2021-10-25 Kähler geometry

Let V be a vector space over $\mathbb{C}$.

$G \subset SL(V)$

<u>Def</u> $p \in V$ $(p \neq 0)$ is stable

$\iff$ (1) the orbit $G \cdot p$ is closed.

(2) the stabilizer

$$\mathrm{Stab}(p) = \{ g \in G \mid gp = p \}$$

is finite subgroup.

<u>Def</u> $p \in V$, $p \neq 0$, is polystable

$\iff$ only (1) holds.

<u>Def</u> $p \in V$ is semi-stable

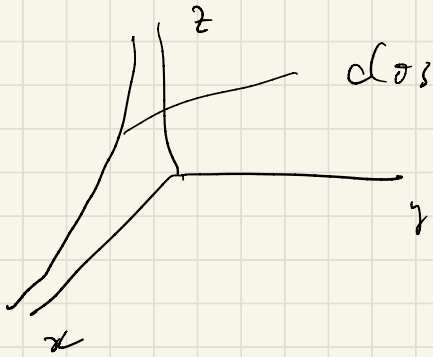
$\iff$ the closure of $G \cdot p$ does not contain $0 \in V$.

(poly stable, stable $\Rightarrow$ semi stable.)

Exaple. $V = \mathbb{C}^3$

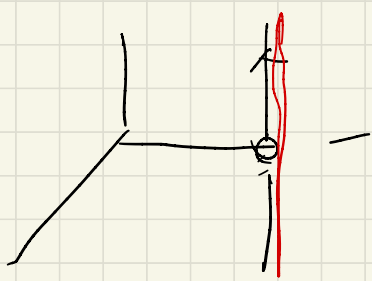
$G = \mathbb{C}^*$, $t(x, y, z) = (t^{-1}x, y, tz)$

(a) If $x \neq 0$, $z \neq 0$



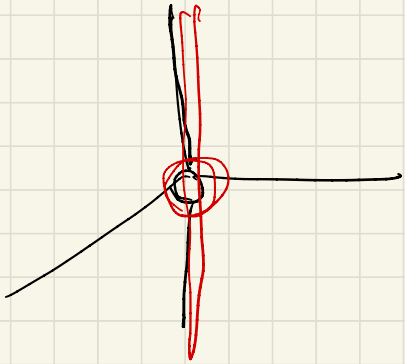
closed orbit

(b) $t(0, 1, z) = (0, 1, tz)$



closure $\not\ni 0$.

(c) $t(0,0,z) = (0,0,tz)$



unstable.

(d)

$t(0,1,0) = (0,1,0)$

$\mathrm{Stab}(0,1,0) = \mathbb{C}^*$

poly stable.

$\mathbb{P}(V)$ = projective space

$= \{\text{lines in } V, \text{ through } 0\} \ni [p]$

$0 \neq v \in p$

$Z \subset \mathbb{P}(V)$ submfd.

$$\mathcal{O}_{\mathbb{P}(V)}(-1) = \{([p], x) \in \mathbb{P}(V) \times V \mid x \in [p]\}$$

$$\longrightarrow \mathbb{P}(V)$$

tautological line bdle (Milnor-Stasheff)

$$c_1(\mathcal{O}(-1)) = \left[-\frac{i}{2\pi} \partial\bar{\partial} \log\left(1 + |t^1|^2 + \cdots + |t^m|^2\right) \right]$$

on $\{z^0 \neq 0\}$, $t^i = \dfrac{z^i}{z^0}$

$\mathcal{O}(-1)$ - zero section $\cong V - \{0\}$.

$$\mathcal{O}_Z(-1) = \mathcal{O}(-1)\big|_Z$$

Suppose $G \subset SL(V)$ preserves $\underline{\underline{Z}}$.

Let (Z, ω) compact Kähler manifold.

$[\omega] \in H^2(M : \mathbb{Z})/\text{torsion}$.

$L \to Z$ holomorphic line bundle with $c_1(L) = [\omega]$.

(such an L is called an ample line bdle.)

Suppose a reductive Lie group G acts on L as bundle isomorphisms.

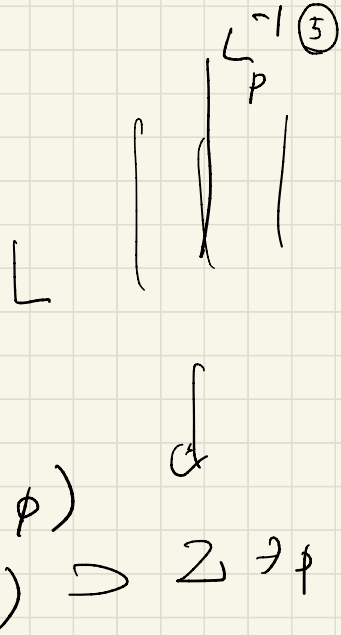
<u>Def</u> $p \in Z$ (semi) stable

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall x \neq 0$ in L_p^{-1}

$G \cdot x$ closed

and finite stabilizer

(resp. $\overline{Gx} \cap$ zero section $= \phi$)



This def is motivated by Kodaira embedding thm.

Let $L \to M$ be an ample line bdle.

$\exists$ ν s.t. $\forall k \geq \nu$, $s_0, \dots, s_{N_k} \in H^0(M, L^k)$ basis

$$\Phi_k : M \longrightarrow \mathbb{P}^{N_k}(\mathbb{C})$$
$$p \longmapsto (s_0(p) : \dots : s_{N_k}(p))$$

gives an embedding. $\left(\mathbb{P}^{N_k}(\mathbb{C}) \cong \mathbb{P}(H^0(M, L^k)^*) \right)$

<u>Def</u> (M, L) polarized mfd

$\underset{\text{def}}{\Longleftrightarrow}$ A pair of complex mfd M and an <u>ample</u> line bundle on M.

$\underline{\underline{c_1(L)}} > 0$.

$\left(\Rightarrow M \text{ algebraic mfd by Kodaira embedding} \right)$

$\boxed{\text{Stability can be interpreted in terms of symplectic geometry using "moment map". (Kempf-Ness).}}$

Let (N, Ω) be a symplectic mfd

i.e. Ω satisfies the following

- Ω closed 2-form.
- $\Omega : TN \times TN \longrightarrow \mathbb{R}$ non-degenerate

i.e. if $\Omega(X, Y) = 0$ for $\forall Y$ then $X = 0$.