_2). \tag{3}$$

Proof. With the chain-level generators in place, along with Lemma 2.11, the proof follows much the same path as that in [16]. For all connect sums $V_1^* \# V_2^*$ there exists the map

$$Kh'(V_1^* \# V_2^*) \xrightarrow{\Delta'} Kh'(V_1 \sqcup V_2) \cong Kh'(V_1) \otimes Kh'(V_2).$$

It sends a canonical generator $\mathfrak{s}_o$ of $Kh'(V_1^* \# V_2^*)$ to a canonical generator of $Kh'(V_1) \otimes Kh'(V_2)$ of the form $\mathfrak{s}^1 \otimes \mathfrak{s}^2$ where $\mathfrak{s}^i$ is a generator of $Kh'(V_i)$ for $i = 1, 2$. As in the classical case, the map is of filtered degree -1 and we obtain

$$\begin{gathered} s(\mathfrak{s}_o) - 1 \leq s(\mathfrak{s}^1 \otimes \mathfrak{s}^2) = s(\mathfrak{s}^1) + s(\mathfrak{s}^2) \\ s_{min}(V_1^* \# V_2^*) \leq s_{min}(V_1) + s_{min}(V_2), \text{ by Eq. (2).} \end{gathered} \tag{4}$$

From this point the proof proceeds as in that of the analogous statement in [16]: utilising the fact that $s_{min}(V) = -s_{max}(\overline{V})$ we are able to obtain from Eq. (4) that

$$s_{min}(V_1^* \# V_2^*) = s_{min}(V_1) + s_{min}(V_2) + 1$$
$$s_{max}(V_1^* \# V_2^*) = s_{max}(V_1) + s_{max}(V_2) - 1$$

as required. □

As a result of Theorem 3.1, the Rasmussen invariant of the connect sum of two virtual knots is independent of the choice of based diagrams used to conduct the connect sum. As mentioned in Section 1.3 it is not known whether $V_1^* \# V_2^*$ is concordant to $V_3^* \# V_4^*$ for V_1^*, V_3^* any two based diagrams of a virtual knot and V_2^*, V_4^* any two based diagrams of another. It is known, however, that neither the Jones polynomial [15] nor the Rasmussen invariant can distinguish them. This leads one to posit whether Khovanov homology can, a question the author intends to return to.

4. Bounds on the virtual Rasmussen invariant

Equipped with chain-level generators we are in a position to formulate the strong slice-Bennequin bounds in the virtual setting. First we define the bounds themselves before describing the case in which they are tight.

4.1. **Formulation.**

Definition 4.1. *Given a virtual link diagram V denote by $O(V)$ the oriented smoothing of V. The* Seifert graph *of V, denoted $T(V)$, is the signed graph with a vertex for each cycle of $O(V)$ and an edge for each classical crossing of V, decorated with the sign of the crossing. The edge associated to a crossing is between the node or nodes associated to the cycles involved in the smoothing site of that crossing. The subgraph of $T(V)$ formed by removing all the edges labelled with $+$ (respectively $-$) is denoted $T^-(V)$ (respectively $T^+(V)$).*

Definition 4.2. *Given a virtual link diagram V the quantities $U(V)$, $\Delta(V) \in \mathbb{Z}$ are given by*

$$U(V) = \#\text{ nodes } (T(V)) - 2\#\text{ components } (T^-(V)) + wr(V) + 1$$
$$\Delta(V) = \#\text{ nodes } (T(V)) - \#\text{ components } (T^+(V)) - \#\text{ components } (T^-(V)) + 1.$$

The quantities $U(V)$ and $\Delta(V)$ are dependent on the diagram V and are not invariants of the virtual link.

Theorem 4.3. *(Theorem 1.2 of Lobb [13]) For V a diagram of a virtual knot*

$$s(V) \leq U(V).$$

Notice that the left hand side is a knot invariant whereas the right is diagram-dependent.

To prove this we require Lemma 2.11, as we have canonical generators in terms of r and g instead of $a = 2r$ and $b = -2g$ and the proof given in [16] relies on the sign of a and b.

Proof. (of Theorem 4.3) The proof is practically identical to that of the classical case in [13]. Form the diagram V^- from V by smoothing all the positive classical crossings of V to their oriented resolution, and suppose that V^- is the disjoint union of l virtual link diagrams. Label these diagrams $V_1^-, V_2^-, \dots, V_l^-$. Then the canonical generator $\mathfrak{s}_o$ splits as a tensor product of canonical generators of $Kh'(V_r^-)$ as

$$\mathfrak{s}_o = \mathfrak{s}_1 \otimes \mathfrak{s}_2 \otimes \cdots \otimes \mathfrak{s}_l.$$

(A) An alternating virtual knot diagram which is not homogeneous. It is virtual knot 3.7 in Green's table [7].

(B) The Seifert graph of virtual knot 3.7.

FIGURE 17

Classically, $\mathfrak{s}_r$ can either be $\mathfrak{s}_{o'}$ or $\mathfrak{s}_{\overline{o'}}$ where o' denotes the induced orientation on V_r^-, as we are possibly altering the number of cycles separating others from infinity. In the virtual case, however, $\mathfrak{s}_r = \mathfrak{s}_{o'}$ by construction as we use abstract link diagrams to produce the canonical generators rather than the method due to Lee.

Where the proof given in [13] invokes Theorem 3.5 of [16] we invoke Lemma 2.11 as given above. □

Theorem 4.4. *(Theorem* 1.10 *of Lobb* [13]*) If* $\Delta(V) = 0$ *then* $s(V) = U(V)$*. In fact*

$$U(V) - 2\Delta(V) \leq s(V) \leq U(V).$$

The proof of Theorem 4.4 is identical to that of the classical case, owing to the identical behaviour of the virtual and classical Rasmussen invariants with respect to the mirror image.

4.2. **The case $\Delta(V) = 0$.** Cromwell defined *homogeneous links* using the Seifert graph [5]. Following this, we make the complementary definition for virtual links.

Definition 4.5. *A* cut vertex *of a graph G is a vertex such that the graph obtained by removing the vertex along with its boundary edges has more connected components than G .*

Definition 4.6. *A* block *of a graph G is a maximal connected subgraph of G containing no cut vertices.*

By signed graph we mean a graph with edges decorated with exactly one of two possible signs, $+$ and $-$.

Definition 4.7. *A signed graph G is* homogeneous *if every block B of G is such that all edges contained in B are decorated with the same sign.*

Definition 4.8. *A virtual link diagram V is* homogeneous *if $T(V)$ (the Seifert graph as defined in Definition 4.1) is homogeneous. A virtual link is homogeneous if there exists a diagram of it which is homogeneous.*

Positive and negative virtual knots are homogeneous trivially (as their Seirfert graphs possess only one kind of decoration). In the classical case, alternating knots are also homogeneous [9]. In the virtual case, however, this no longer holds. For example, the virtual knot diagram given in Fig. 17(A) is alternating but not homogeneous.

Abe showed that for a classical knot diagram D, $\Delta(D) = 0$ if and only if D is homogeneous [1]. As this result is obtained using the Seifert graph only, and by recalling that $U(D)$ and

$\Delta(D)$ are defined identically in the classical and virtual cases we are able to state the following:

Theorem 4.9. *(Virtual analogue of Theorem 3.4 of Abe [1]) A virtual knot diagram V is homogeneous if and only if $\Delta(V) = 0$. Hence, for a homogeneous diagram V of a virtual knot K*

$$U(V) = s(K).$$

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