Lecture No 26 June 7, 2022 (Tue)

[Recall what we discussed last time]

4.2.1. $\boldsymbol{d = 1}$:

- Consider the SPDE

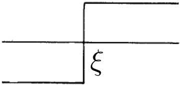
$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \varepsilon^\gamma a(x) \dot{W}(t,x), \quad x \in \mathbb{R},$$

which has a unique solution: $u^\varepsilon(t,x) \in C^{\frac{1}{4}-,\frac{1}{2}-}((0,\infty)\times\mathbb{R})$

  - $\gamma > \frac{19}{2}$ (smallness of noise)
  - $a \in C_0^2(\mathbb{R})$: intensity of noise with spatial cut-off
  - $\dot{W}(t,x)$: space-time Gaussian white noise (only in 1D)
  - Assume f is symmetric (i.e., $f(u) = -f(-u)$), in particular, it satisfies the balance condition $A(f) = 0$ with two stable solutions ± 1.

- We denote

$$\chi_\xi(x) := 1_{(\xi,\infty)}(x) - 1_{(-\infty,\xi)}(x)$$



[Theorem 1] (F, PTRF 1995, Proc. Taniguchi sympo 1997)

If the initial value has the form $u^\varepsilon(0,x) = m((x-\xi)/\varepsilon)$, then

$$\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-1}t,x) \Longrightarrow \chi_{\xi_t}(x) \quad (\varepsilon \downarrow 0),$$

(i.e. convergence in law). The phase separation point ξ_t moves according to the SDE:

$$d\xi_t = \alpha_1 a(\xi_t)dB_t + \alpha_2 a(\xi_t)a'(\xi_t)dt, \quad \xi_0 = \xi, \tag{1}$$

where B_t is a 1D Brown motion, $\alpha_1 = ||m'||^{-1}_{L^2(\mathbb{R})}$,

$$\alpha_2 = -||m'||^{-2}_{L^2(\mathbb{R})} \int_0^\infty dt \int_{\mathbb{R}^2} xp(t,x,y)^2 f''(m(y))m'(y)dxdy,$$

and $p(t,x,y)$ is a fundamental solution of the linearized operator $\partial/\partial t - \{\partial^2/\partial y^2 + f'(m(y))\}$. □

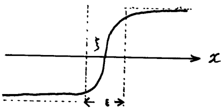
- The diffusion coefficient (mobility) α_1^2 coincides with the inverse of the surface tension $||m'||^{-2}_{L^2(\mathbb{R})}$
- We showed some numerical results.

- The intuitive reason that $\varepsilon^{-2\gamma-1}$ is the proper time scale is explained as follows: $\bar{u} = \bar{u}^{\varepsilon}$ satisfies (in law):

$$\frac{\partial \bar{u}}{\partial t} = \varepsilon^{-2\gamma-1}\left\{\Delta \bar{u} + \frac{1}{\varepsilon^2} f(\bar{u})\right\} + \left(\varepsilon^{-2\gamma-1}\right)^{1/2} \cdot \varepsilon^{\gamma} a(x)\dot{W}(t,x).$$

- Noise term is $a(x)\varepsilon^{-1/2}\dot{W}(t,x)$.
- The strong drift $\varepsilon^{-2\gamma-1}$ pushes $\bar{u}$ to the neighborhood of

$$M^{\varepsilon} := \left\{\bar{u};\ \Delta \bar{u} + \frac{1}{\varepsilon^2} f(\bar{u}) = 0,\ \bar{u}(\pm\infty) = \pm 1\right\}$$
$$= \{m((x-\xi)/\varepsilon);\ \xi \in \mathbb{R}\}$$



so that $\bar{u}^{\varepsilon}(t,x) \sim m((x-\xi_t)/\varepsilon)$.

- In particular, the width of the interface is $O(\varepsilon)$.
- The contribution of the noise $\dot{W}(t,x)$ comes only from this region, therefore its order is $O(\varepsilon^{1/2})$ by self-similarity.
- This balances with the factor $\varepsilon^{-1/2}$ in front of the noise.

[These are what we stated last time]

Different scaling in t for asymmetric f, but still $A(f) = 0$

- If the symmetry condition (oddness of f) is violated ($A(f) = 0$ is still assumed), we can show the LLN:

$$u^\varepsilon(\varepsilon^{-2\gamma}t, x) \Longrightarrow \chi_{\xi_t}(x), \qquad \dot{\xi}_t = \alpha_3 a^2(\xi_t)$$

  with the constant

$$\alpha_3 = -\frac{1}{2||m'||^2_{L^2(\mathbb{R})}} \int_0^\infty dt \int_{\mathbb{R}^2} p(t,x,y)^2 f''(m(y)) m'(y) dx dy$$

- The centering condition implies $\alpha_3 = 0$, so that we get CLT under longer time scale.
- $^\exists f$: asymmetric, $A(f) = 0$ and $\alpha_3 \neq 0$.
  Traveling wave produced by noise.

The proof of Theorem 1 consists of two steps

Step 1:

- To show that $\bar{u}^\varepsilon$ stays near
  $M^\varepsilon := \{ m((\cdot - \xi)/\varepsilon); \xi \in \mathbb{R} \}$,
  we take Ginzburg-Landau-Wilson free energy

$$\mathcal{H}^\varepsilon(u) := \int_{\mathbb{R}} \left\{ \frac{1}{2} |\nabla u|^2(x) + \frac{1}{\varepsilon^2} V(u(x)) \right\} dx$$

  as a Lyapunov function, where V is the integral of f (potential, i.e., $f = -V'$); recall $m'' + f(m) = 0$ and $\frac{\delta H^\varepsilon}{\delta u(x)} = -\Delta u - \frac{1}{\varepsilon^2} f(u(x))$ with $f = -V'$.
- However, since $\bar{u}^\varepsilon$ is not differentiable, we cannot insert $\bar{u}^\varepsilon$ into $\mathcal{H}^\varepsilon$ and this requires some extra trick (we actually consider convolution: $\bar{u}^\varepsilon * \rho^\delta$ with convolution kernel ρ).
- See next pages for the method of Lyapunov function.

[Simple example to apply the method of Lyapunov function]

Consider SDE on $\mathbb{R}^d$:

$$dX_t = -\frac{1}{\varepsilon^2}\nabla H(X_t)dt + \varepsilon^\alpha dB_t,$$

where $\alpha > 0$ and $H(x) = \frac{1}{2}\sum_{i=1}^{d-1} x_i^2, x = (x_1, \ldots, x_d)$. Then,

$$\{H \equiv 0\} = \{(0, \ldots, 0, x_d)\} \cong \mathbb{R} \quad (\leftrightarrow M^\varepsilon)$$

and $\nabla H = (x_1, \ldots, x_{d-1}, 0)$ so that $|\nabla H|^2 = 2H$ and $\Delta H = d - 1$.
Apply Itô's formula for $H(X_t)$ to obtain

$$\begin{aligned} dH(X_t) &= \nabla H(X_t) \cdot dX_t + \frac{\varepsilon^{2\alpha}}{2}\Delta H(X_t)dt \\ &= -\frac{1}{\varepsilon^2}|\nabla H(X_t)|^2 dt + \varepsilon^\alpha \nabla H(X_t) \cdot dB_t + \frac{\varepsilon^{2\alpha}}{2}\Delta H(X_t)dt. \end{aligned}$$

Let $\sigma = \inf\{t \geq 0; |(X_{t,1}, \ldots, X_{t,d-1})| \geq c\}$, i.e. the time X_t leaves the c-neighborhood of $\{H \equiv 0\}$.

Then, by Doob's optional sampling theorem,

$$E[H(X_{t\wedge\sigma})] + \frac{1}{\varepsilon^2} E\Big[\int_0^{t\wedge\sigma} |\nabla H(X_s)|^2 ds\Big] = E[H(X_0)] + \frac{\varepsilon^{2\alpha}}{2} E\Big[\int_0^{t\wedge\sigma} \Delta H(X_s) ds\Big],$$

which shows (assuming $H(X_0) = 0$ for simplicity),

$$E[H(X_{t\wedge\sigma})] \le \frac{\varepsilon^{2\alpha}}{2}(d-1)t.$$

Thus,

$$P(\sigma \le t) \le \frac{1}{c^2/2} E[H(X_{t\wedge\sigma})] \le \frac{\varepsilon^{2\alpha}}{c^2}(d-1)t \to 0$$

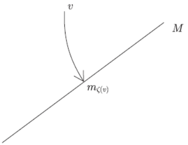
as $\varepsilon \downarrow 0$. This means that X_t stays in the c-neighborhood of $\{H \equiv 0\}$ before time t with high probability. □

Step 2:

- We introduce a nice coordinate in a tubular neighborhood of M^ε (or on M^1 under spatial scaling $x = \varepsilon y$).
- Consider the PDE:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + f(v), \quad t > 0,\ y \in \mathbb{R}. \tag{2}$$

- If its initial data v_0 is in an L^2-tubular neighborhood of M^1, the solution $v = v(t, y)$ converges to a certain $m_\zeta(y) := m(y - \zeta)$ in M^1 as $t \to \infty$.
- The limit ζ depends on the initial value v_0 so that we denote it by $\zeta = \zeta(v_0) \in \mathbb{R}$. This defines a nice coordinate in an L^2-tubular neighborhood of M^1.
- If we compute the time derivative of $\zeta(u^\varepsilon(t))$, the diverging factor cancels (see below).



Outline of the derivation of the limit SDE (4).

- First introduce $v^\varepsilon(t,y) := \bar{u}^\varepsilon(t,\varepsilon y)$ observing under the microscopic spatial variable y.
- Then, from the SPDE for $\bar{u}^\varepsilon$, v^ε satisfies the SPDE in law:

$$\partial_t v = \varepsilon^{-2\gamma-3}\{\Delta v + f(v)\} + \varepsilon^{-1}a(\varepsilon y)\dot{W}(t,y). \qquad (3)$$

- The coordinate $\zeta(v) \in \mathbb{R}$ defined in the L^2-tubular neighborhood of M^1 introduced above enjoys the properties in the following lemma.
- We denote its first and second Fréchet derivatives by $D\zeta(y,v)$ and $D^2\zeta(y_1,y_2,v)$, respectively. The shifted standing wave m is defined by $m_\eta(y) = m(y-\eta), y \in \mathbb{R}$ for $\eta \in \mathbb{R}$.

$$d\xi_t = \alpha_1 a(\xi_t)dB_t + \alpha_2 a(\xi_t)a'(\xi_t)dt, \quad \xi_0 = \xi, \qquad (4)$$

[Lemma 2] (1) For every v in the neighborhood of M^1, we have that

$$\langle D\zeta(\cdot, v), \Delta v + f(v)\rangle_{L^2} = 0.$$

(2) For every $\eta \in \mathbb{R}$, $D\zeta(y, m_\eta) = -\frac{m'_\eta(y)}{||m'||^2_{L^2}}$.
(3) For every $\eta \in \mathbb{R}$,

$$D^2\zeta(y, y, m_\eta) = -\frac{1}{||m'||^2_{L^2}} \int_0^\infty dt \int_{\mathbb{R}} p(t, y, z; m_\eta)^2 f''(m_\eta(z)) m'_\eta(z)\, dz,$$

where $p(t, y, z; m_\eta)$ is the fund. sol. of $\partial_t - \{\partial_y^2 + f'(m_\eta(y))\}$. □

- (1) follows from "$\zeta(v(t)) =$ const along the sol. $v(t)$ of the PDE (2)." In fact,

$$0 = \frac{d}{dt}\zeta(v(t)) = \langle D\zeta(\cdot, v(t)), \Delta v(t) + f(v(t))\rangle, \ t > 0.$$

- For (2) and (3), see (7.6) and (7.9) in [F, 1995], which were shown after some (rather long) computation.

- $\xi_t^\varepsilon := \varepsilon\zeta(v^\varepsilon(t))$: macroscopic phase separation point of $v^\varepsilon(t)$
- Then, applying Itô's formula and from the microscopic SPDE (3),

$$d\xi_t^\varepsilon = \int_{\mathbb{R}} D\zeta(y, v^\varepsilon(t))a(\varepsilon y)W(dtdy) + \frac{1}{2}\varepsilon^{-1}\int_{\mathbb{R}} D^2\zeta(y, y, v^\varepsilon(t))a^2(\varepsilon y)dy\ dt. \tag{5}$$

- Diverging factor (first term in (3)) appearing in $\varepsilon\langle D\zeta(\cdot, v^\varepsilon(t)), dv^\varepsilon(t)\rangle$ cancels due to Lemma 2-(1).
- The quadratic variation of the first term in (5) is given by

$$\int_{\mathbb{R}} D\zeta(y, v^\varepsilon(t))^2 a^2(\varepsilon y)dy\ dt.$$

- By Step 1, we can assume that $v^\varepsilon(t)$ is close to $m_{\varepsilon^{-1}\xi_t}$ for some ξ_t, and thus, from Lemma 2-(2), this integral is close to

$$a^2(\xi_t)\int_{\mathbb{R}} \frac{(m'_{\varepsilon^{-1}\xi_t}(y))^2}{||m'||_{L^2}^4}dy\ dt = a^2(\xi_t)\alpha_1^2\ dt,$$

which leads to the first term in SDE (4); recall $\alpha_1 = ||m'||_{L^2}^{-1}$.

- On the other hand, in the second term in (5), the contribution of $D^2\zeta(y, y, m_{\varepsilon^{-1}\xi_t})$ comes only from the vicinity of $y = \varepsilon^{-1}\xi_t$.
- Therefore, we may expand $a^2(\varepsilon y)$ as

$$a^2(\varepsilon y) = a^2(\xi_t) + \frac{1}{2}(a^2)'(\xi_t) \cdot \varepsilon(y - \varepsilon^{-1}\xi_t) + \cdots .$$

- However, the contribution of the first $a^2(\xi_t)$ vanishes under the integration in y, since

$$\int_{\mathbb{R}} D^2\zeta(y, y, m_\eta)\, dy = 0.$$

  (To see this, we may assume $\eta = 0$ and then $\int p(t, y, z; m)^2 dy$ $= p(2t, z, z; m)$ is even in z while $f''(m(z))m'(z)$ is odd.)
- The contribution of the second term, after cancellation of ε^{-1} and ε, gives

$$\frac{1}{2}(a^2)'(\xi_t)\alpha_2\, dt,$$

  from Lemma 2-(3), and this is just the second term in SDE (4).
- Thus, the SDE (4) is derived. □

Related results

- (F 1997) We can also study a self-similar Gaussian space-time (colored) noise $\{W_h, 1/2 \leq h \leq 1\}$ with the covariance structure:

  $$E[\dot{W}_h(t,x)\dot{W}_h(s,y)] = \delta_0(t-s)Q_h(x-y)$$

  where Q_h is the Riesz potential kernel of $(2h-1)$ the order:

  $$Q_h(x) = \begin{cases} h(2h-1)|x|^{2h-2}, & 1/2 < h \leq 1, \\ \delta_0(x), & H = 1/2. \end{cases}$$

- Brassesco-De Masi-Presutti (1995), Bertini-Brassesco-Buttà (2008) at microscopic level.

Multi-kinks ($d = 1$):

- S. Weber (2014): Several interfaces (multi-kinks) on $[0, 1]$ with periodic boundary conditions. Noise is $\varepsilon^\gamma \dot{W}(t, x)$ (space-time white noise). Annihilating BMs are obtained in the limit.
- Method: (1) Consider approximate slow manifold $\mathcal{M}$ and coordinate system around $\mathcal{M}$ (PDE case: Carr-Pego, Xinfu Chen).
  (2) Use the idea of expansion in stochastic case due to Antonopoulou-Blömker-Karali (2012) when u is close to $\mathcal{M}$.
  (3) Show annihilation when two interfaces touch.
- Fatkullin-Kovačič-Vanden Eijnden (2010) for a related heuristic argument for stochastic multi-kink motion.

Cahn-Hilliard ($d = 1$)

- Antonopoulou-Blömker-Karali (2012): Cahn-Hilliard eq with smooth noise (TDGL eq of conservative type with Q-BM in place of space-time white noise) on $[0, 1]$ with no-flux boundary conditions of Neumann type:

$$\partial_x u = \partial_x^3 u = 0 \quad \text{at } x = 0, 1.$$

- SDEs are obtained in the sharp interface limit for multi-kinks before collisions (result is infinitesimal in time). Bertini-Brassesco-Butta (2014)

- Numerical simulation (by K. Lee):
  (1) multi-kinks
  (2) boundary condition