Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 4. Explicit computations for examples in the last lectures

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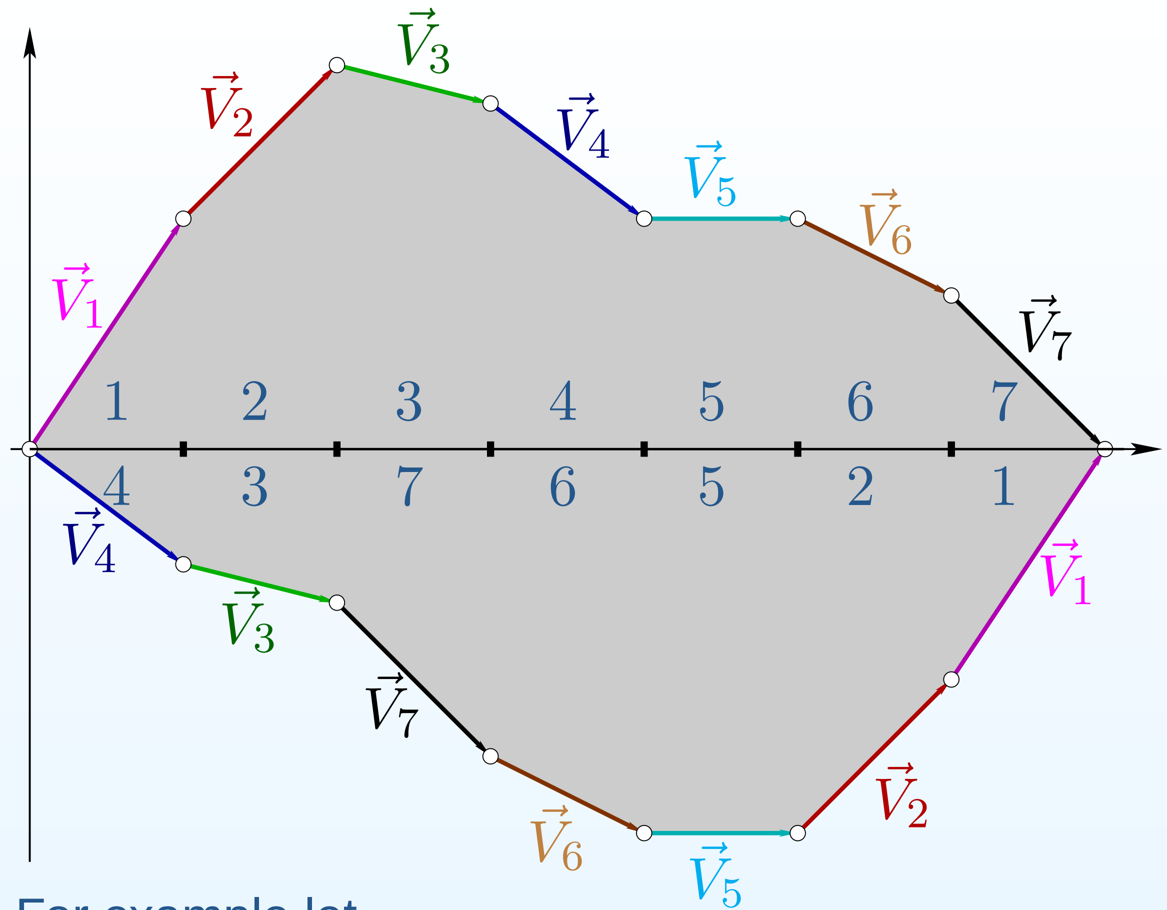
YMSC, Tsinghua University, October 13, 2022

Detecting the stratum associated to an interval exchange transformation

Canonical suspension

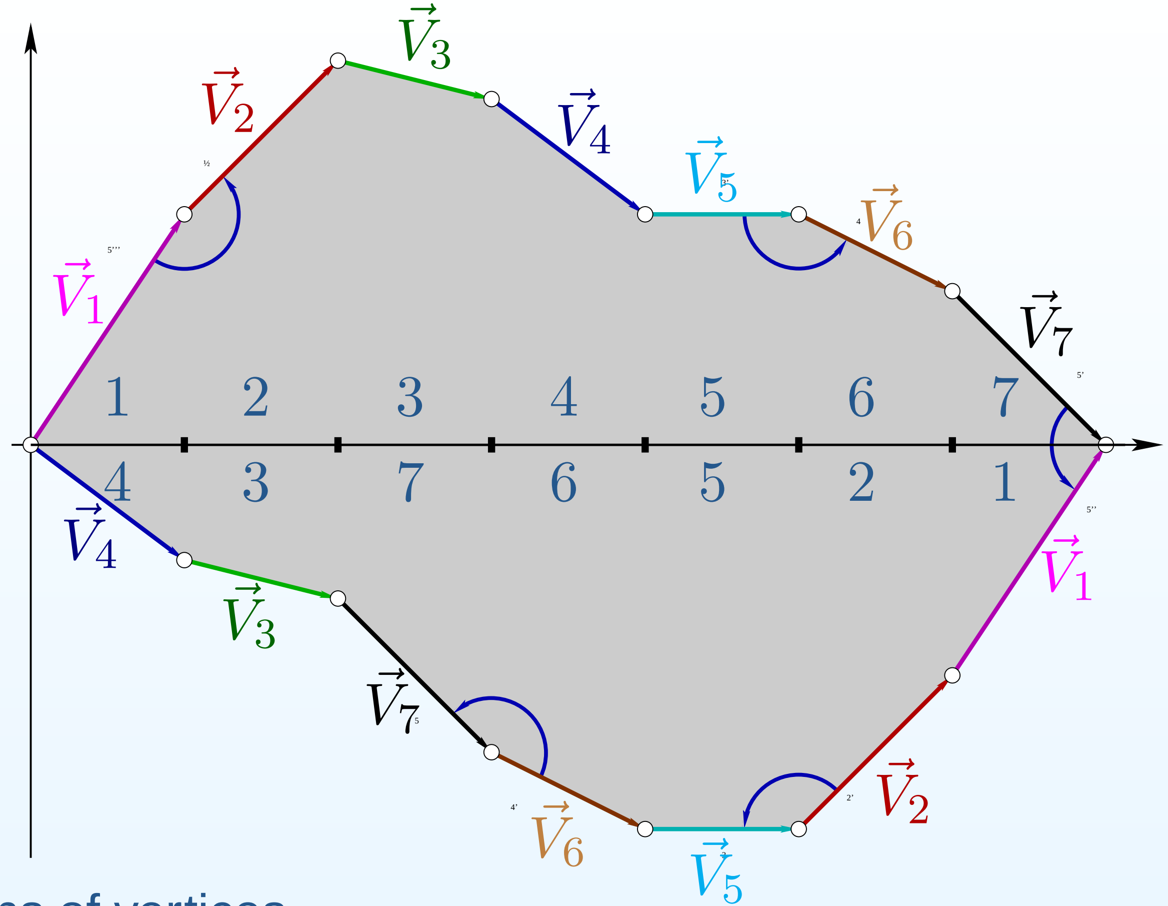


Consider some permutation. For example let

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 7 & 6 & 5 & 2 & 1 \end{pmatrix} .$$

Ambient stratum depends on the permutation but not on a particular choice of coordinates of vectors of the suspension. We can choose $\vec{V}_j = (1, \pi(j) - j)$.

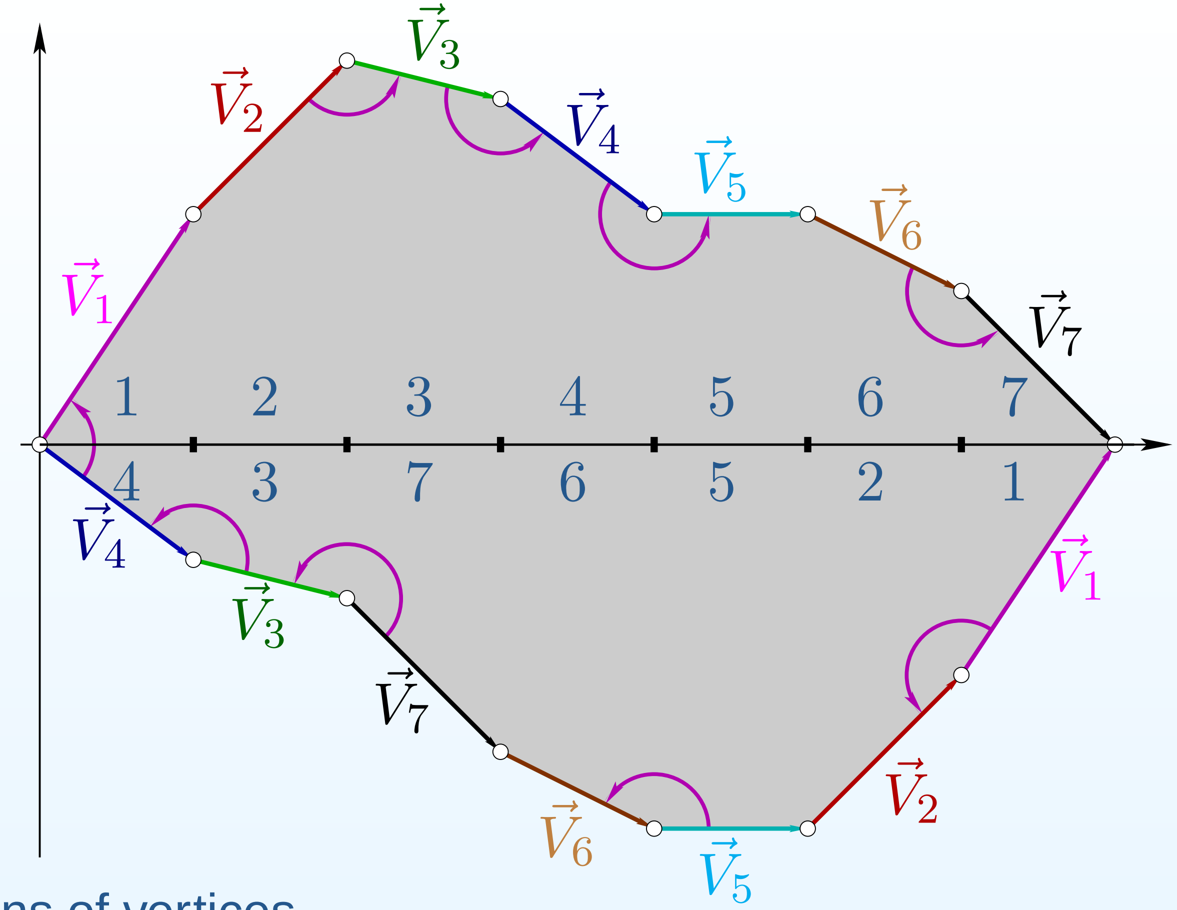
Canonical suspension



Now let us trace identifications of vertices.

We turn around a conical point on the surface before we complete a loop.

Canonical suspension



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Then we proceed with the next loop.

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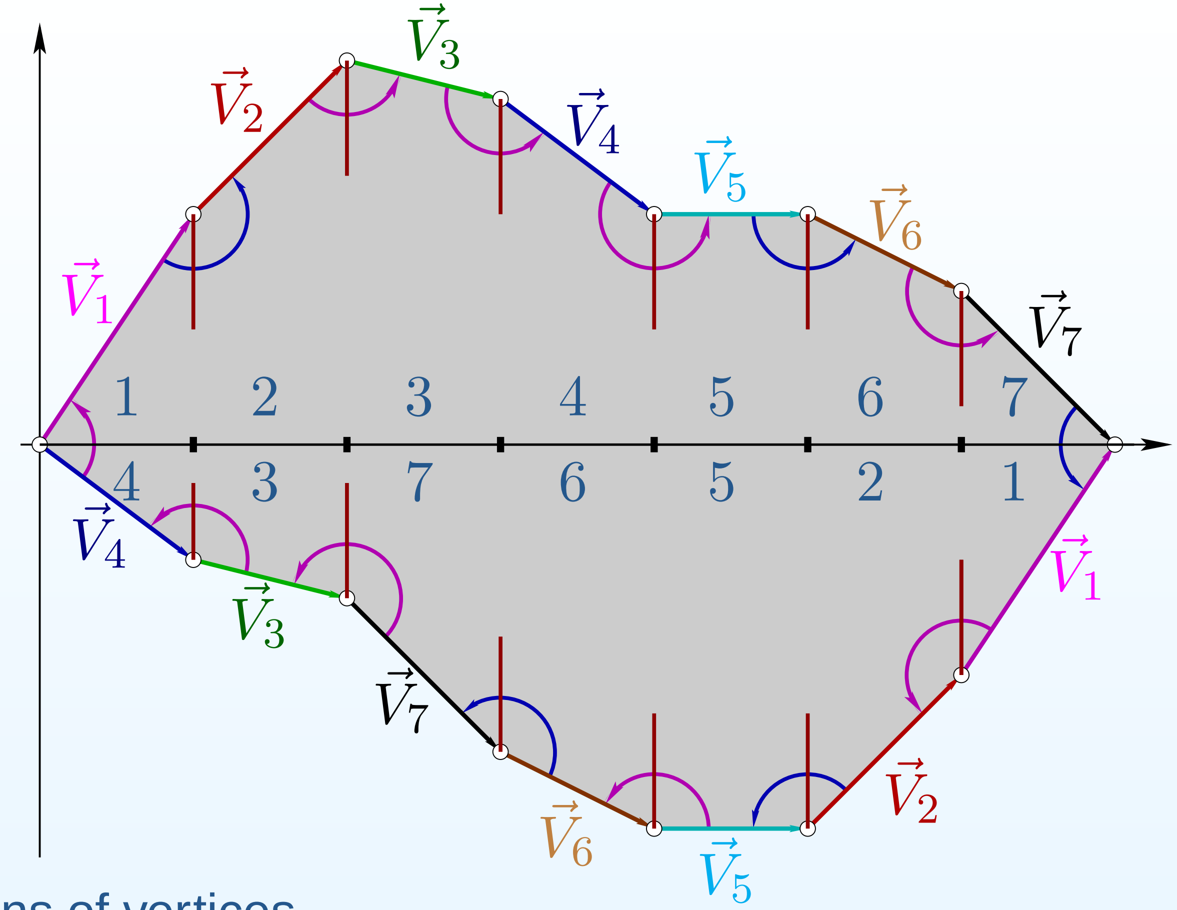

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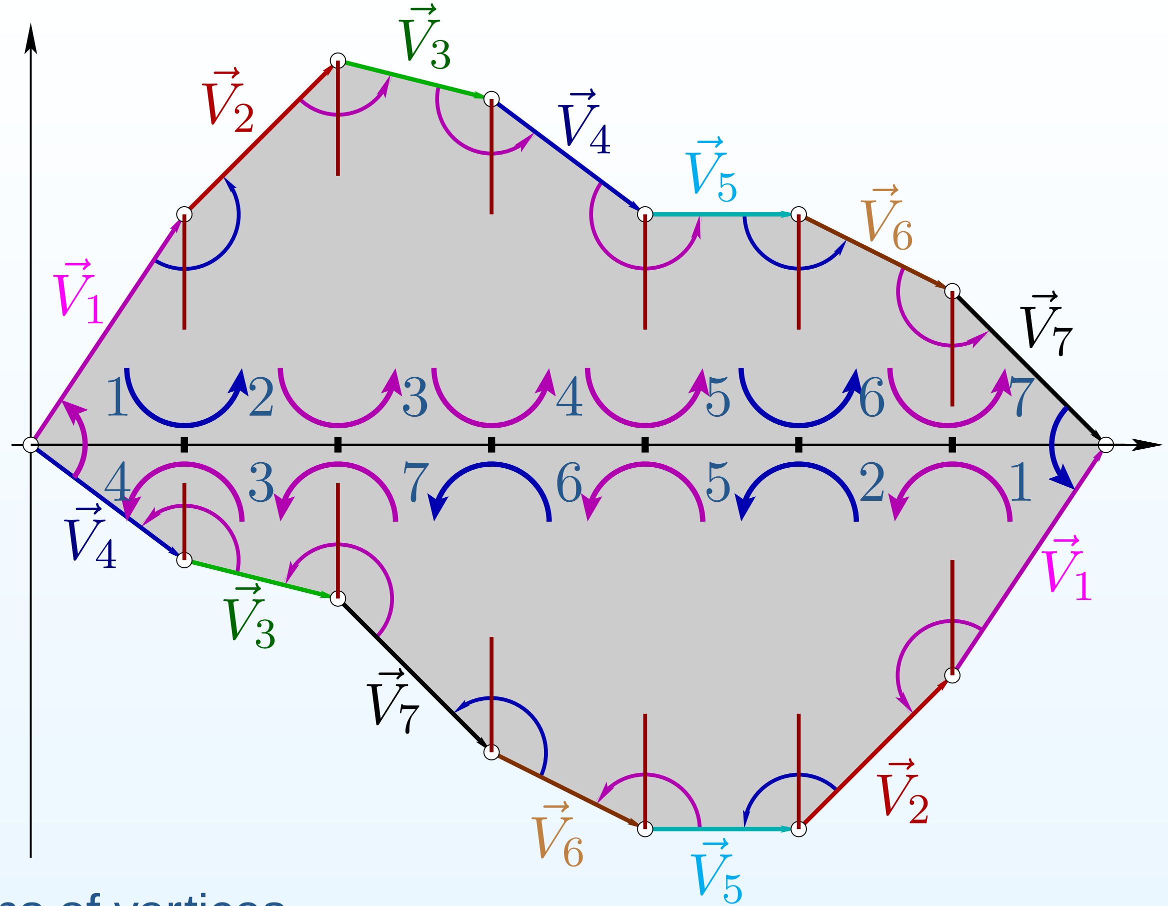
We turn around a conical point on the surface before we complete a loop.

Then we proceed with the next loop.

To compute cone angles we count how many times our loops cross the vertical direction: 4 times the blue cycle and 8 times the purple one. We are in $\mathcal{H}(3,1)$.

Canonical suspension



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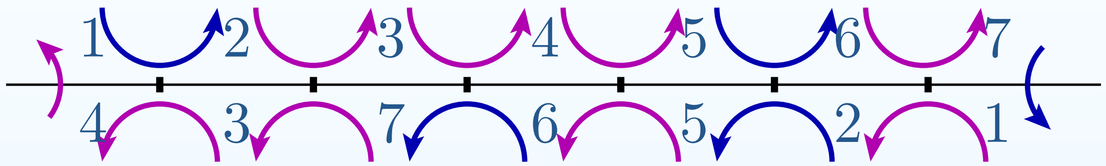

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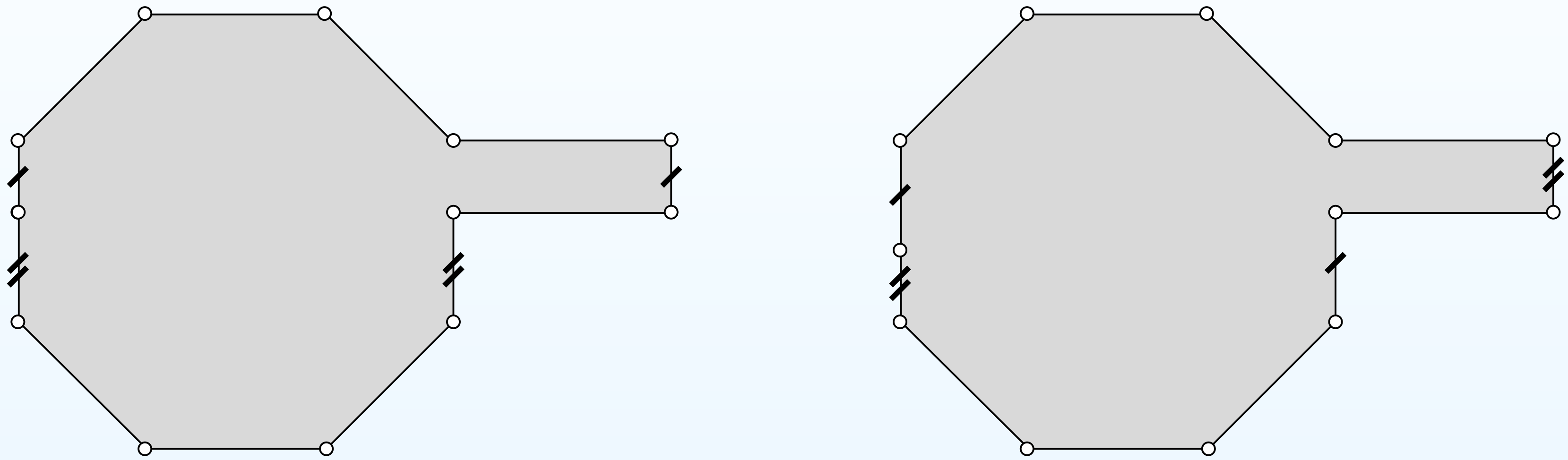
Exercise with representatives of the two components of $\mathcal{H}(4)$

Exercise

- Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.

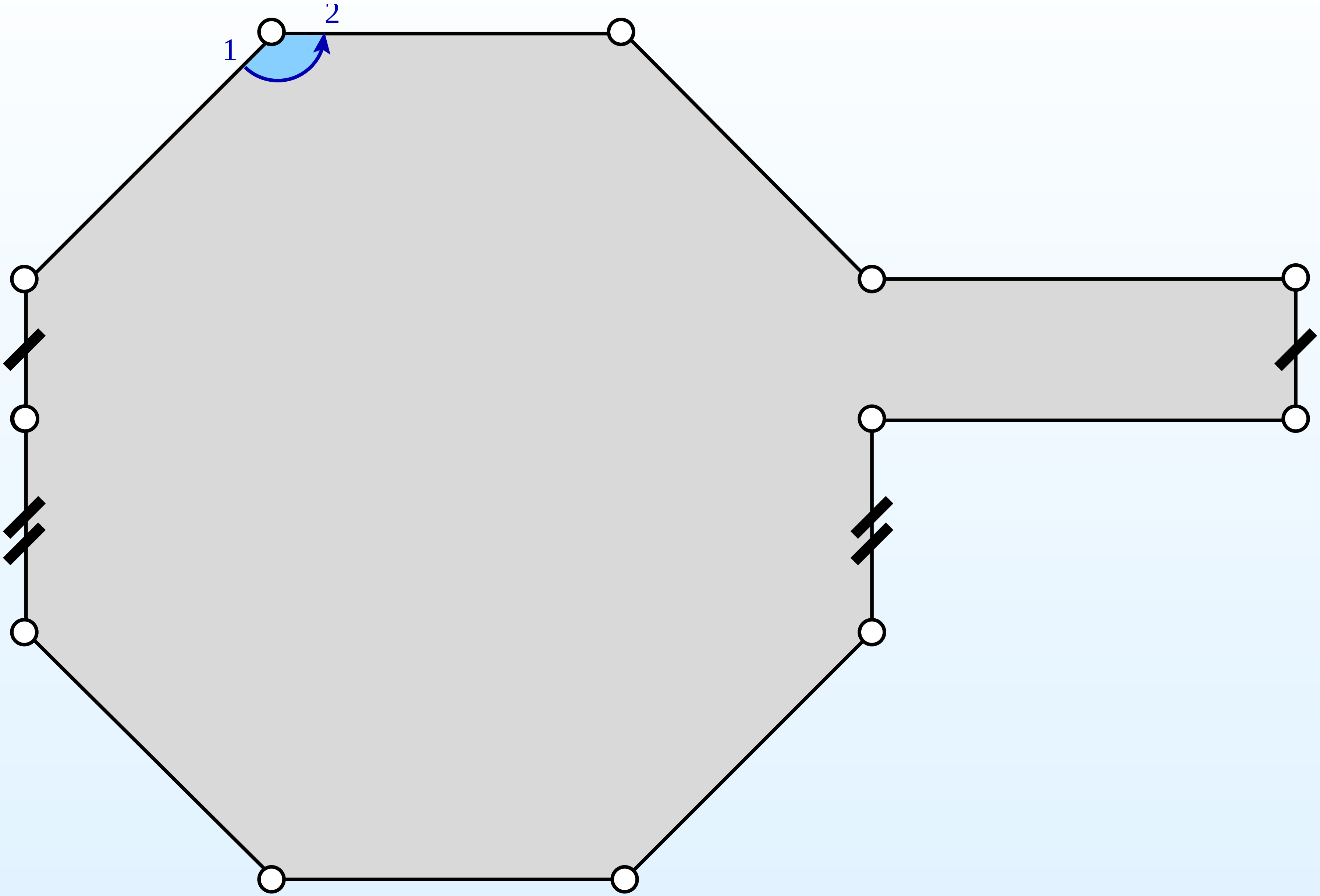


- Compute the parity of the spin structure for these surfaces (and notice that it is not the same).

- Determine which of the two surfaces is hyperelliptic.

- Find the hyperelliptic involution of this surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.

- Proof that the following flat surface belongs to the stratum $\mathcal{H}(4)$.



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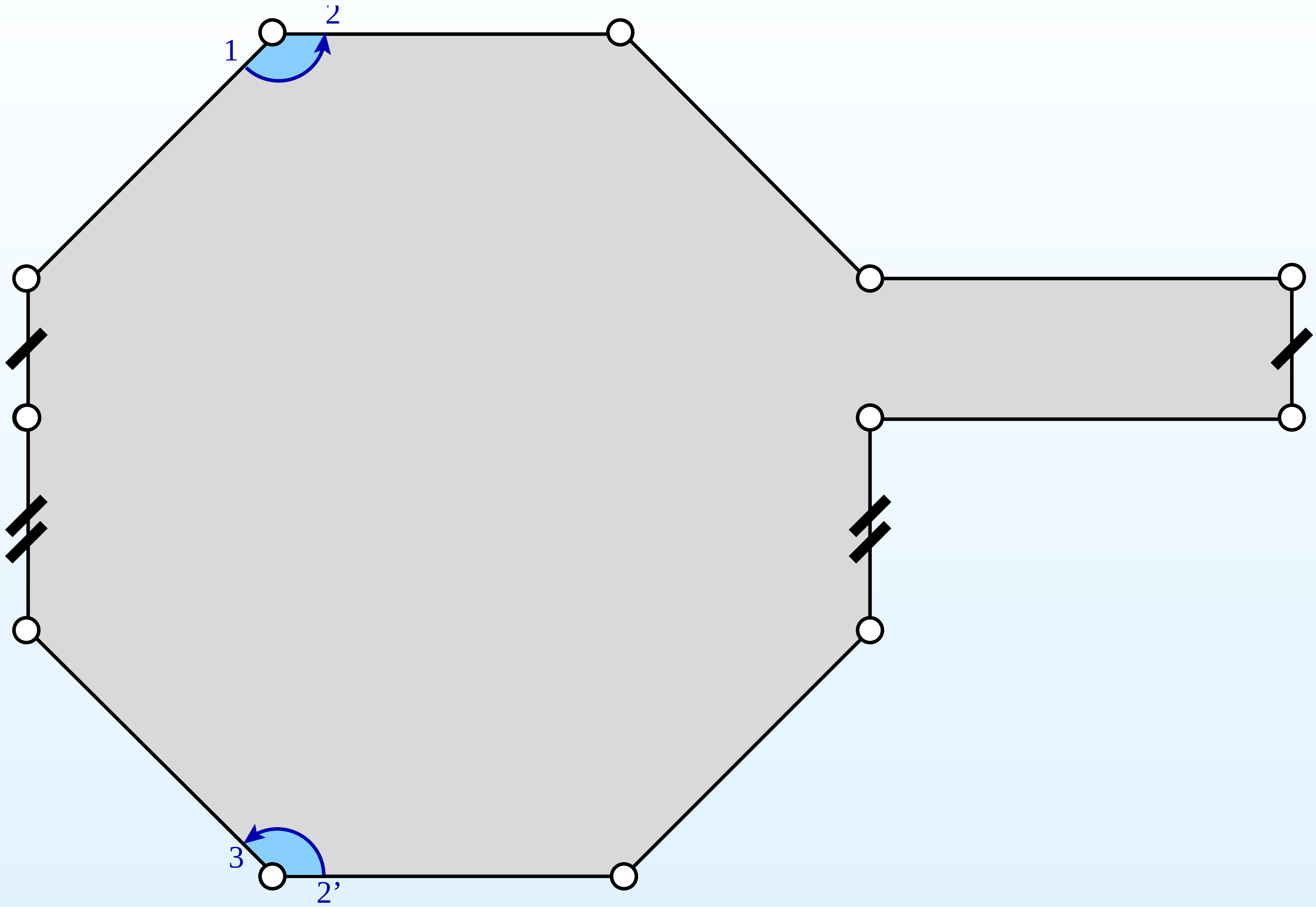

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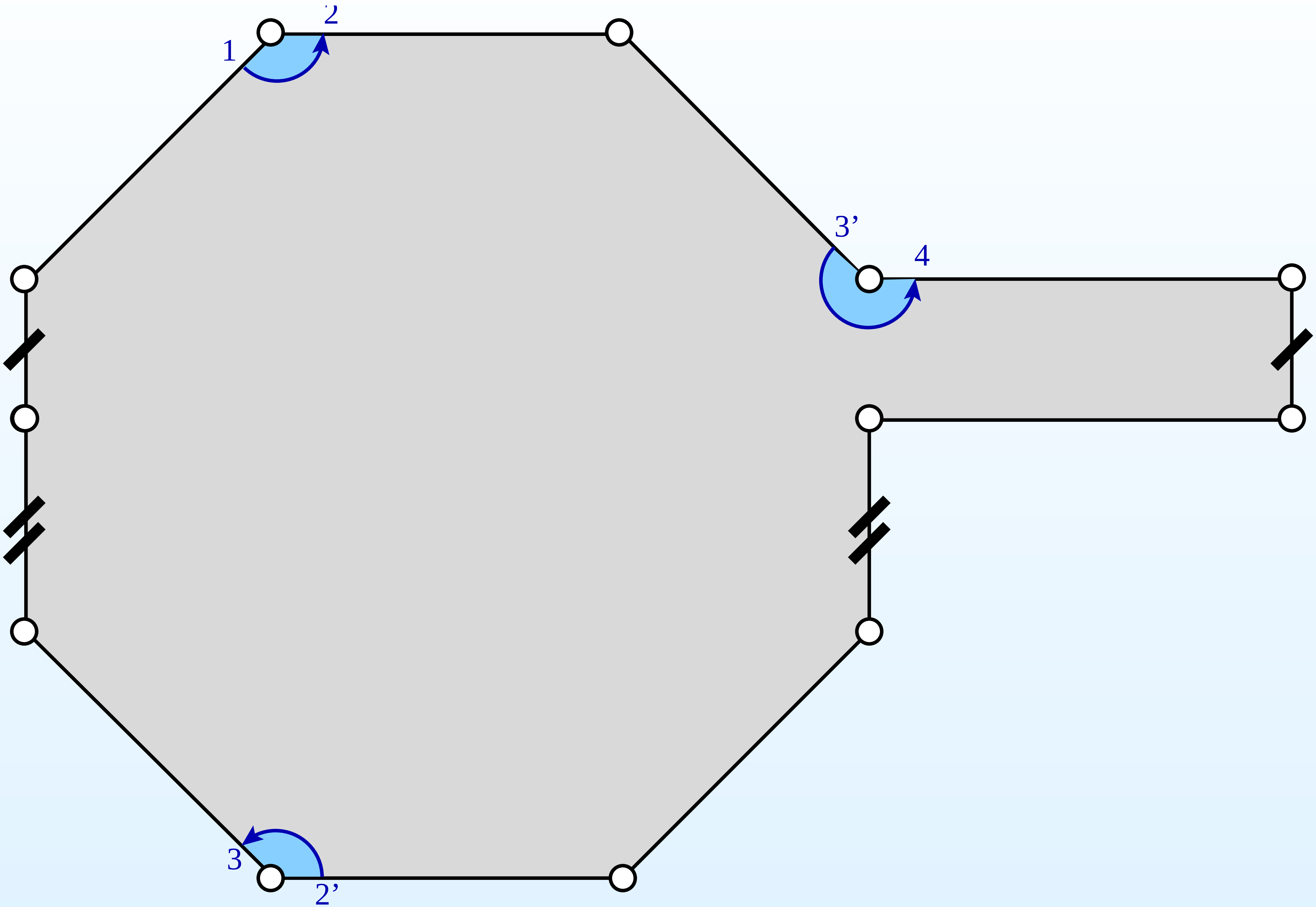

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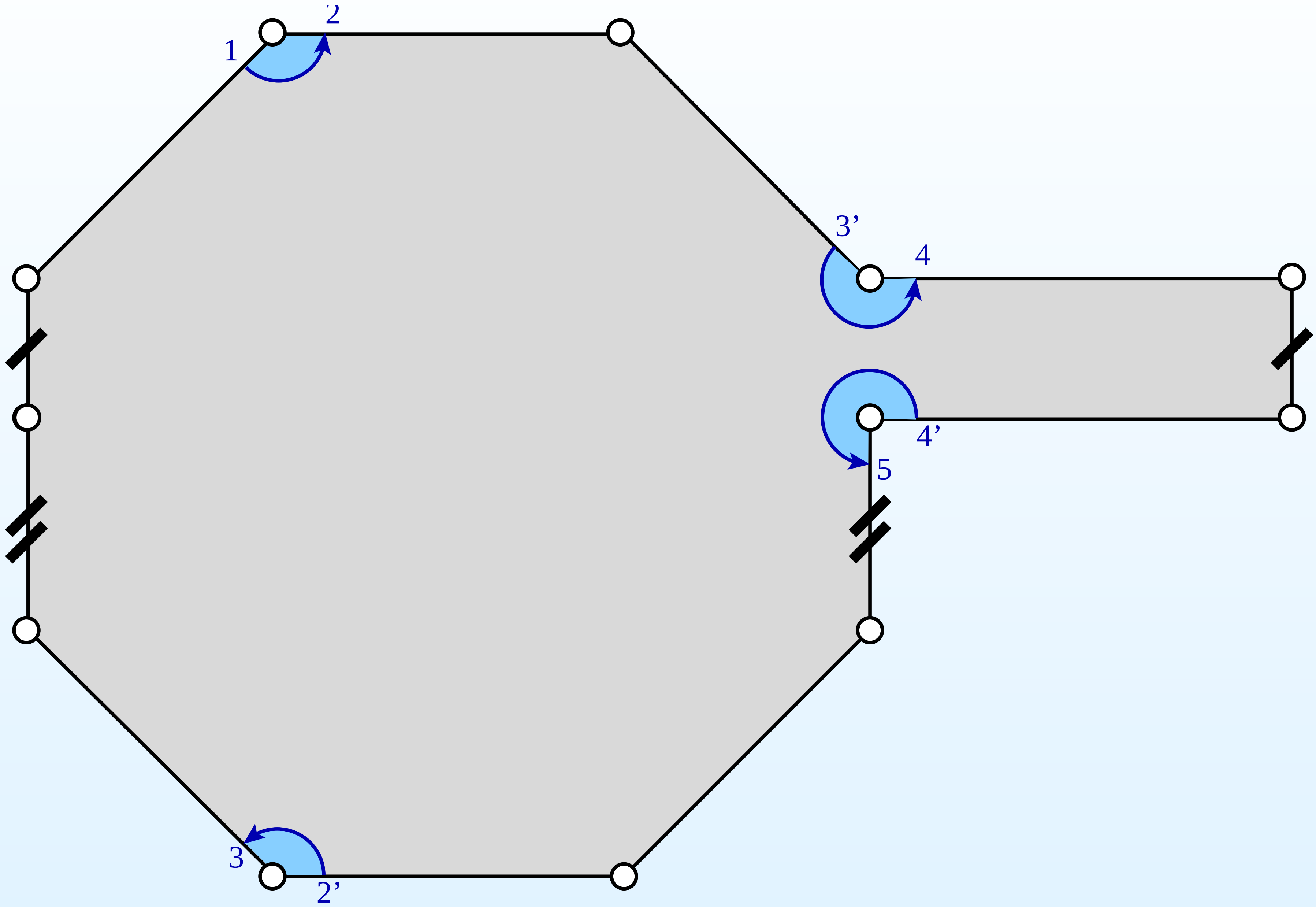

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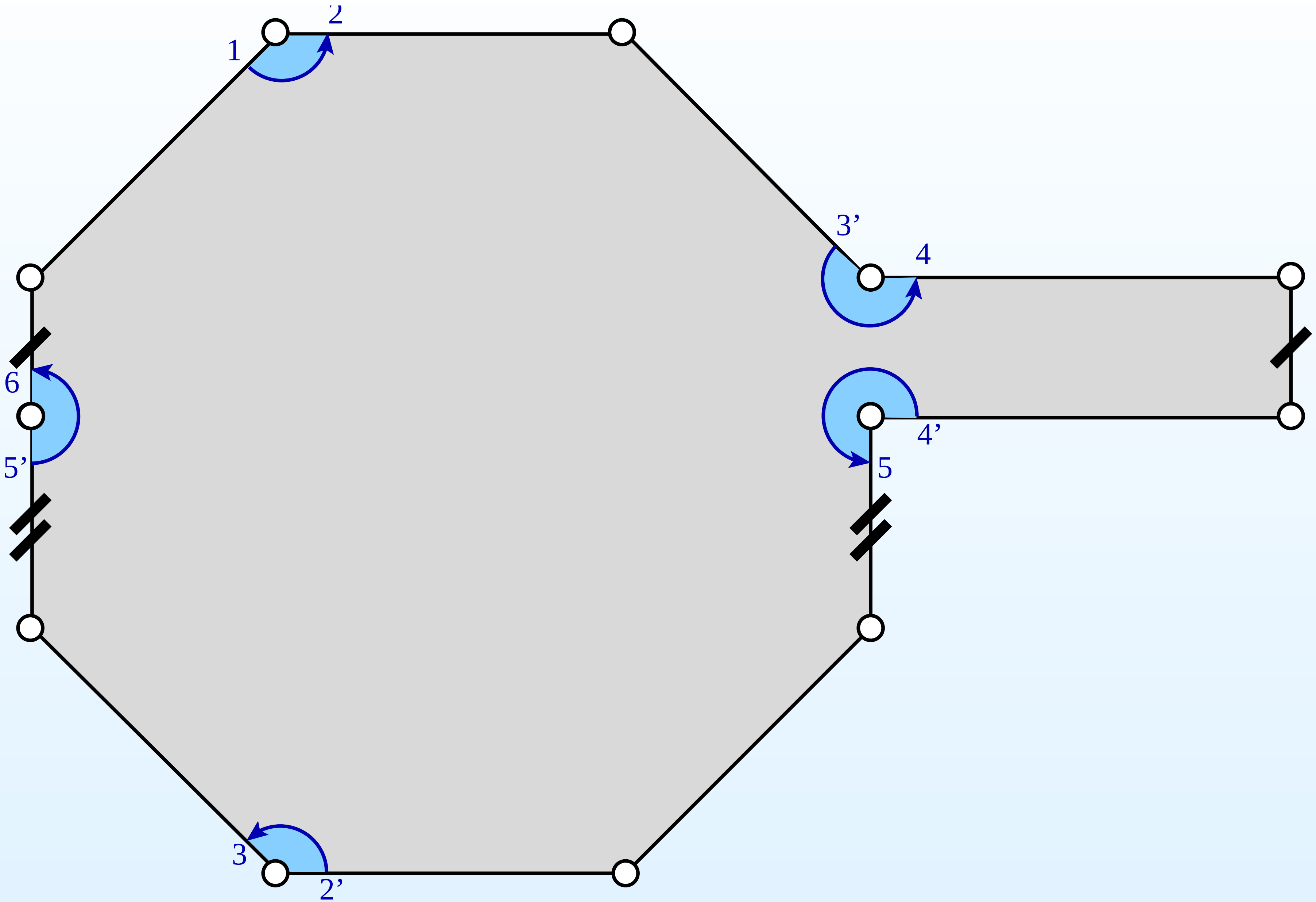

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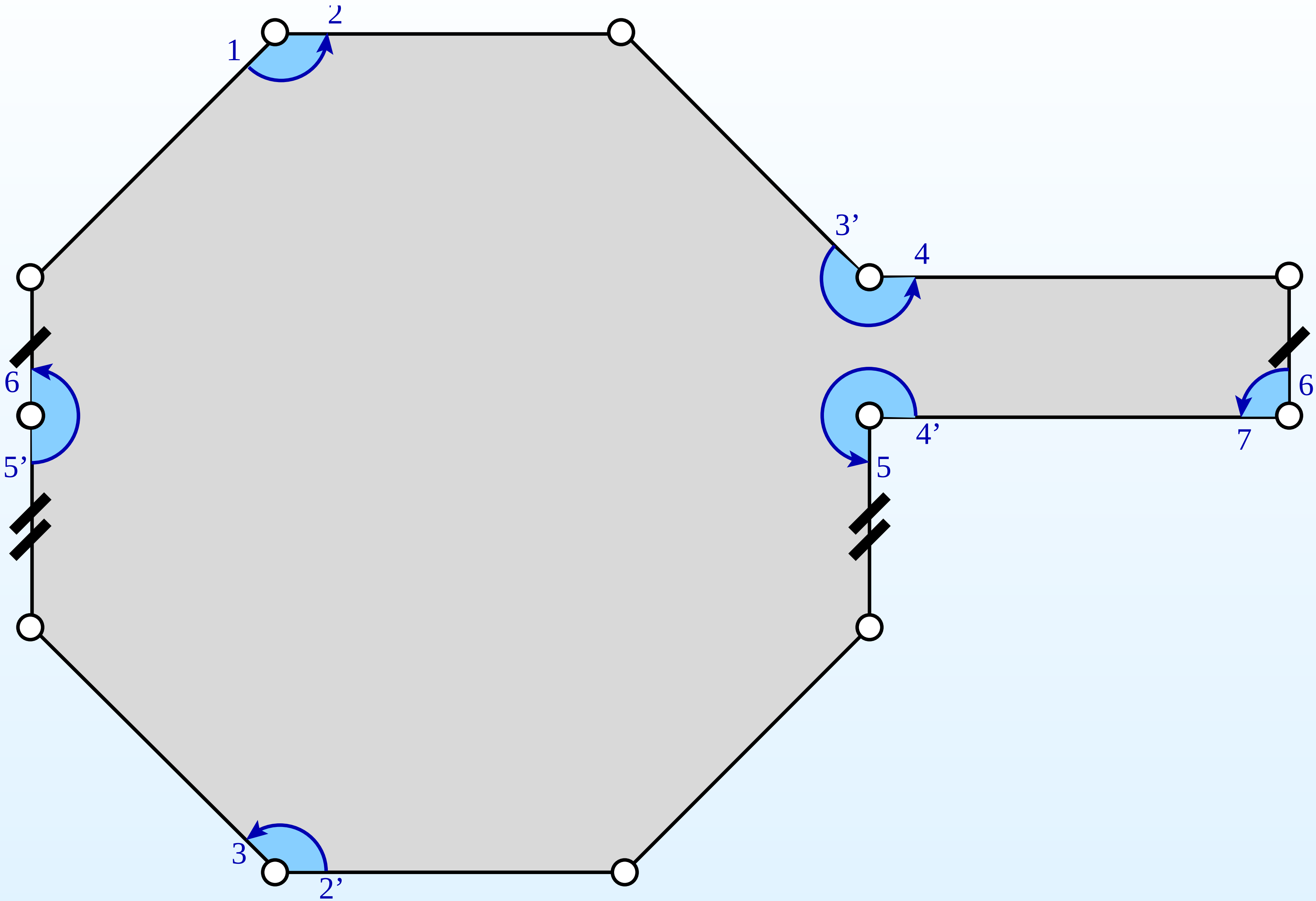

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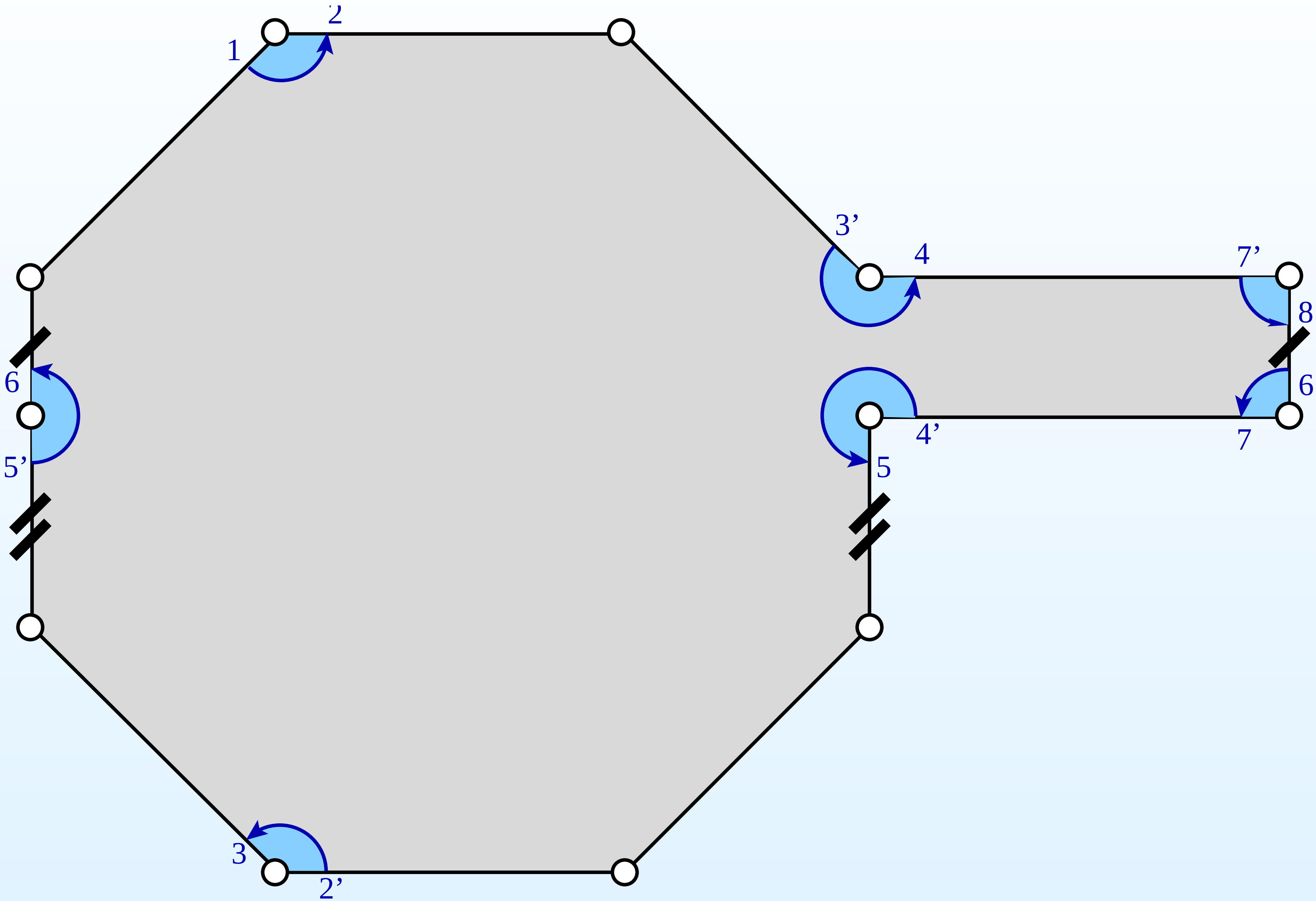

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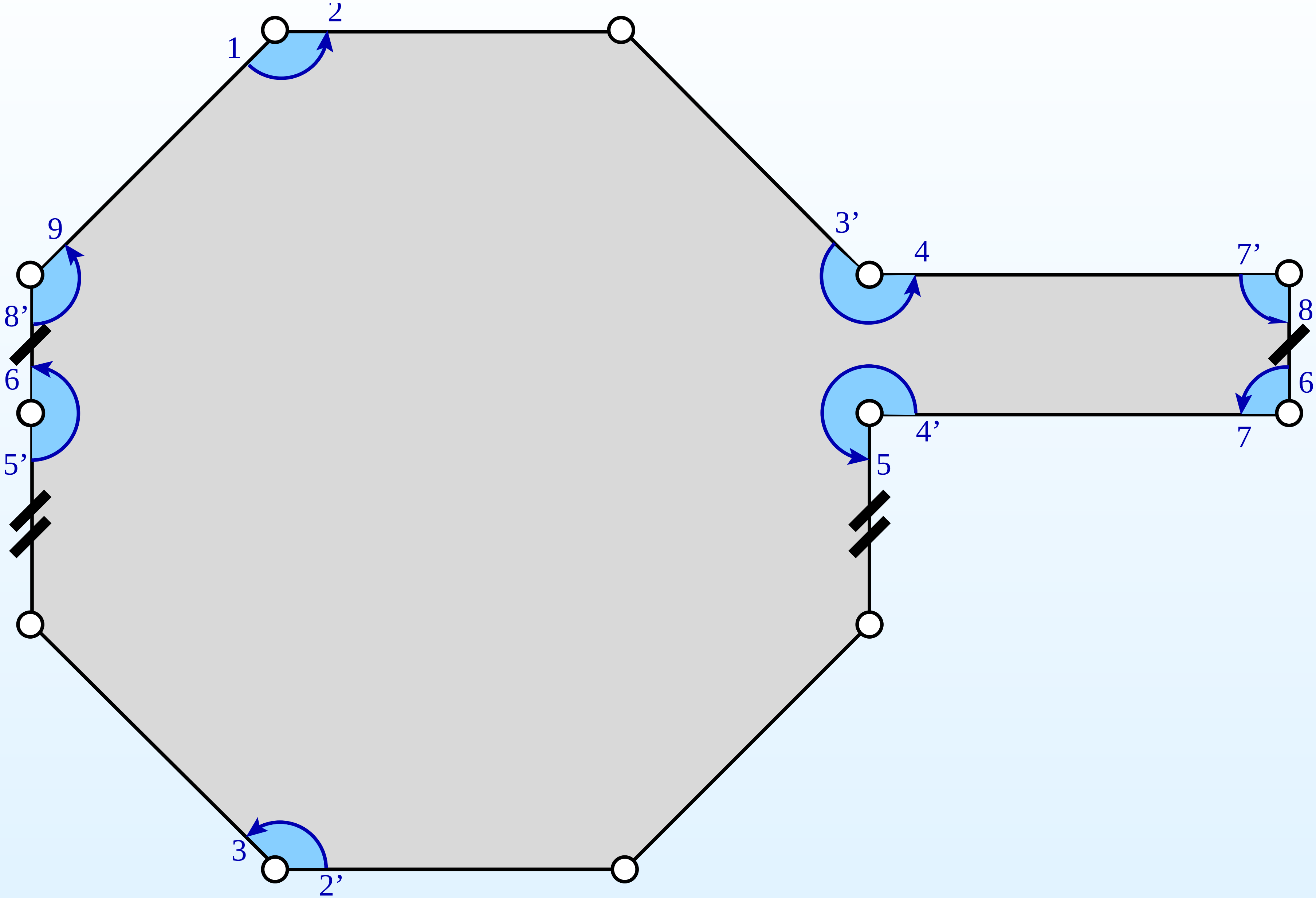

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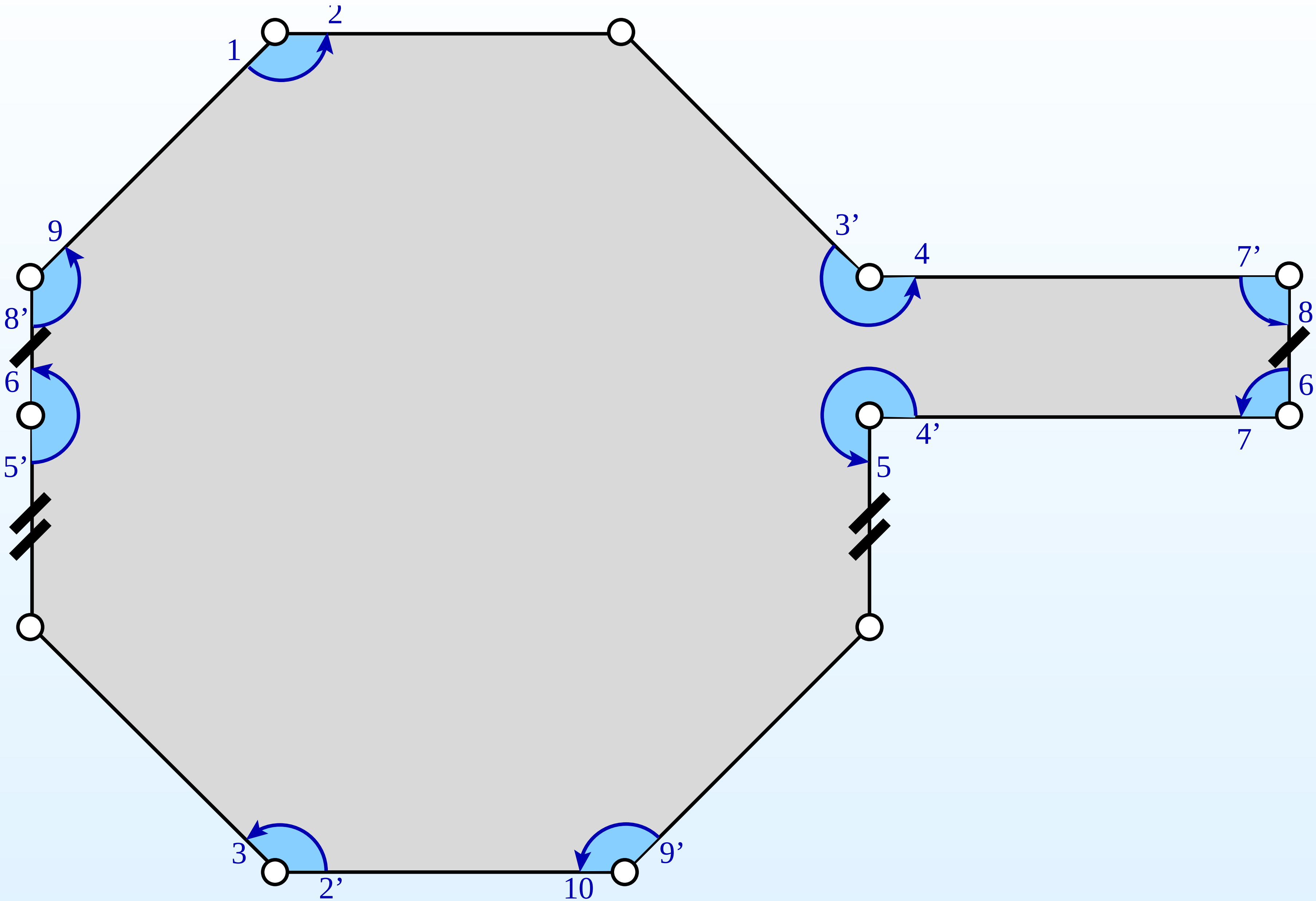

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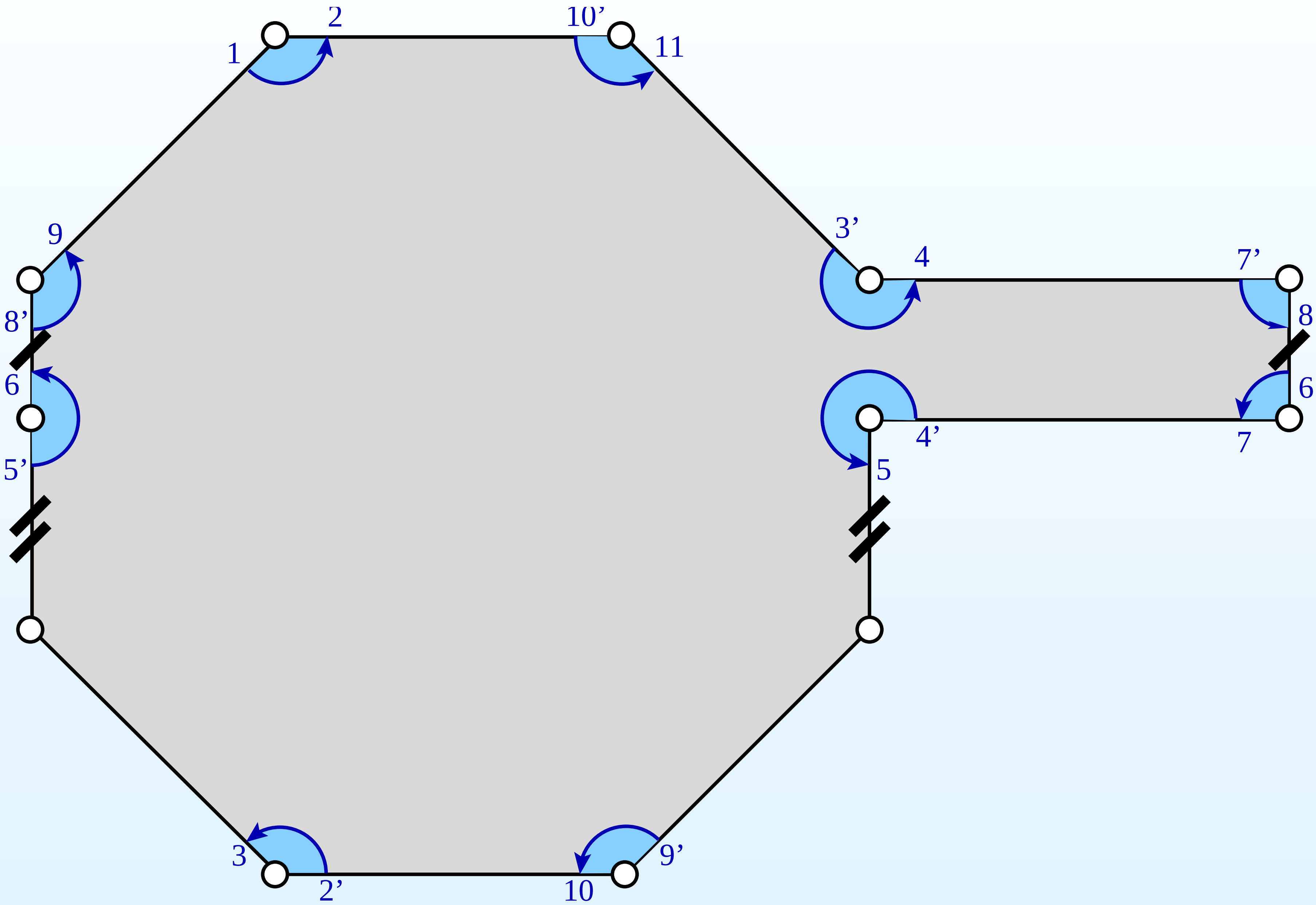

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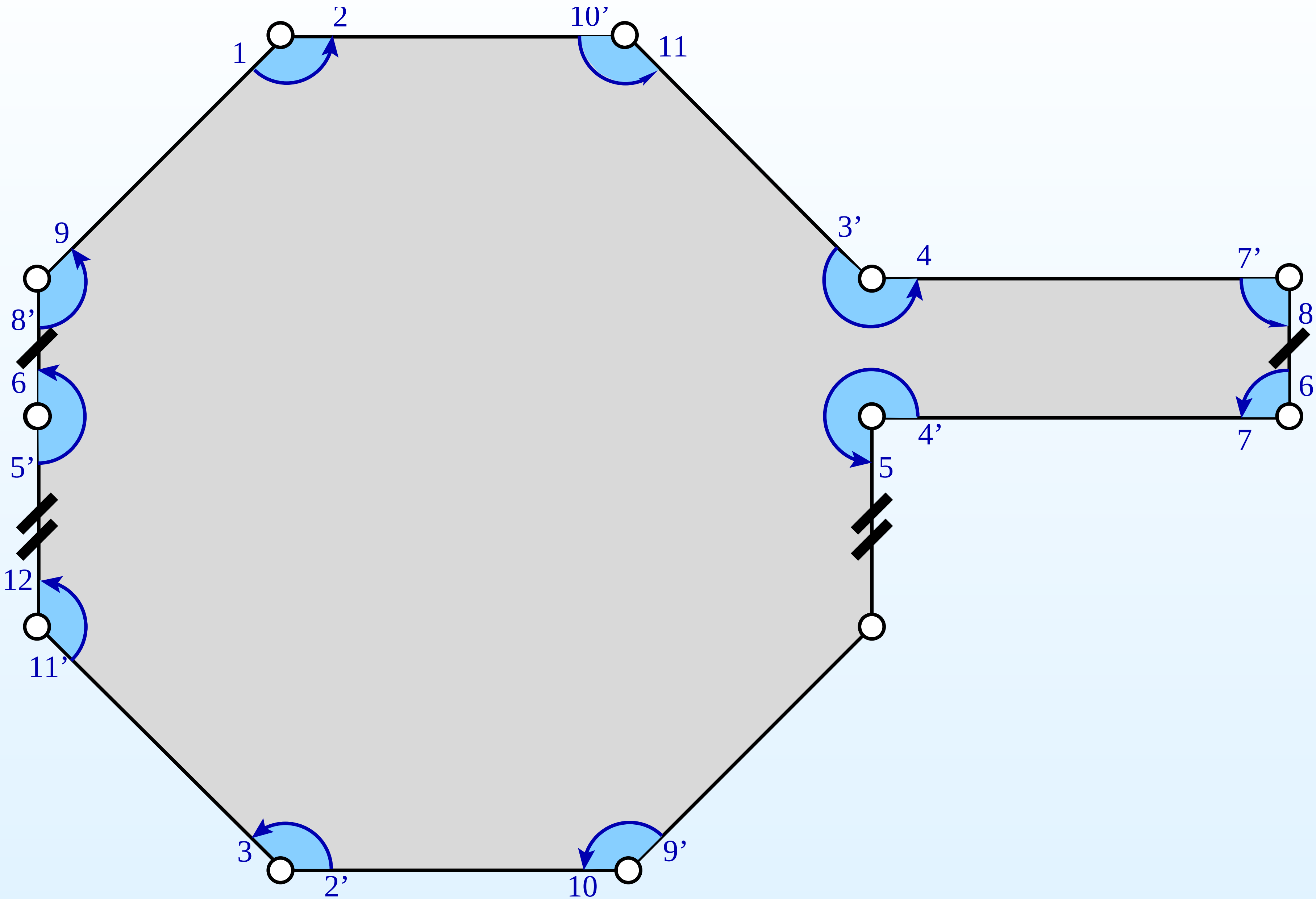

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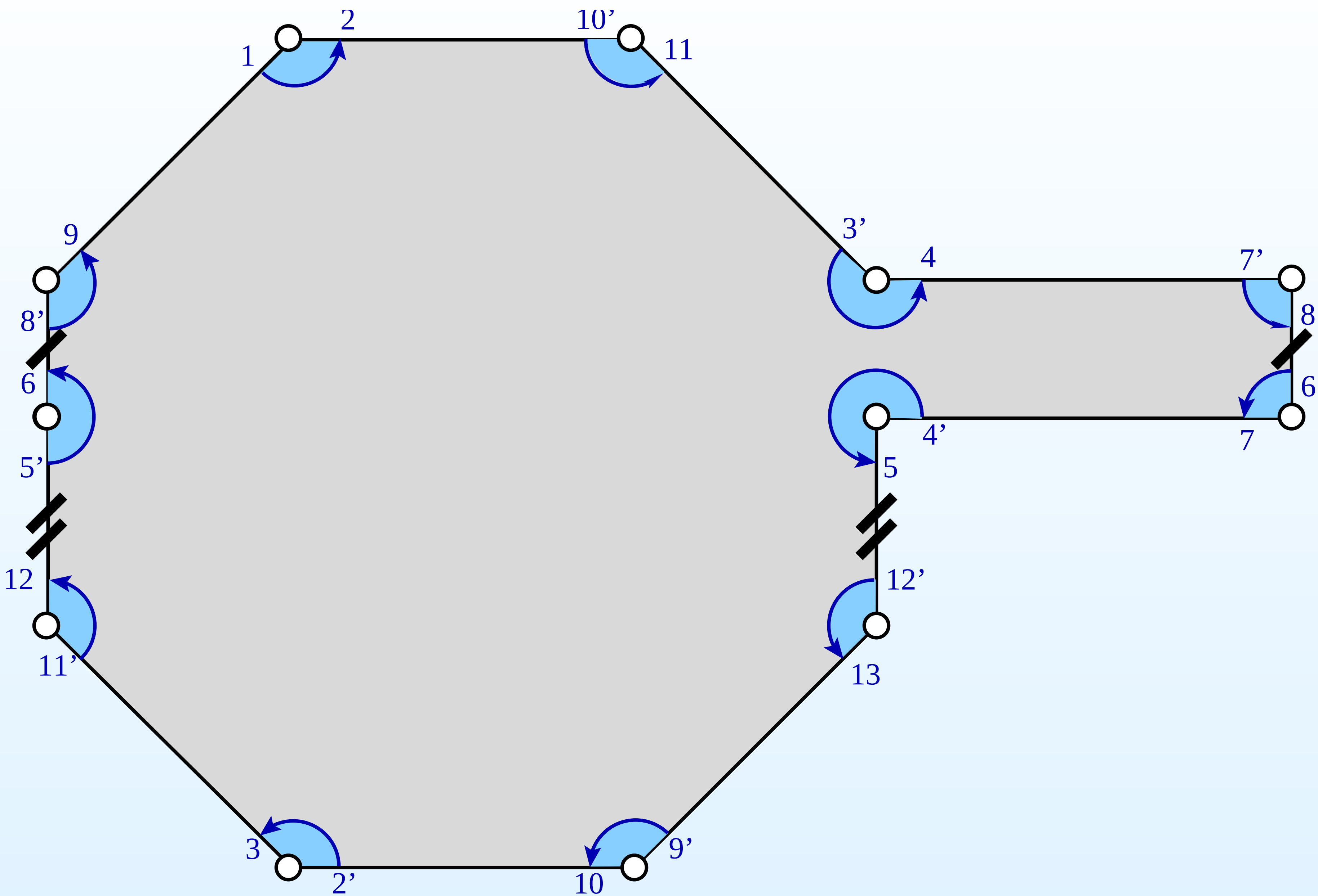

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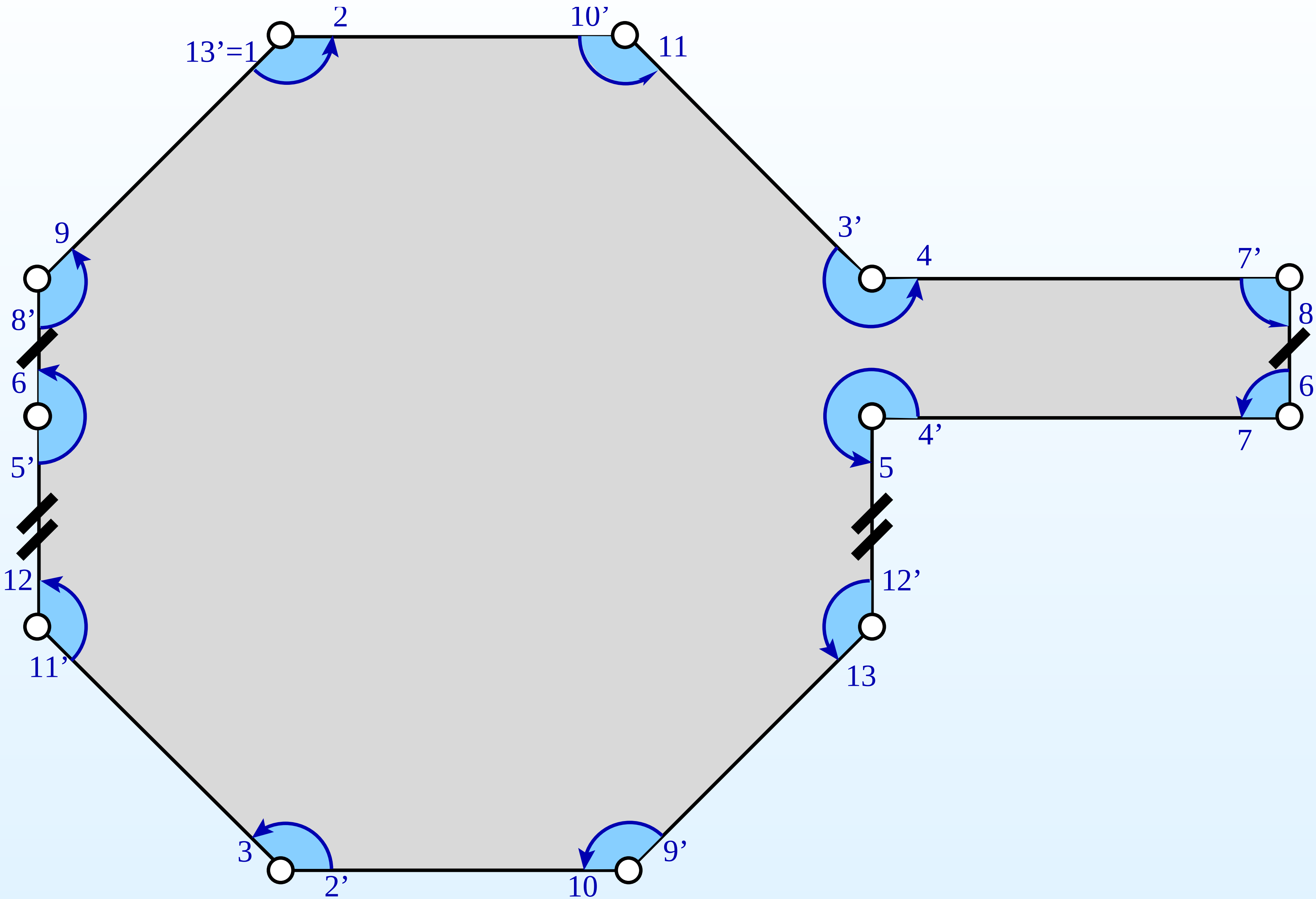
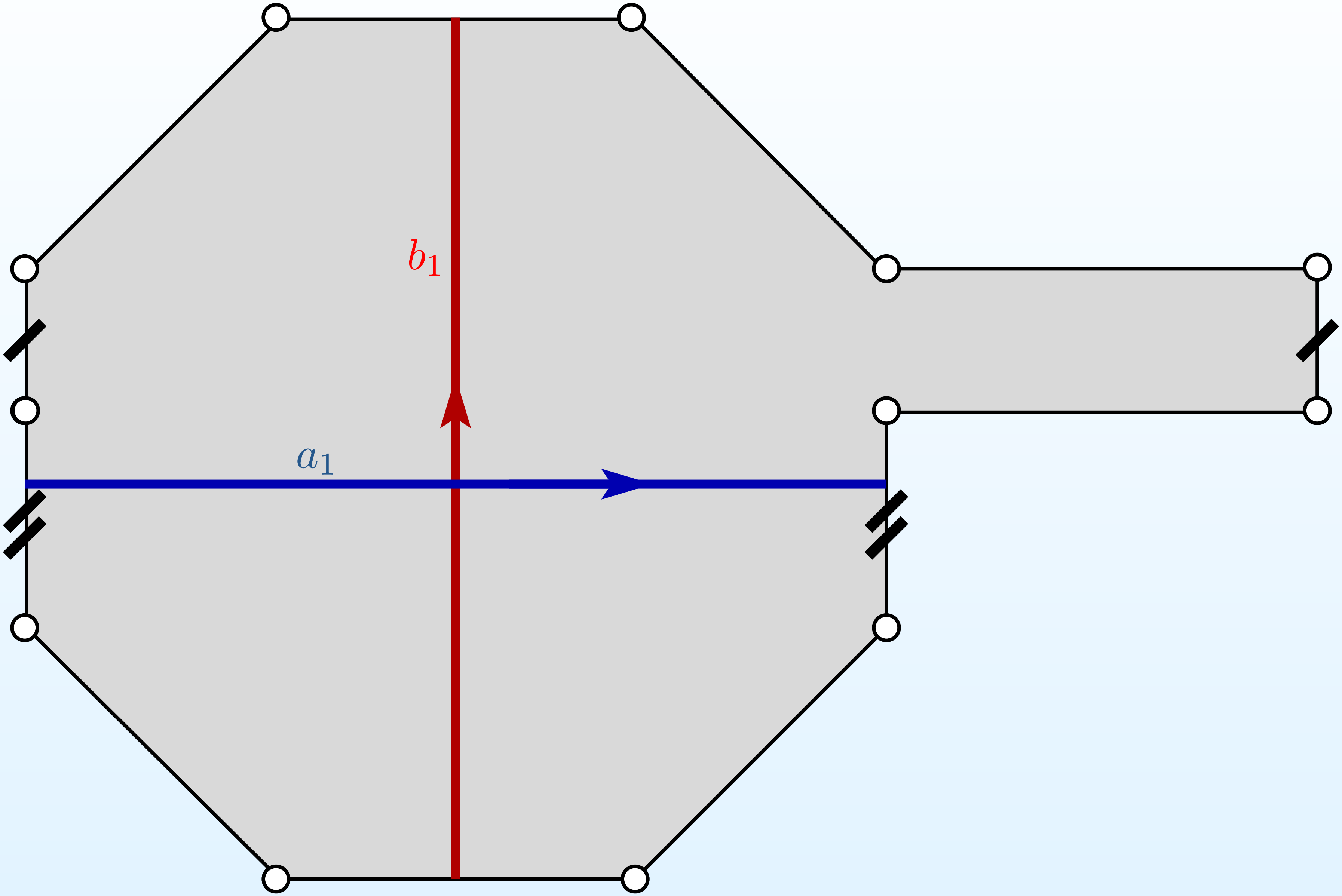


- Construction of a canonical basis of cycles.



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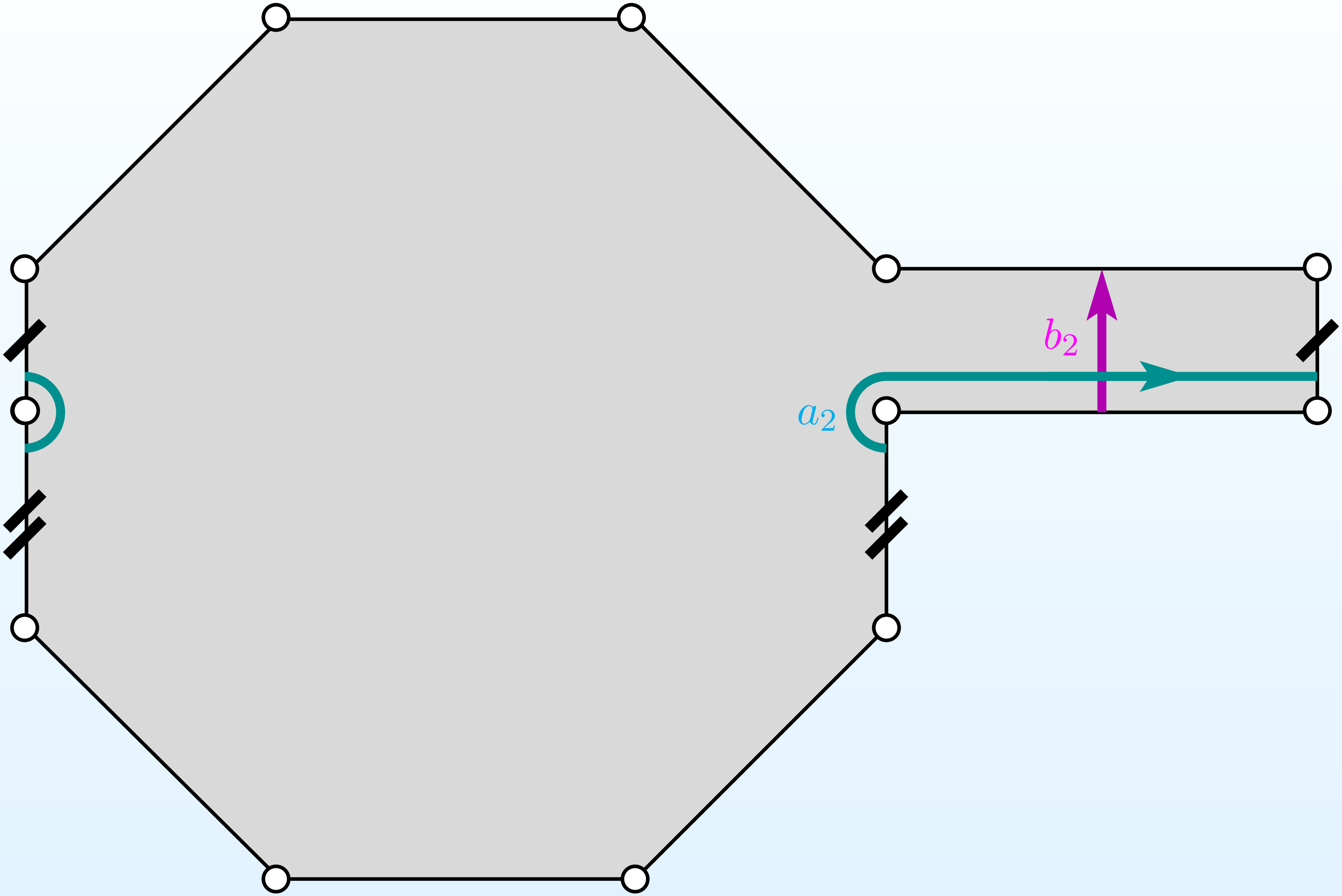

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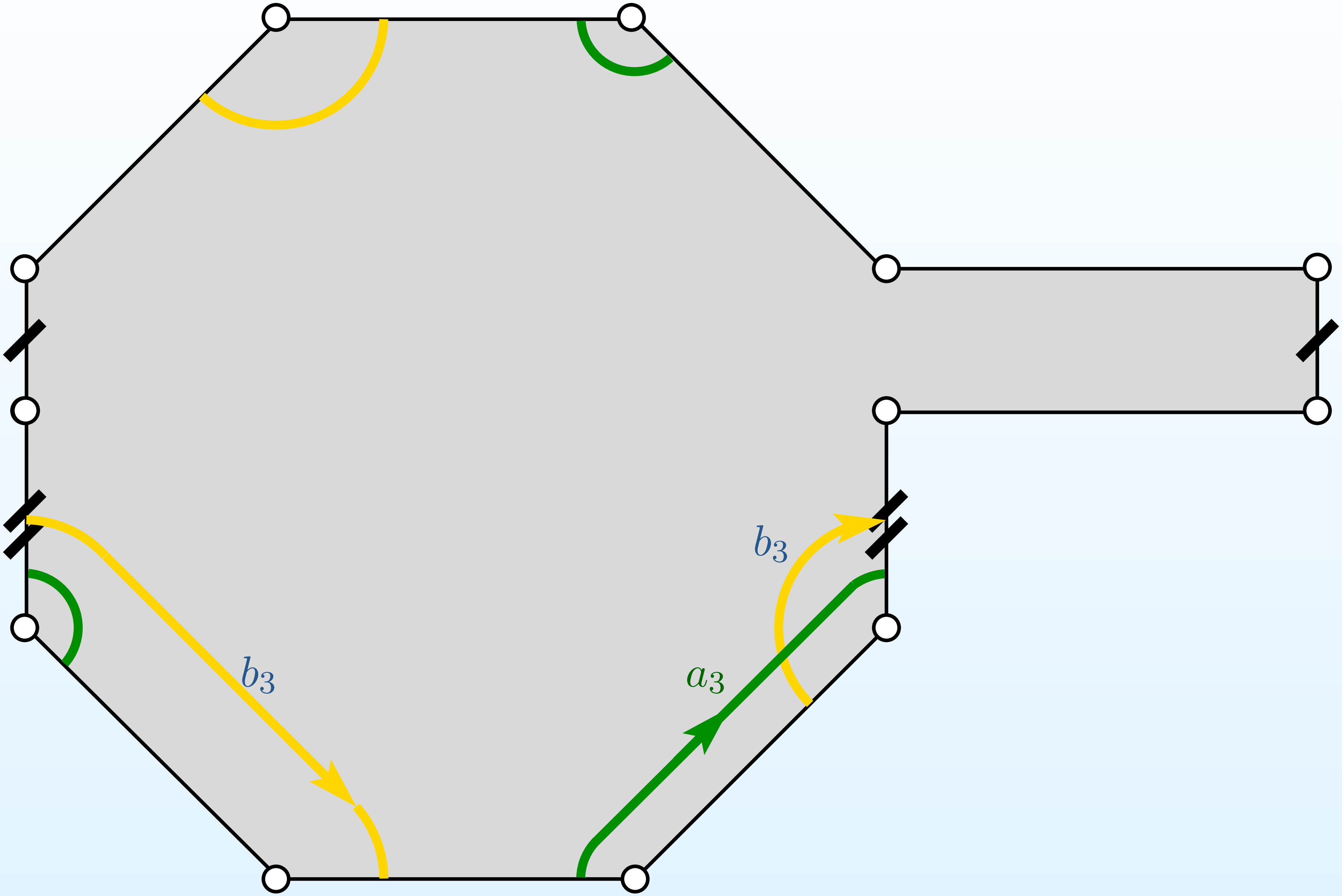

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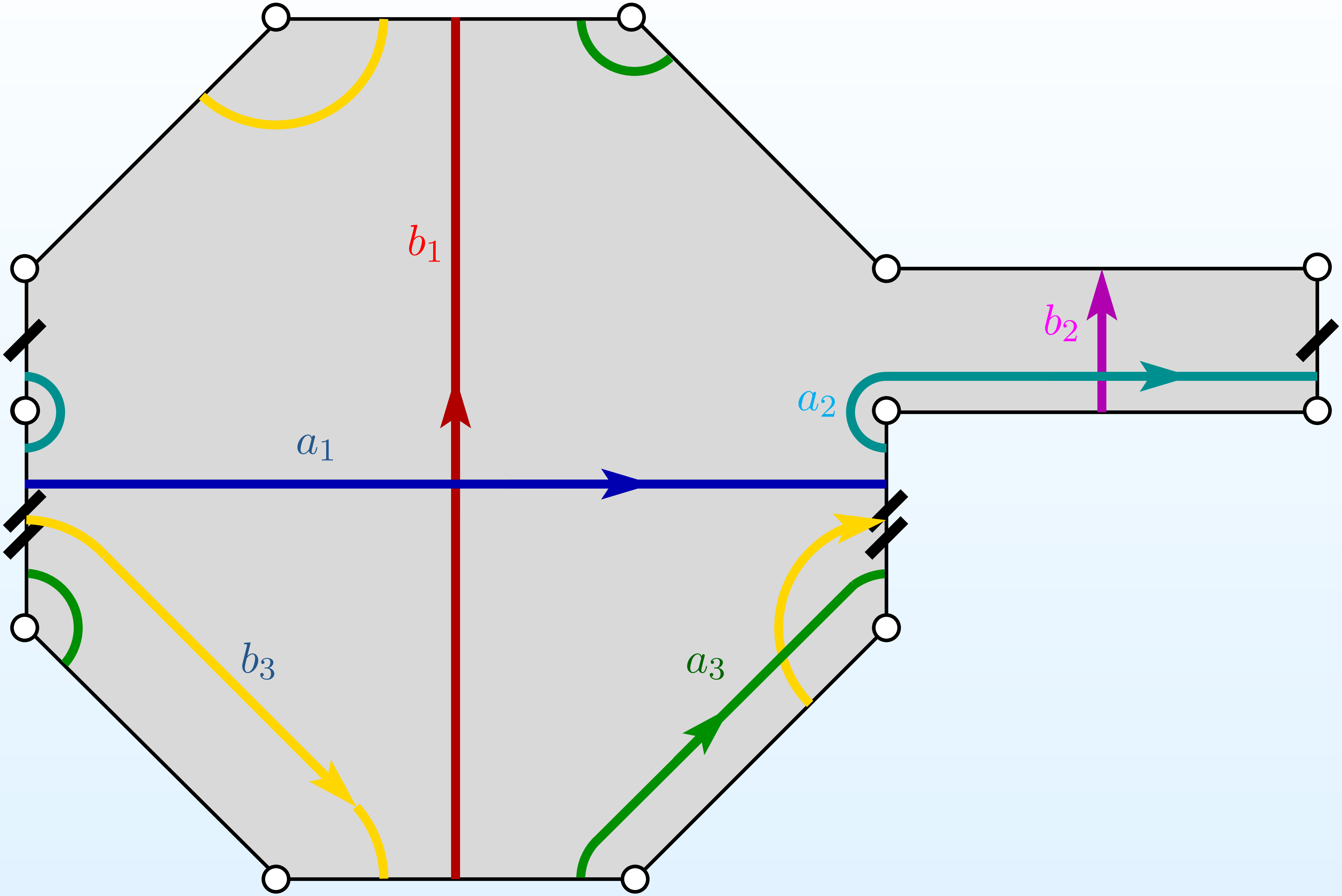
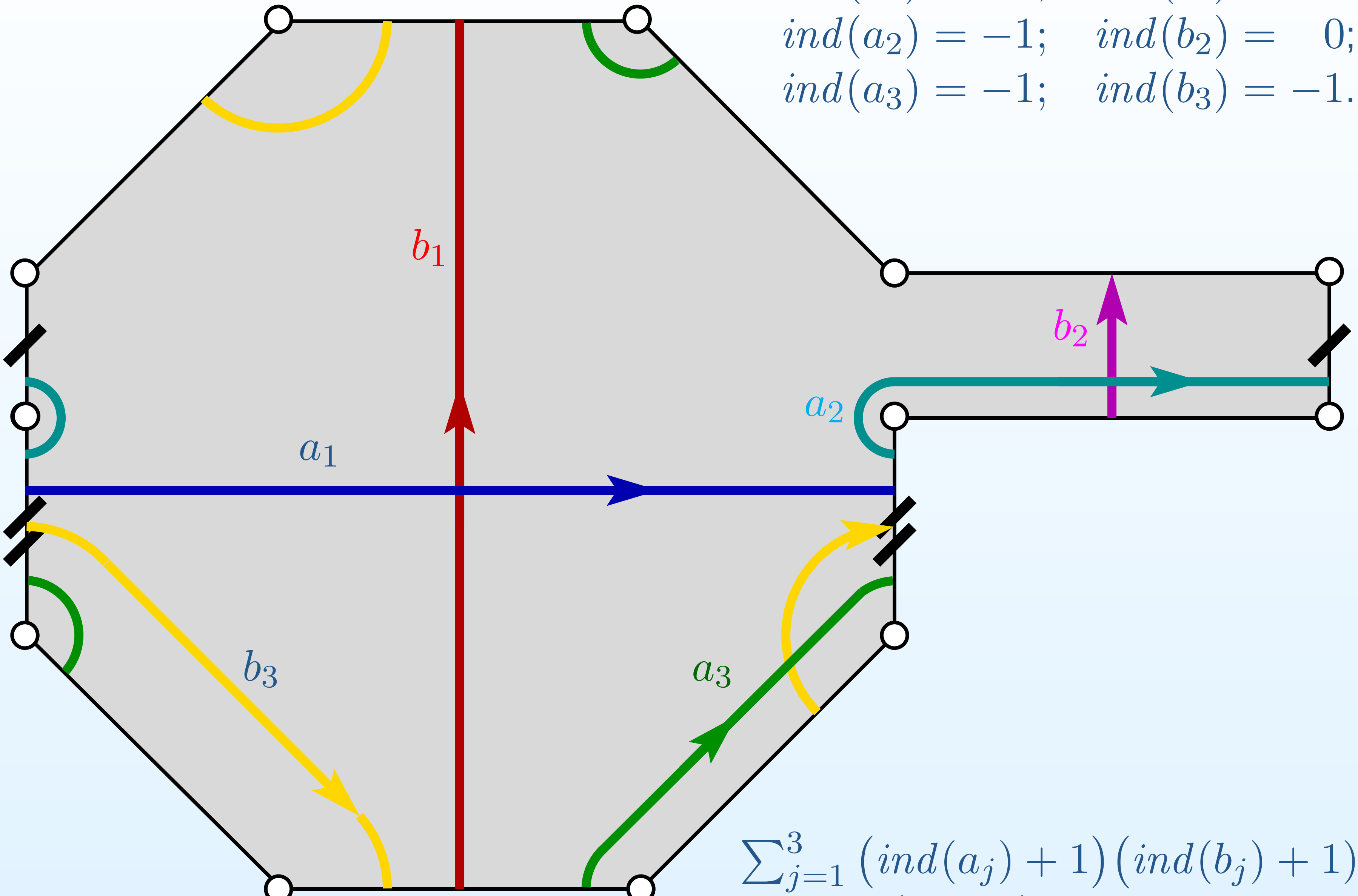


- Evaluation of the parity of a spin-structure.

$$\begin{aligned} ind(a_1) &= 0; & ind(b_1) &= 0; \\ ind(a_2) &= -1; & ind(b_2) &= 0; \\ ind(a_3) &= -1; & ind(b_3) &= -1. \end{aligned}$$



$$\sum_{j=1}^{3} \big(ind(a_j)+1\big)\big(ind(b_j)+1\big) = 1 (\mathrm{mod}\, 2)$$

Spin-structure of minimal hyperelliptic components

Theorem (M. Kontsevich, A. Zorich, 2003) *Parity of the spin structure determined by an Abelian differential from the hyperelliptic component* $\mathcal{H}^{hyp}(2g-2)$ *equals*

$$\varphi(\mathcal{H}^{hyp}(2g-2)) \equiv \left[\frac{g+1}{2}\right] \left(\bmod 2\right).$$

In particular,

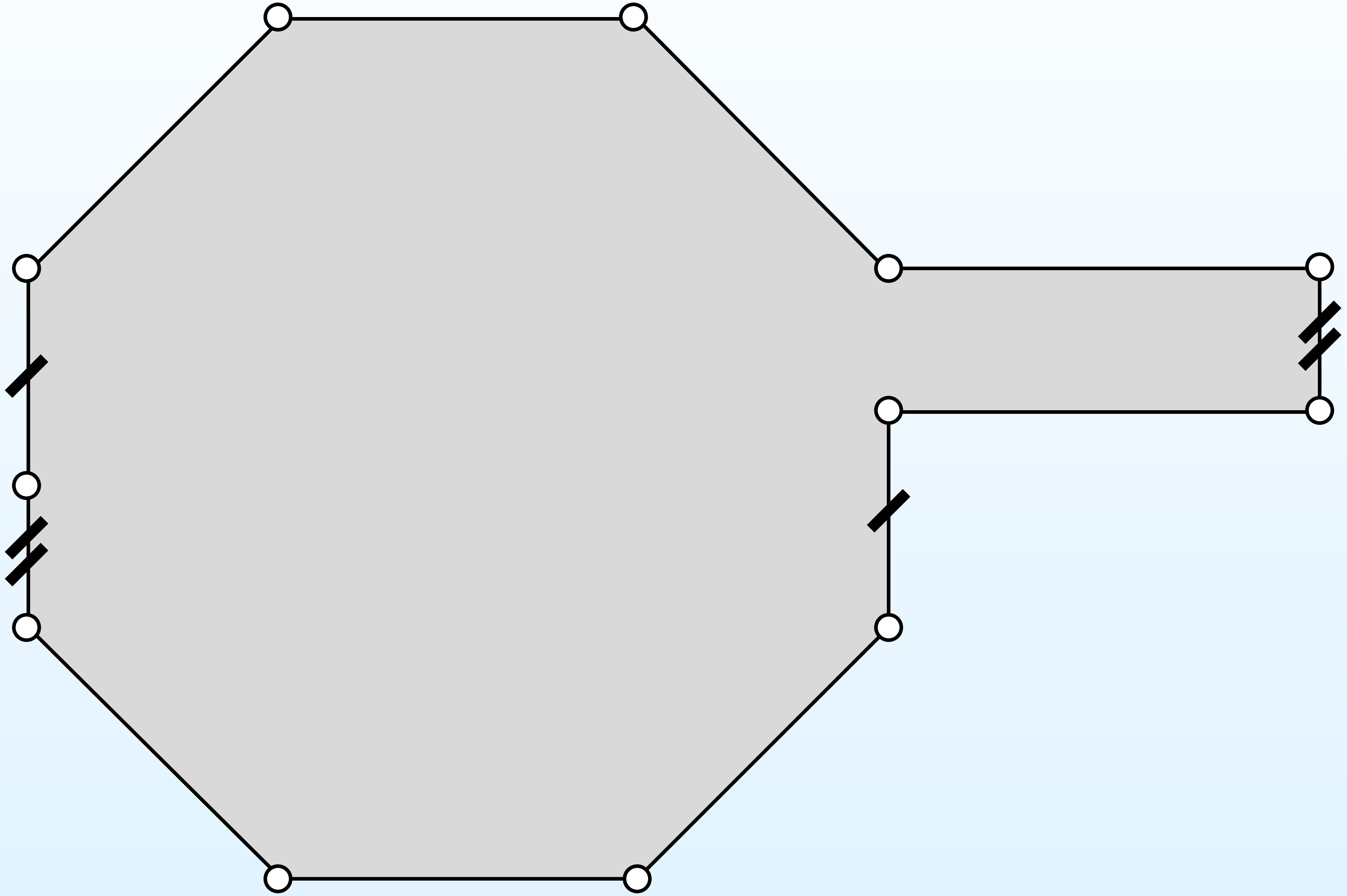
$$\varphi(\mathcal{H}^{hyp}(0)) = 1 \qquad \varphi(\mathcal{H}^{hyp}(2)) = 1 \qquad \varphi(\mathcal{H}^{hyp}(4)) = 0$$

and we conclude that the surface constructed above lives in $\mathcal{H}^{odd}(4)$.

- Find the hyperelliptic involution of the remaining surface in geometric terms.

Find the Weierstrass points (the fixed points of the hyperelliptic involution).

Check that there are $2g + 2 = 2 \cdot 3 + 2 = 8$ such points.



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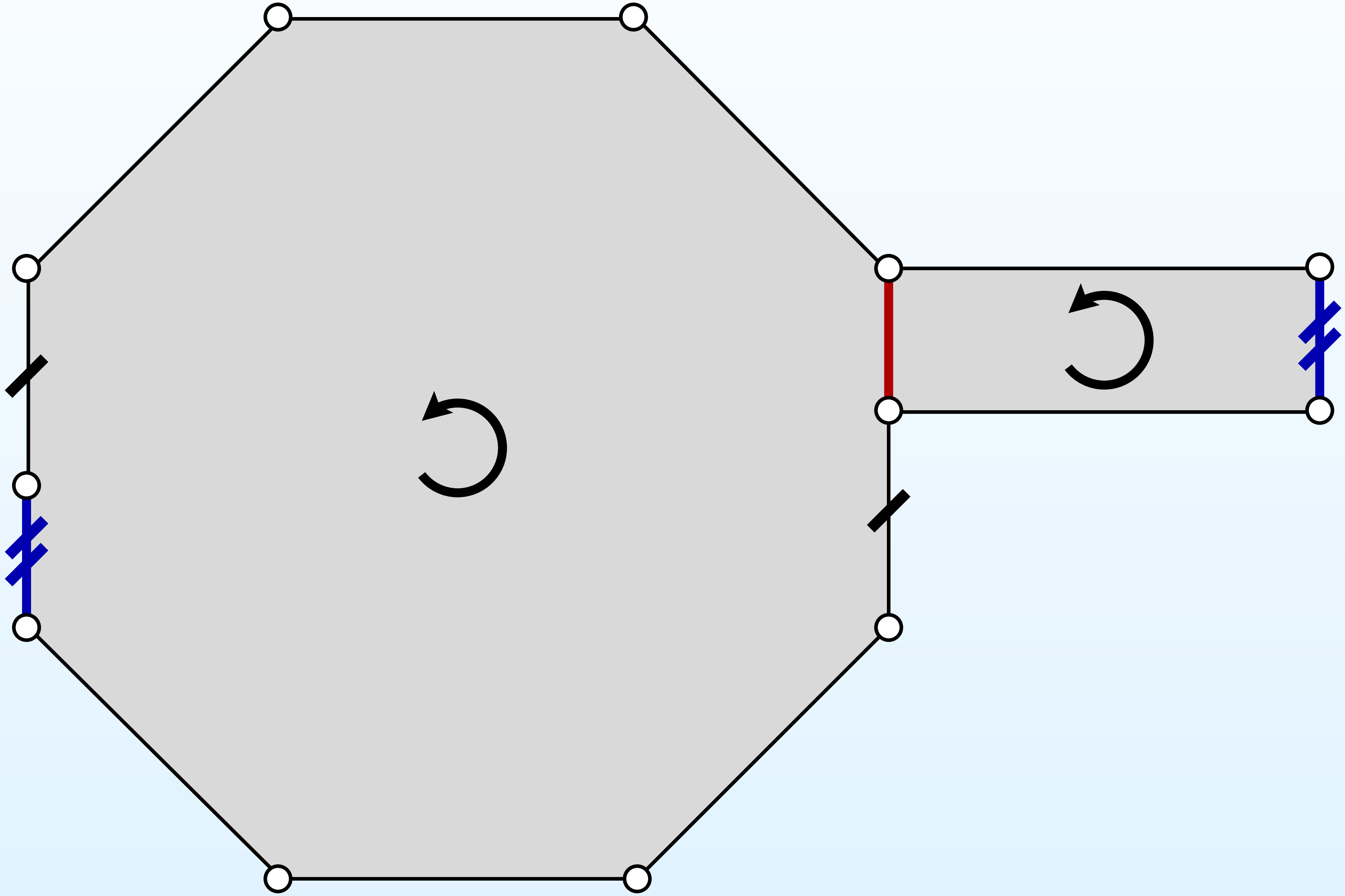

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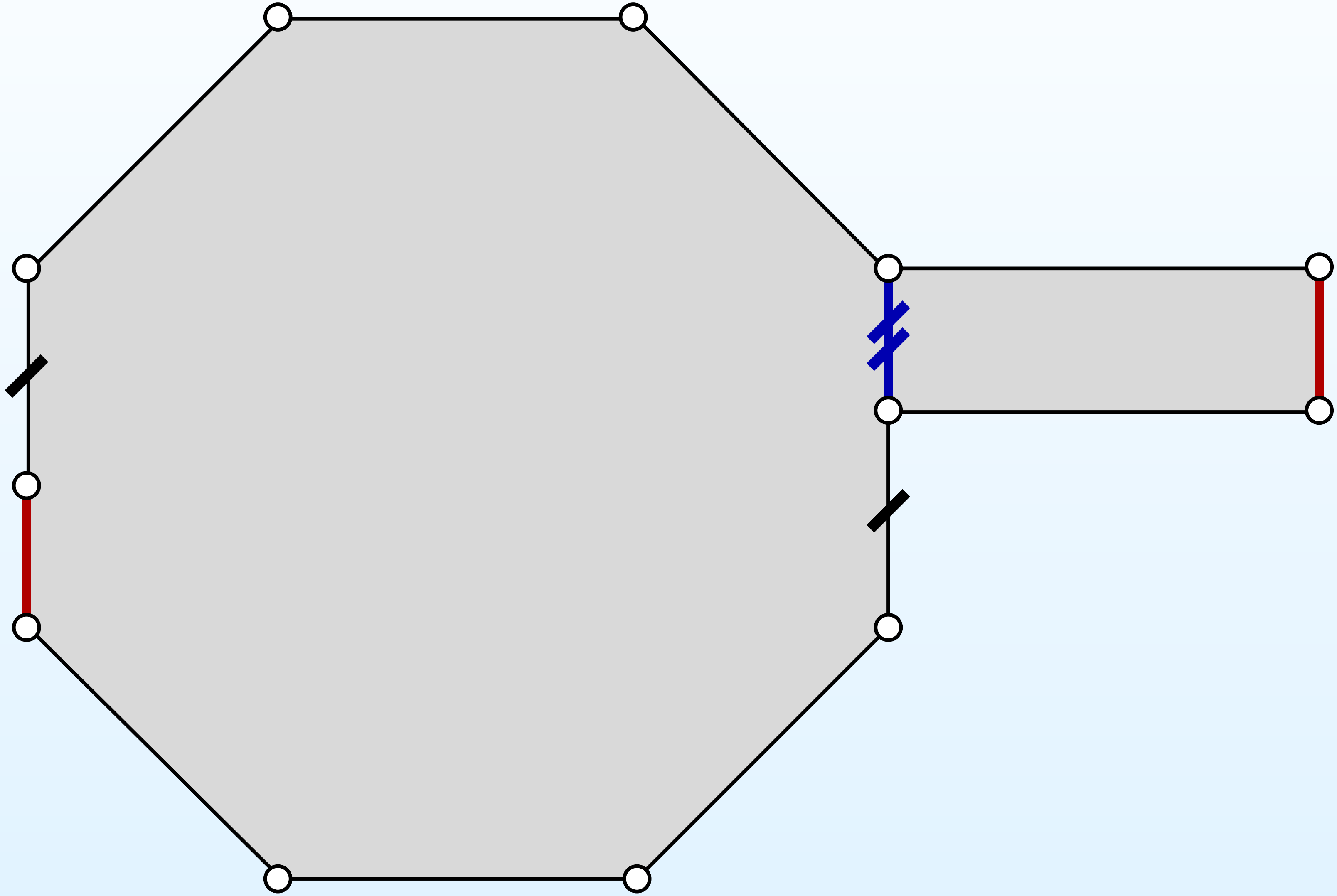

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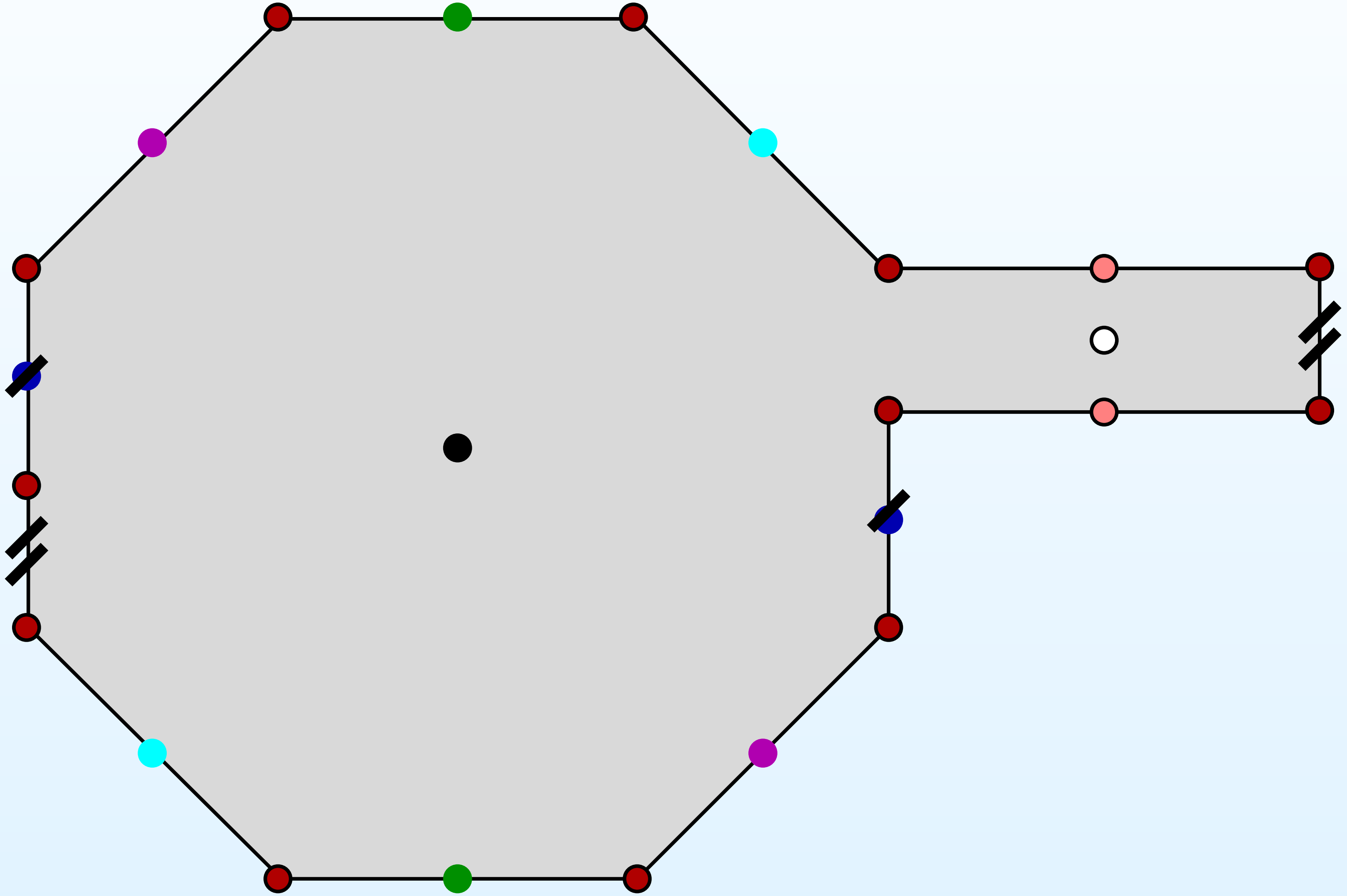
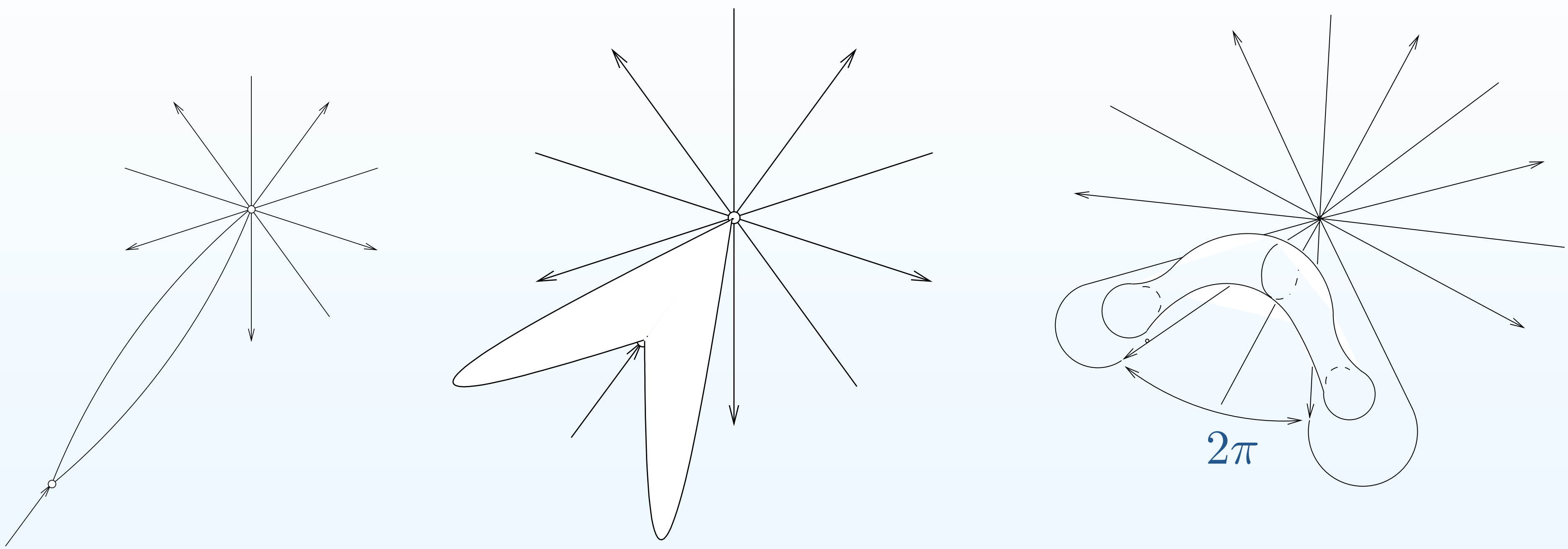
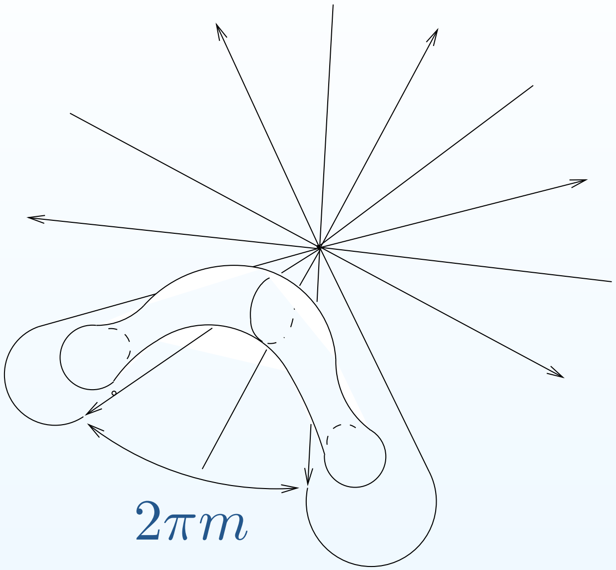


Bubbling a handle



Here is a way to paste a small flat handle (flat cylinder) at a zero of a flat surface by a surgery changing the flat metric only in a small neighborhood of the zero. Break the zero into two zeroes joined by a short saddle connection. Slit the saddle connection and join the endpoints thus creating a pair of loops. Paste a flat cylinder to the resulting pair of holes. Such a local constructions allows to get all possible angles $2\pi, 4\pi, \ldots$ between the "petals".

Parity of spin under bubbling a handle

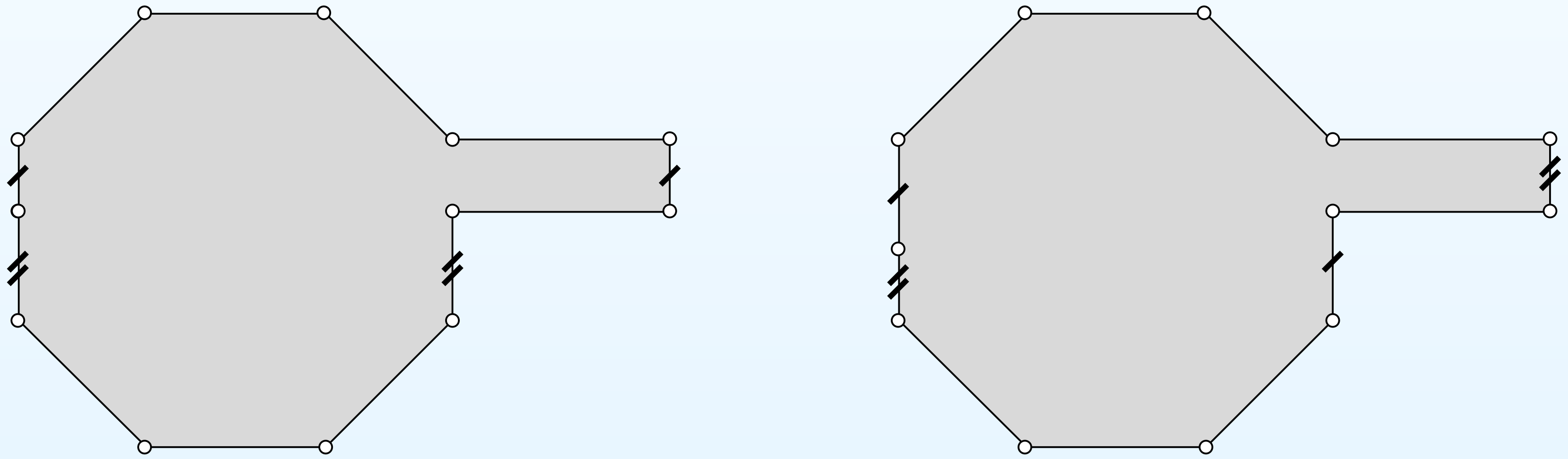


Lemma. *Let an Abelian differential* $\hat{\omega} \in \mathcal{H}(2(l_1+1), 2l_2, ..., 2l_n)$ *on a surface of genus* $g+1$ *be obtained from an Abelian differential* $\omega \in \mathcal{H}(2l_1, 2l_2, ..., 2l_n)$ *on a surface of genus* g *by "bubbling a handle". Let* $2\pi m$ *be the angle of one of the two sectors complementary to the handle. The parities of the spin structures determined by* ω *and by* $\hat{\omega}$ *are related in the following way:*

$$\varphi(\hat{\omega}) - \varphi(\omega) = m + 1 \pmod 2$$

Exercise

- Prove the Lemma above.
- Check that the following two flat surfaces are obtained from the same flat surface in $\mathcal{H}(2)$ (the one which one gets by identifying opposite sides of a regular octagon by parallel translations) by "bubbling a handle" in two different ways.



- Compute the parity of the spin structure for these surfaces using the Lemma above (and notice that it is not the same).

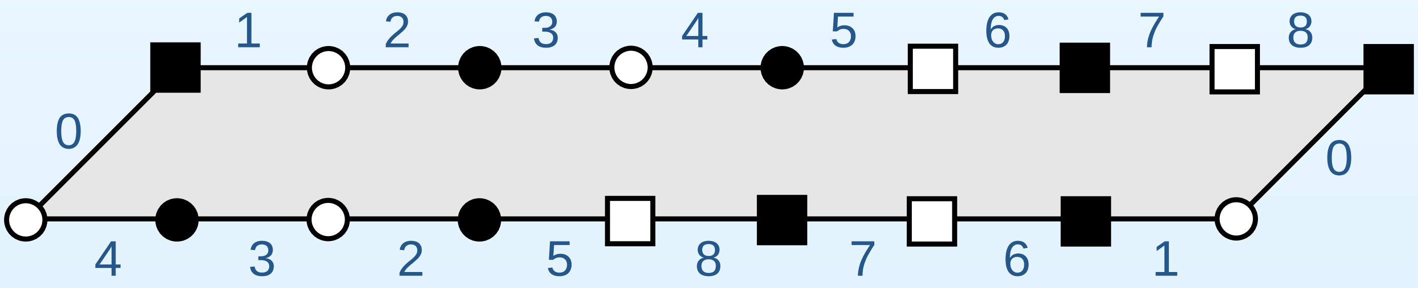
Explicit representatives of connected components

Polygon which mimics a parallelogram

In case when the permutation $\pi \in S_n$ defining the polygon has the property $\pi^{-1}(n) = 1$, we can choose all vectors $\vec{V}_2, \ldots, \vec{V}_n$ horizontal. The resulting polygon will have the shape of a parallelogram with extra vertices added to horizontal sides.

It would be convenient to shift enumeration of elements in the permutation π starting it from 0. Under this convention the polygon below corresponds to the permutation π, where

$$\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 5 & 8 & 7 & 6 & 1 & 0 \end{pmatrix}$$



Idea of construction

We construct a representative of any connected component of any stratum in two steps. We first use an alternative representation of one-cylinder differentials now cutting the single cylinder filled with horizontal circles not by a singular horizontal leaf on the boundary of the cylinder (i.e. not by a chain of horizontal saddle connections), but by a regular horizontal leaf — by a “waist curve” of the cylinder. We construct representatives of the strata $\mathcal{H}(1,\dots,1)$, $\mathcal{H}^{even}(2,\dots,2)$, $\mathcal{H}^{odd}(2,\dots,2)$ explicitly. Given any connected component of any stratum we contract appropriate saddle connections of appropriate differential from the above list thus merging groups of zeroes. We shall see that in this way we can get to any component of any stratum.

Consider an explicit example in which we first construct a representative of the stratum $\mathcal{H}(1,1,1,1)$ and then deform it into a representative of the stratum $\mathcal{H}(3,1)$, by merging three simple zeroes into one.

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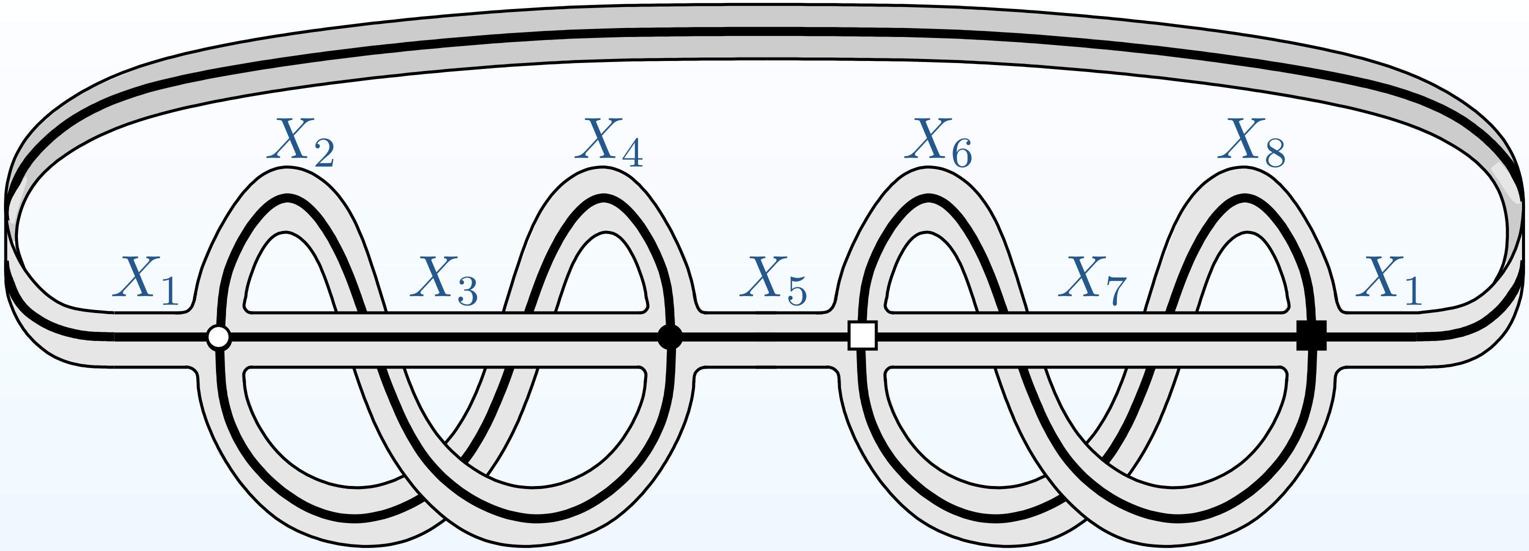
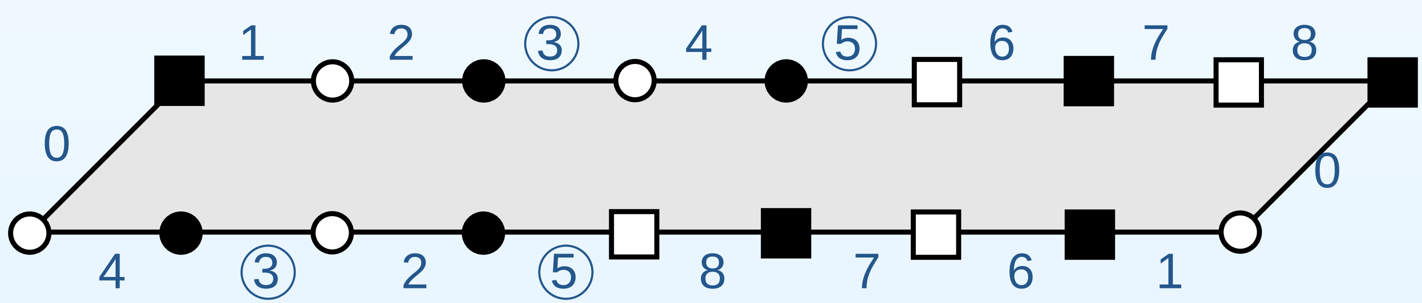
Getting from $\mathcal{H}(1,1,1,1)$ to $\mathcal{H}(3,1)$





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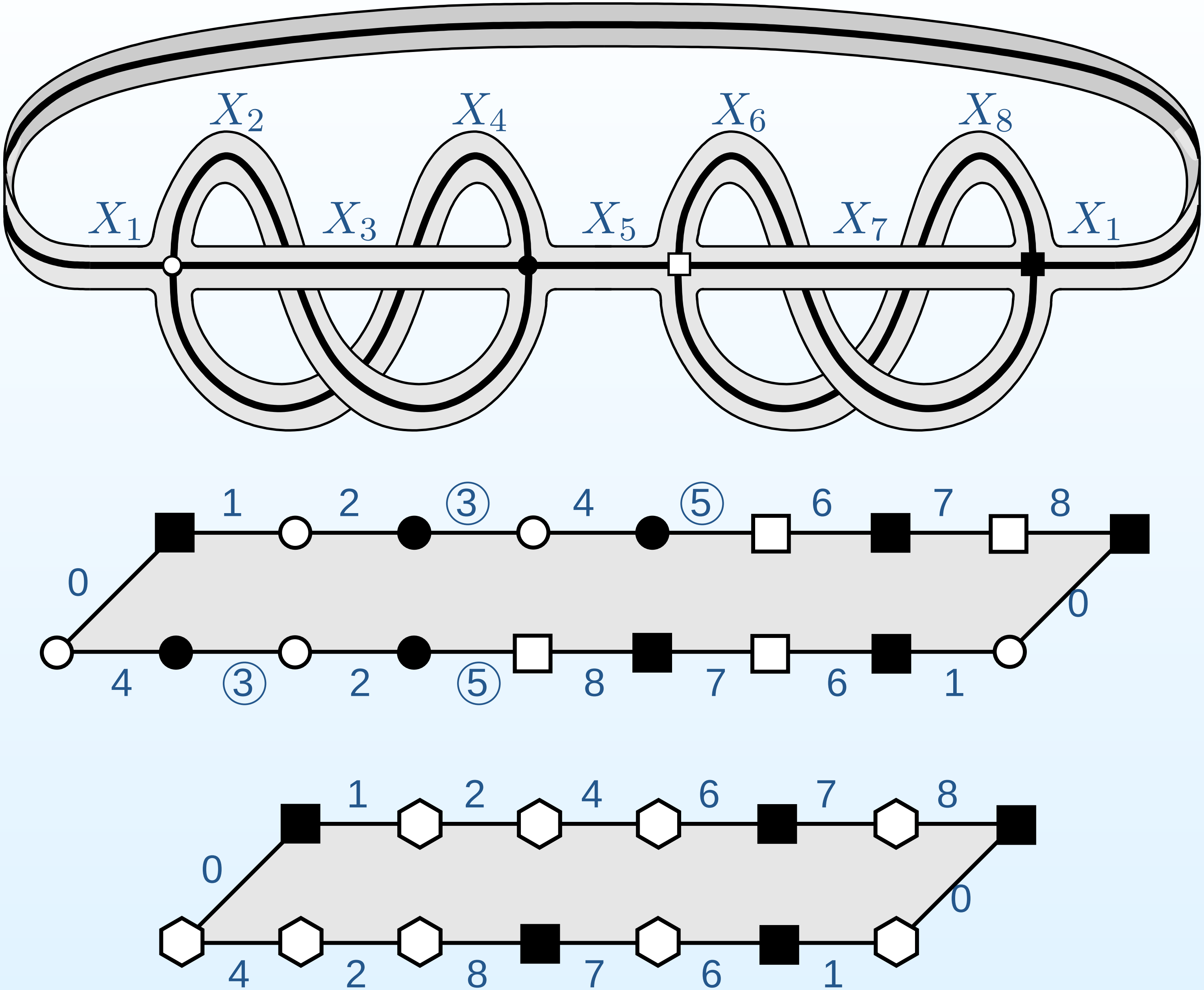
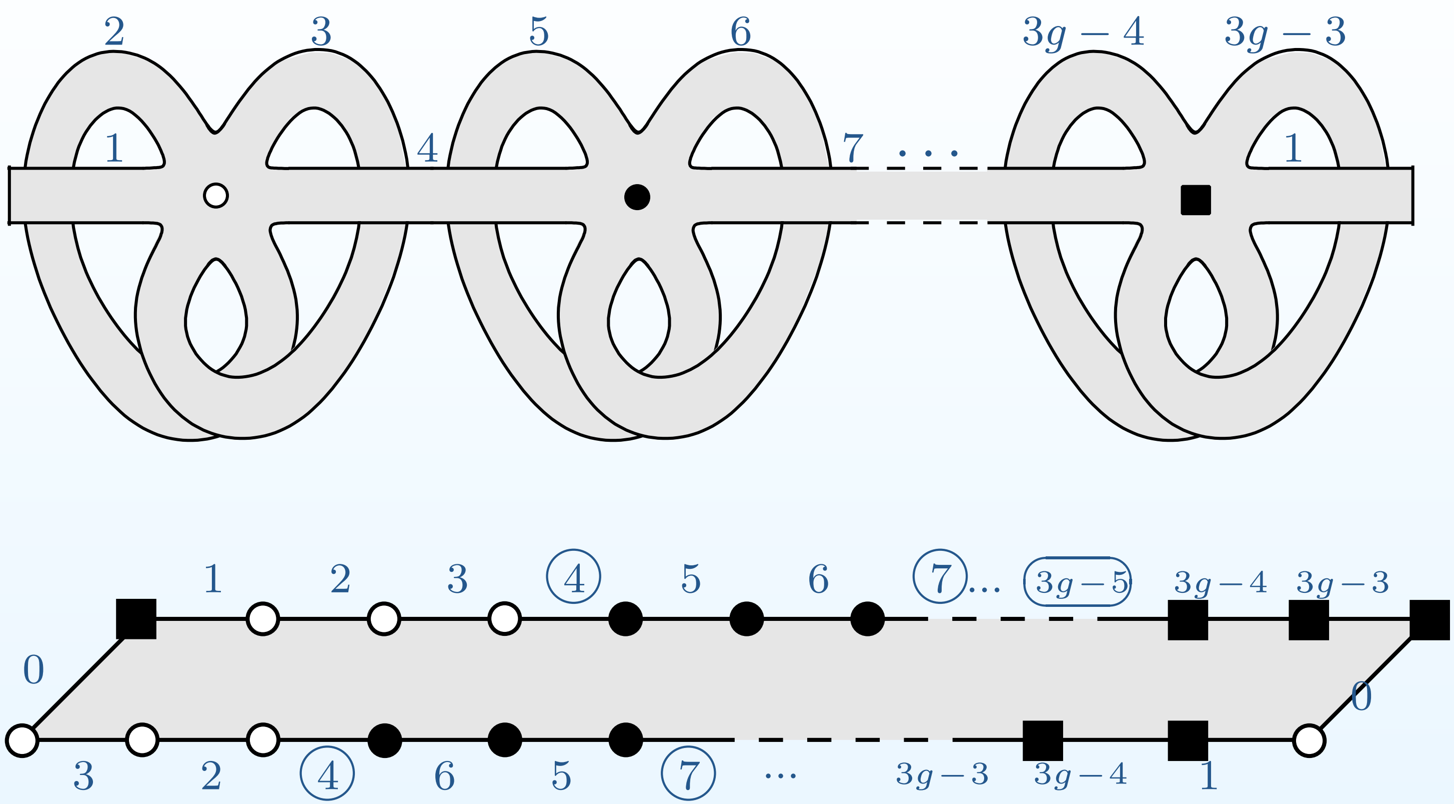


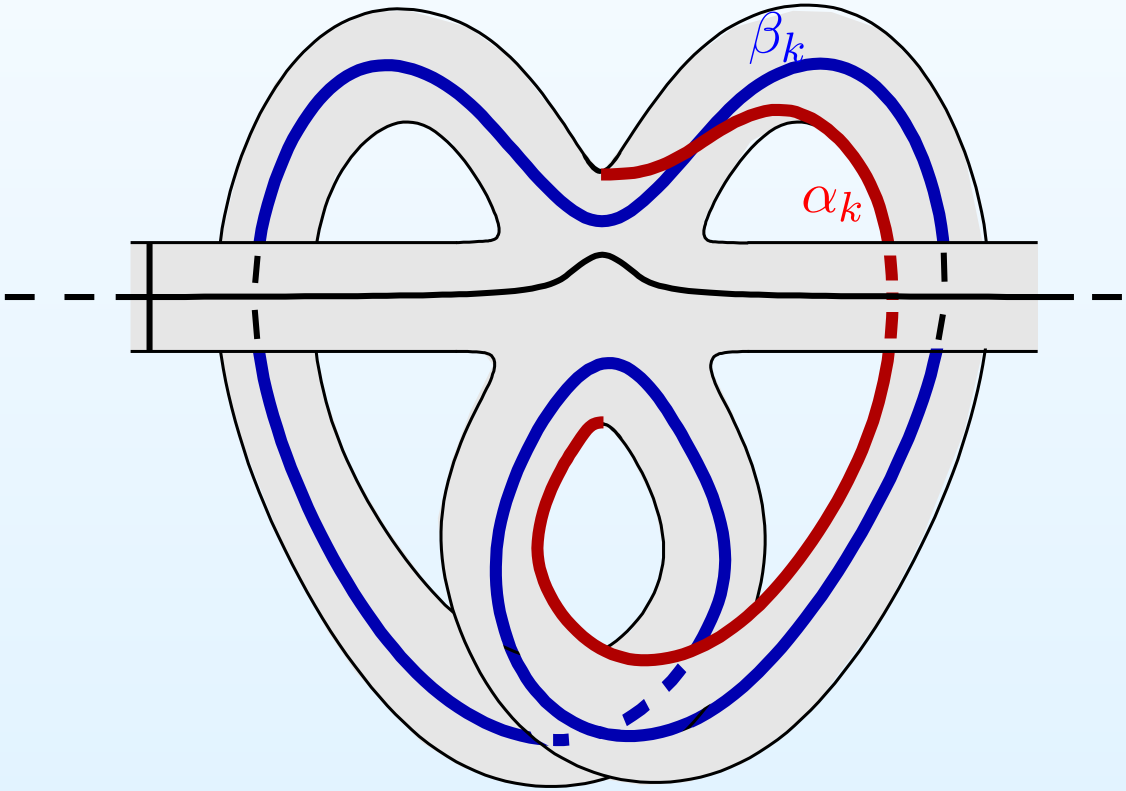
Representative of $\mathcal{H}^{odd}(2,\ldots,2)$



A one-cylinder Strebel differential from the component $\mathcal{H}^{odd}(2,\ldots,2)$ is represented by a ribbon graph on top and by a cylinder (on the bottom). Any subcollection of saddle connections with marked indices $4, 7\ldots, 3g-5$ is suitable for contraction.

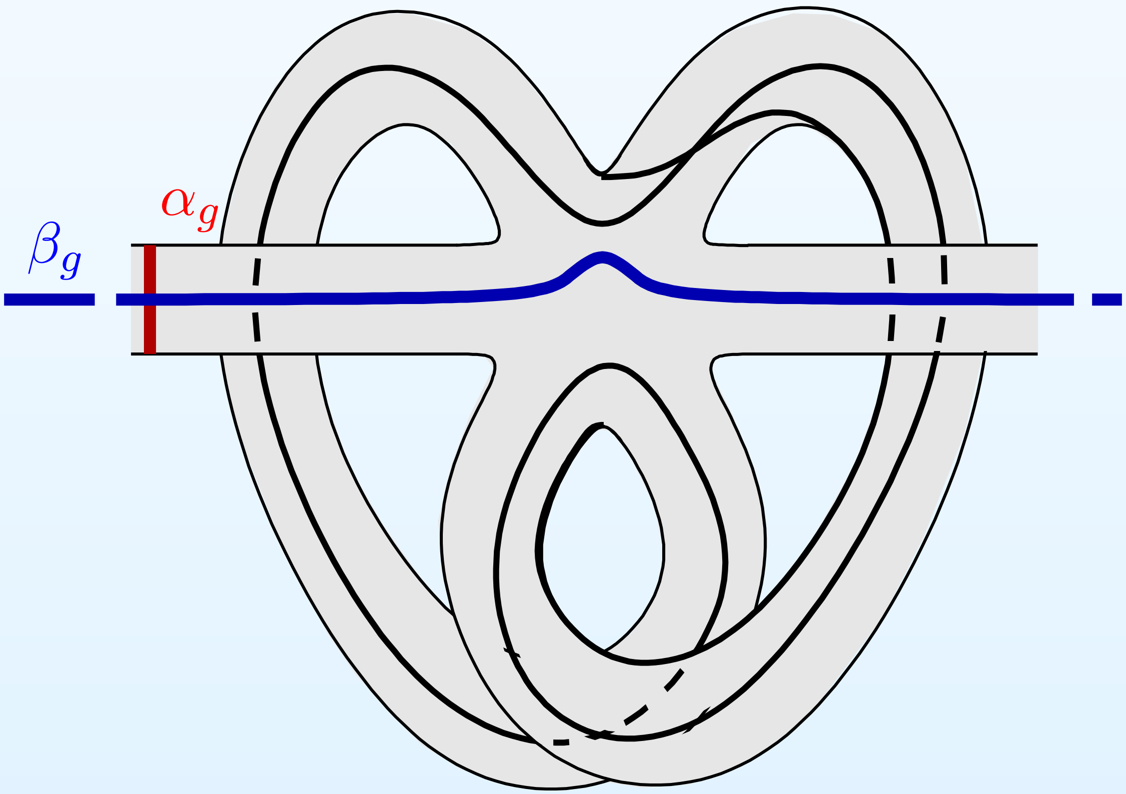
Basis of cycles

For each of the $g - 1$ repetitive fragments of our surface we construct a pair of smooth closed curves α_k, β_k as in the picture. By construction each of the curves is everywhere transverse to the vertical foliation and curves α_k and β_k have a single transverse intersection.

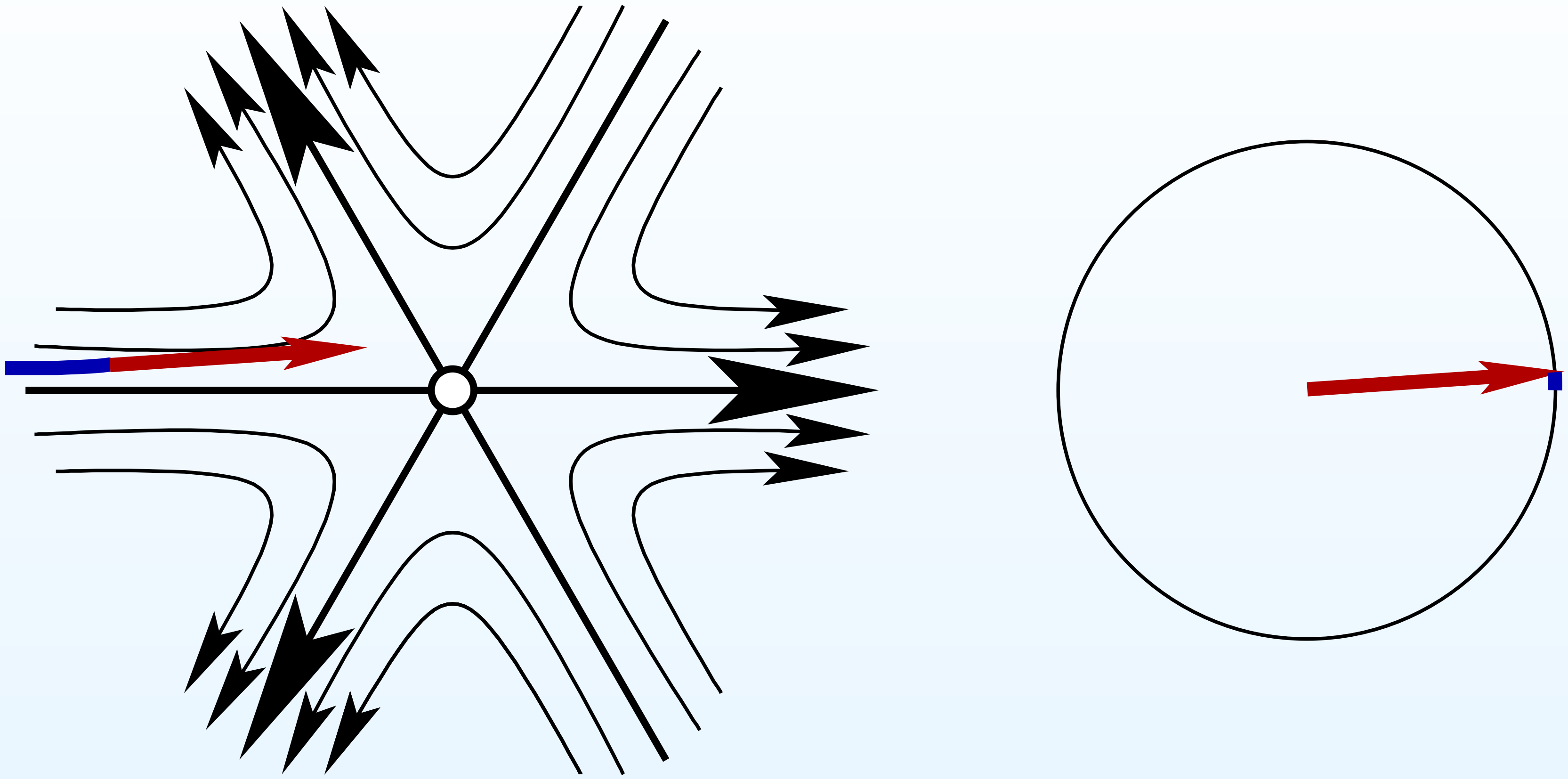


Basis of cycles

We construct one more pair of smooth closed curves α_g, β_g as in the picture. We can choose α_g to be a closed leaf of the vertical foliation and make β_g follow the core horizontal leaf in the complement of neighborhoods of zeroes and bypass the zeroes as in the picture.

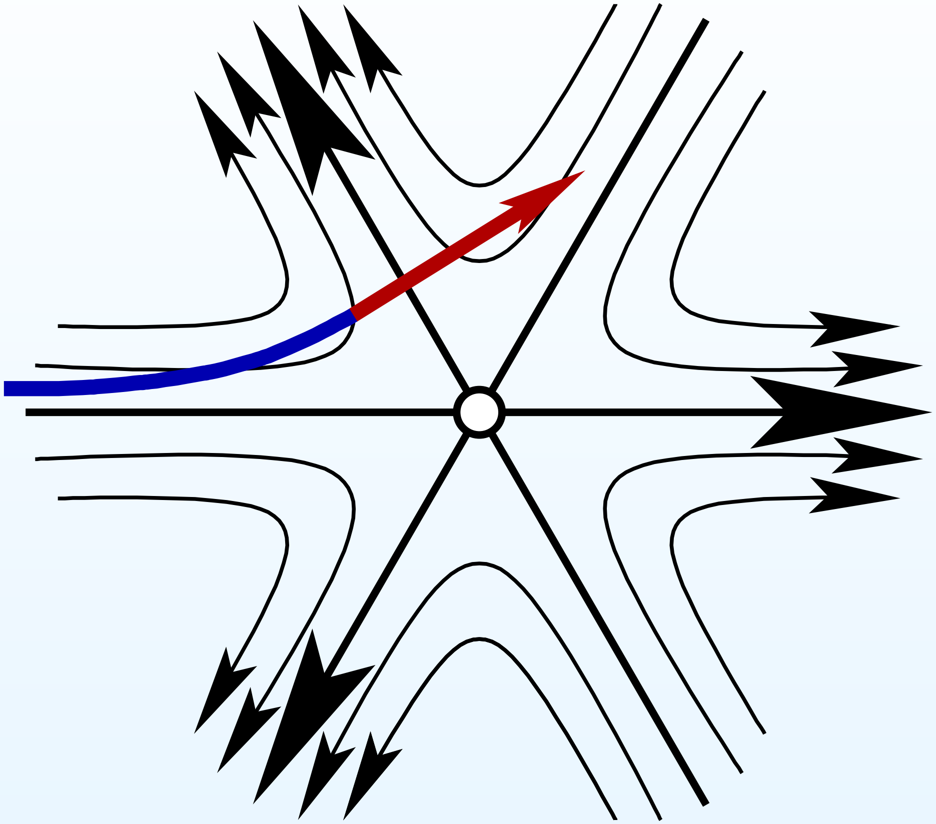


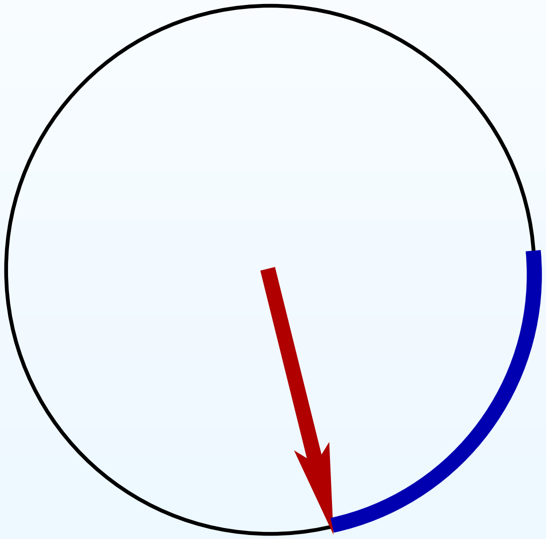
Index of the curve β_g



Following the blue path we make its image under the Gauss map perform a complete turn around the unit circle in the counterclockwise direction.

Index of the curve β_g





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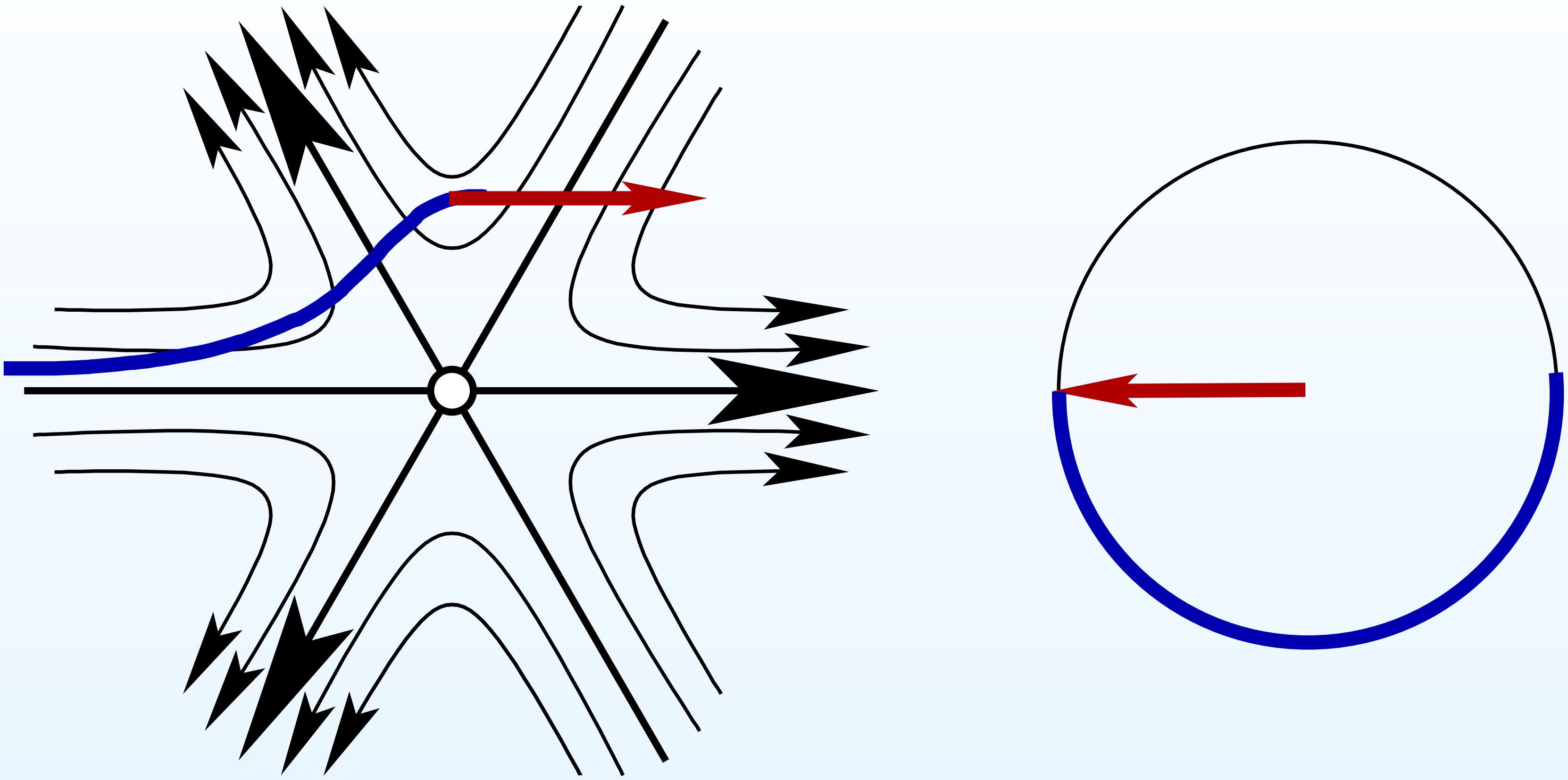

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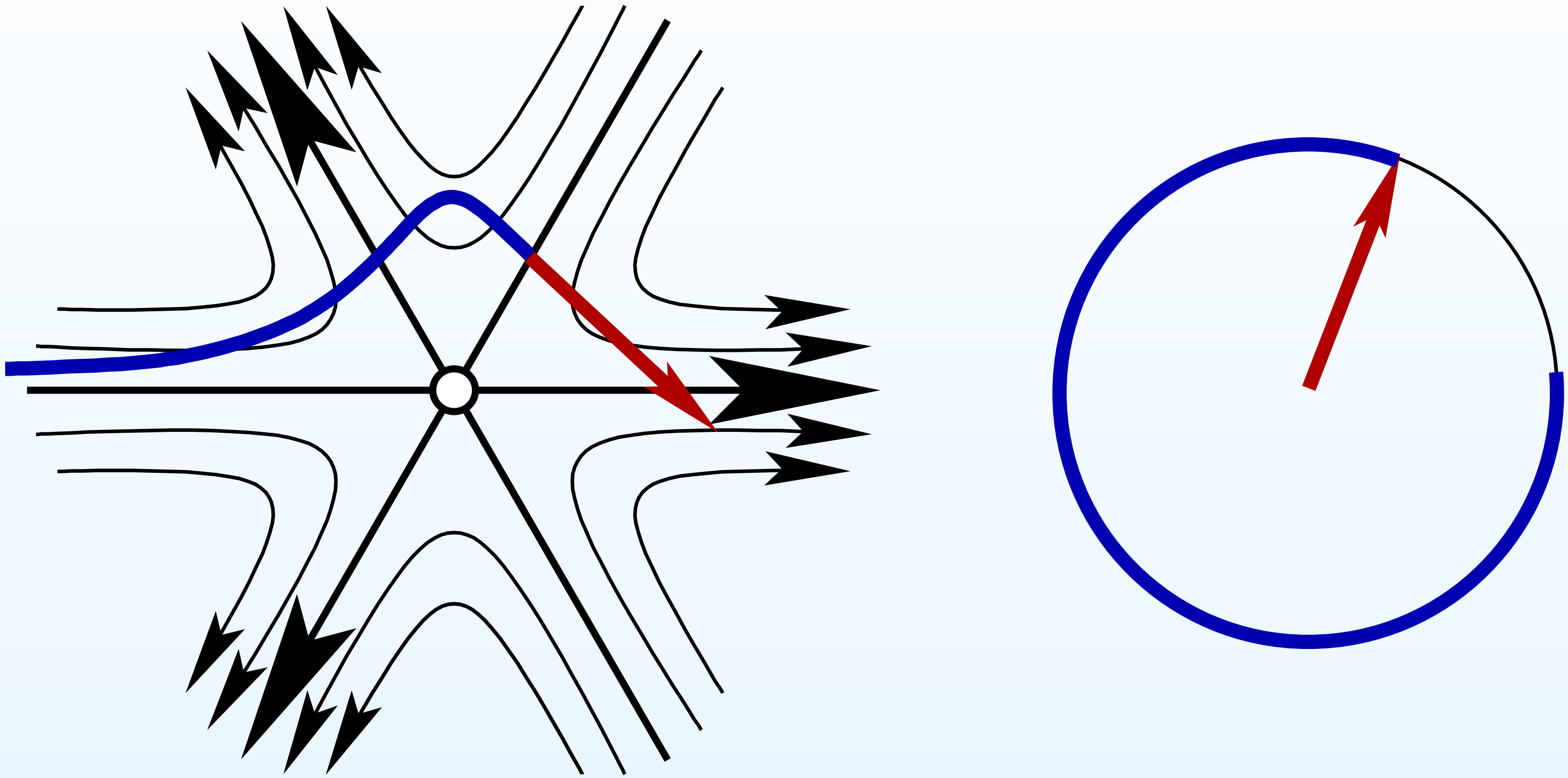

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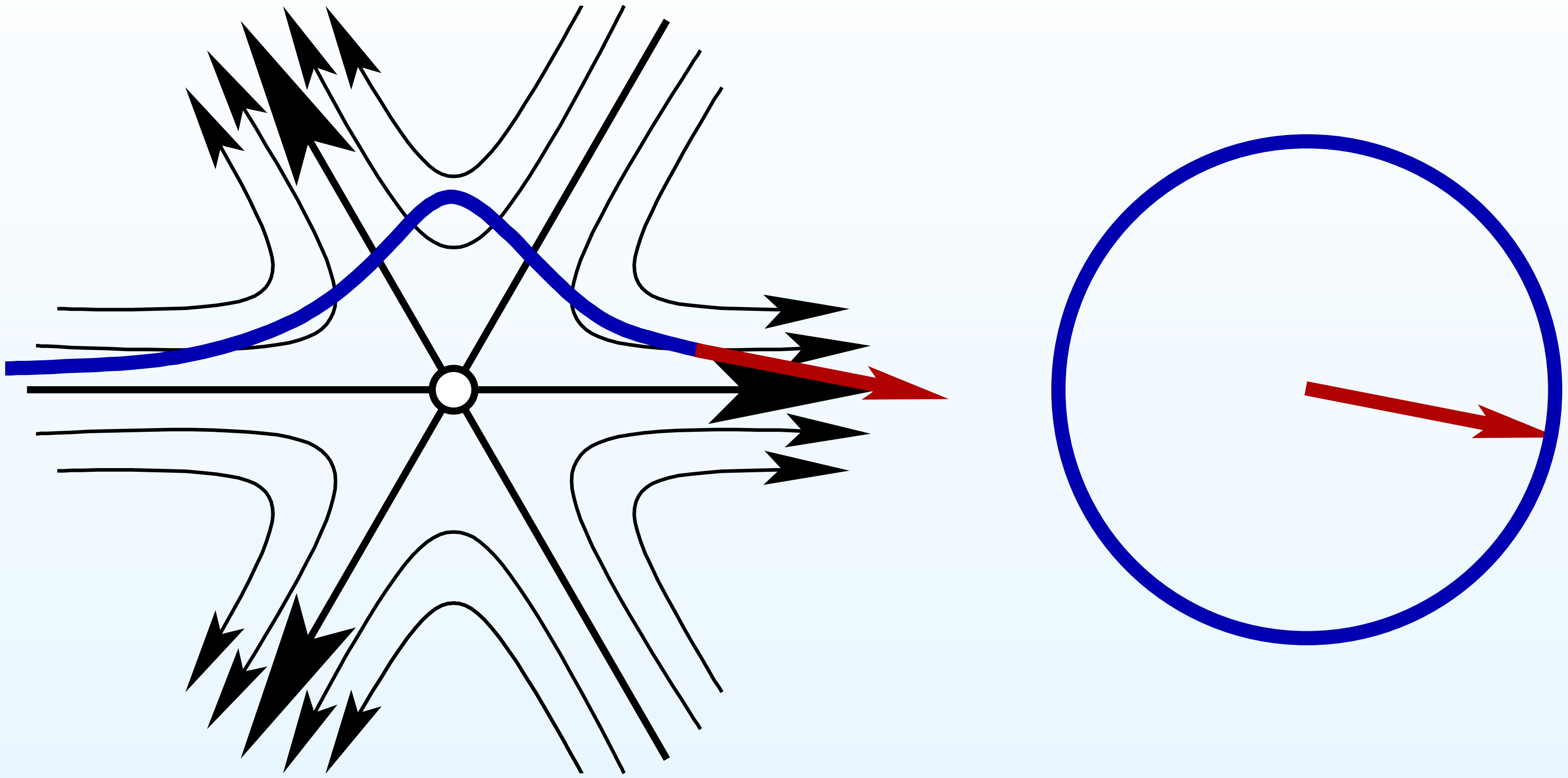

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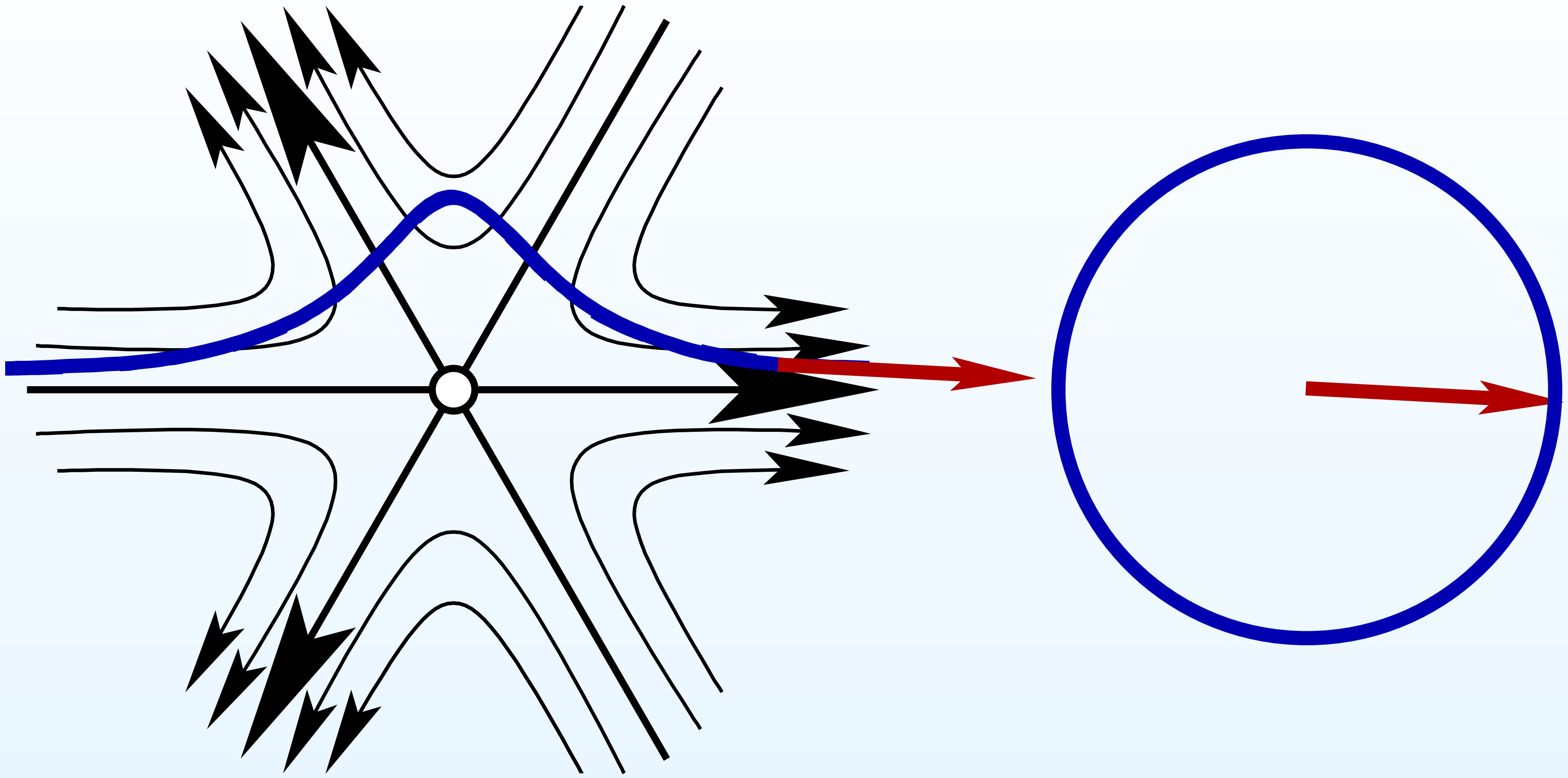

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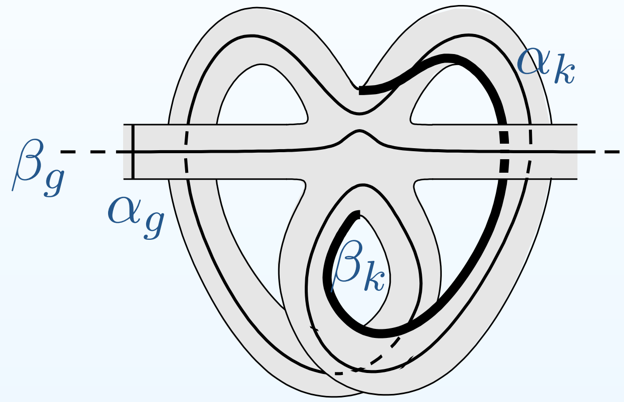
Computation of the parity of the spin-structure

Parity of the spin-structure: $\varphi(S) := \sum_{i=1}^{g} \big(ind(\alpha_i)+1\big)\big(ind(\beta_i)+1\big) \mod 2\,,$

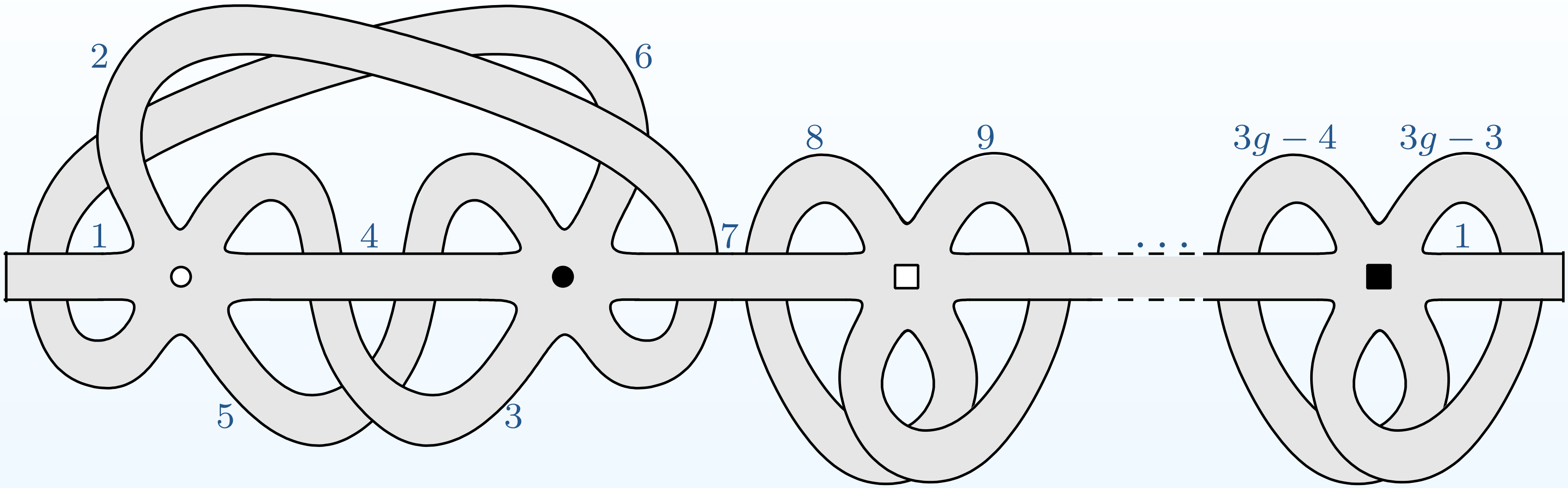
Cycles α_j, β_j for $j = 1, \ldots, g$ form a "canonical" basis of cycles.

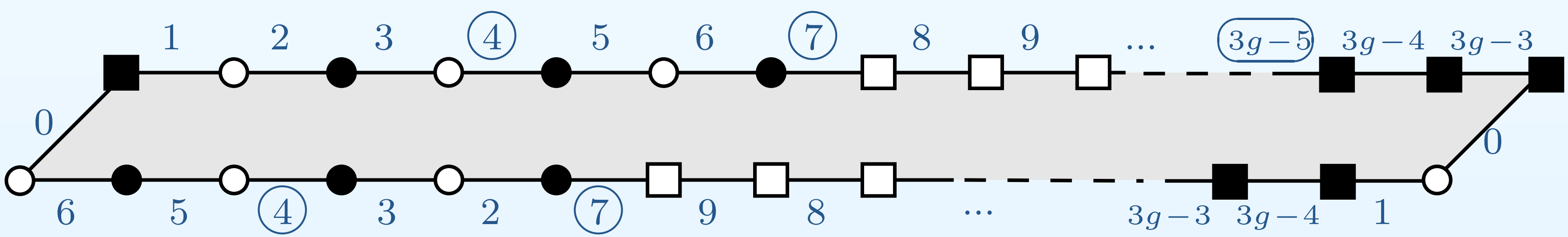


All curves except β_g are transverse to the vertical foliation, and, hence, $ind(\alpha_1) = ind(\beta_1) = \cdots = ind(\alpha_g) = 0$. We have, $ind(\beta_g) = -(g-1)$, since each time when β_g turns around one of the $g-1$ zeroes, the image of the Gauss map makes a complete clockwise turn around a circle. Hence,

$$\varphi(S) = \sum_{i=1}^{g-1}(0+1)(0+1) + (0+1)\big((1-g)+1\big) = (g-1)+(2-g) = 1 \mod 2\,.$$

Representative of $\mathcal{H}^{even}(2,\ldots,2)$





A one-cylinder Strebel differential from the component $\mathcal{H}^{even}(2,\ldots,2)$ is represented by a ribbon graph on top and by a cylinder (on the bottom). Any subcollection of saddle connections with marked indices $4, 7\ldots, 3g-5$ is suitable for contraction.

Hyperelliptic components

Theorem. *Let ω be an Abelian Jenkins–Strebel differential with a single cylinder. If it belongs to a hyperelliptic connected component, then a natural cyclic structure on the set of horizontal saddle connections of ω has the following form:*

$$
\begin{array}{ccccc}
1 & 2 & \dots & k-1 & k \\
k & k-1 & \dots & 2 & 1
\end{array}.
$$

The Abelian differential ω belongs to $\mathcal{H}^{hyp}(2g-2)$ when $k = 2g-1$ is odd and to $\mathcal{H}^{hyp}(g-1, g-1)$ when $k = 2g$ is even.

Idea of the proof. A hyperelliptic involution τ acts on any Abelian differential ω as $\tau^*\omega = -\omega$. Hence, the flat isometry τ acts on the horizontal cylinder by an orientation-preserving involution interchanging the boundaries of the cylinder. Since zeroes are mapped to zeroes, horizontal saddle connections are isometrically mapped to horizontal saddle connections. Since their lengths might be different, every saddle connection X_i is mapped to X_i for $i = 1, \dots, k$. This proves that the cyclic orders of the horizontal saddle connections on two components of the cylinder are inverse to each other.