Lecture 6 Kirby calculas (I)

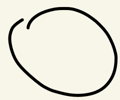
Kirby calculas: a way to represent 4-mfds via knots/links.

A knot is a smooth embedding $S^1 \hookrightarrow S^3$

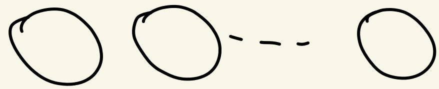
A link is a smooth embedding $\sqcup S^1 \hookrightarrow S^3$

Examples

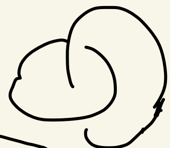
- The unknot



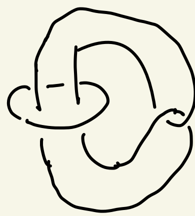
- The unlink



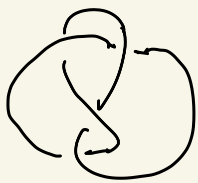
- The Hopf link



- Whitehead link



- figure eight knot



- torus knot (link) $T_{p,q}$

$T_{3,2}$

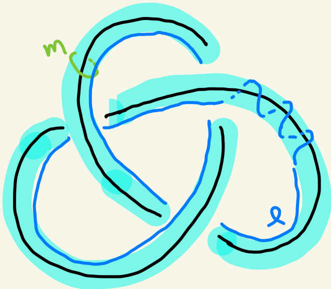
$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

$$T_{p,q} = S^3 \cap \{z_1^p + z_2^q = 0\}$$

Given $K \subset S^3$, let $\nu(K)$ be an tubular neighborhood.

Then $S^3 - \mathring{\nu}(K)$ is a 3-manifold with boundary

$$T^2 = \partial \nu(K) = \partial(S^3 - \mathring{\nu}(K))$$



on T^2, there are two loops:

m: meridian

$[m]$ generates $H_1(S^3 - \mathring{\nu}(K)) = \mathbb{Z}$

ℓ: longitude, a parallel copy of K s.t. $[\ell] = 0 \in H_1(S^3 - \mathring{\nu}(K))$

Note: If we orient K, then we can orient m (using a right-hand rule) and orient ℓ.

Given $p, q \in \mathbb{Z}$ with $(p,q)=1$, we define the $\frac{p}{q}$-surgery of S^3 along K

$$S^3_{\frac{p}{q}}(K) = (S^3 - \mathring{\nu}(K)) \underset{\partial\nu(K)}{\cup} (D^2 \times S^1)$$

where $\partial D^2 \times S^1 \xrightarrow{\cong} \partial \nu(K)$

$[\partial D^2 \times *] \longrightarrow [pm + q\ell] \in H_1(\partial \nu(K))$

Example: • $S^3_{\infty}(K) = S^3_{\frac{1}{0}}(K) = S^3 \quad \forall K$

• $S^3_{+1}(T_{2,3})$ = the Poincaré homology 3-sphere

$= \Sigma(2,3,5) := \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0, \, |x|^2 + |y|^2 + |z|^2 = 1\}$

Brieskorn sphere

$H_*(S^3_{+1}(T_{2,3})) = H_*(S^3)$ but $|\pi_1(S^3_{+1}(T_{2,3}))| = 120$.

(Note $H_1(S^3_{\frac{p}{q}}(K)) = \mathbb{Z}/p\mathbb{Z} \quad \forall K$.)

Similarly, given a n-component link $L = K_1 \sqcup \cdots \sqcup K_n \hookrightarrow S^3$ and $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \cdots, \frac{p_n}{q_n} \in \mathbb{Q} \cup \{\infty\}$, we can talk about the $\left(\frac{p_i}{q_i}\right)_{1 \le i \le n}$ surgery along L.

Theorem (Lickorish-Wallace) Every closed, orientable, connected 3-manifold can be obtained from a surgery on a link with coefficient ± 1.

- Seifert surface

For any oriented link $L \hookrightarrow S^3$, there exists an embedded orientable surface $\Sigma \hookrightarrow S^3$ with $\partial \Sigma = L$.

Seifert genus of K $(g(K))$:= minimal genus of Seifert surface

There is anoter quantity

Slice genus of K $(g_s(K))$:= minimal genus of $\underset{\text{orientable}}{\Sigma} \overset{\text{smooth}}{\hookrightarrow} D^4$ with $\partial \Sigma = K$.

Example: $g(T_{p,q}) = \left| \frac{(p-1)(q-1)}{2} \right|$

$g_s(T_{p,q}) = \left| \frac{(p-1)(q-1)}{2} \right|$ (Milnor conjecture, proved by Kronheimer-Mrowka)

- linking number

Let K_1, K_2 be oriented knots in S^3. We define the linking number $lk(K_1,K_2) \in \mathbb{Z}$ by

$$[K_2] = lk(K_1,K_2)\cdot[\underset{\text{meridian}}{m_1}] \in H_1(S^3 \setminus K_1) \cong \mathbb{Z}$$

Fact: $lk(K_1,K_2) = lk(K_2,K_1)$, $lk(-K_1,K_2) = -lk(K_1,K_2)$

Alternative definitions: • Σ_1 : Seifert surface of K_1

Then $lk(K_1,K_2) = \Sigma_2 \bullet K_1$

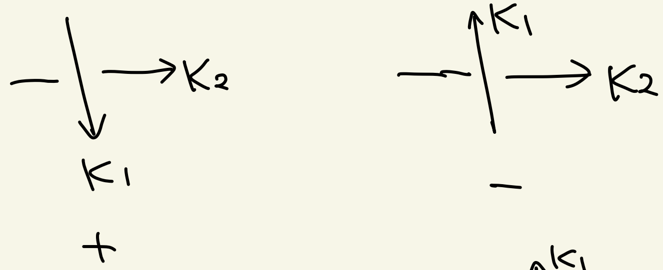
• $\Sigma_1 \hookrightarrow D^4$, $\Sigma_2 \hookrightarrow D^4$, $\partial\Sigma_1 = K_1$ $\partial\Sigma_2 = K_2$

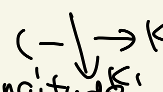
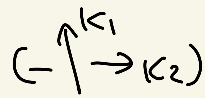
Then $lk(K_1,K_2) = K_1 \bullet K_2$

- combinatorial definition

consider all crossings where K_1 is above K_2

two types:



Then $lk(K_1,K_2) = \#($  $) - \#($  $)$

Note: $lk(K, \underset{\text{longitude}}{l})$ is always 0, this helps us to determine how many extra twist we need to find l.

framed knot

Recall that a framing of an embedded $M \hookrightarrow N$ is a trivilization of the normal bundle

Now consider $K \hookrightarrow S^3$. A framing of K is essentially a no-where vanishing normal vector along K. By pushing K along this normal vector field, we get a parallel K'.

We can record the framing by $lk(K, K') \in \mathbb{Z}$.

Similarly, a framing on $L = K_1 \sqcup \cdots \sqcup K_n$ can be recorded by $lk(K_1, K_1')$, $lk(K_2, K_2')$, $\cdots$, $lk(K_n, K_n')$.

Kirby Calculas

X : Smooth 4-mfd, connected, orientable, $\partial X = Y$

$X = X_0 | X_1 | X_2 | X_3 | X_4$ monotone handle decomposition.

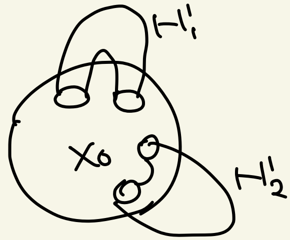
single 0-handle, so $X_0 = D^4$

Attach 1-handles $H_1^1, H_2^1, \cdots H_n^1$,

$H_i^1 = D^1 \times D^3$, attached to D^4 along $\partial D^1 \times D^3 = D^3 \sqcup D^3$

(In the orientable setting, there is essentially a unique framing, since $SO(4)$ is connected.)

So $X_0 \cup X_1 = (S^1 \times D^3) \natural \cdots \natural (S^1 \times D^3)$

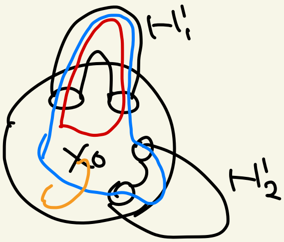


Next, we attach 2-handles

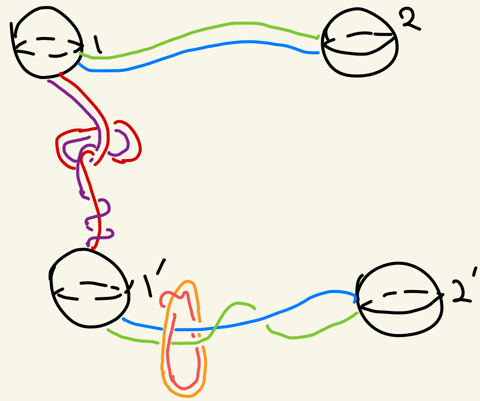
$$X_2 = H^2_1 \sqcup H^2_2 \sqcup \cdots \sqcup H^2_{n_2}$$

along framed link $L \hookrightarrow \partial(X_0 \cup X_1) = \#^{n_1}(S^1 \times S^2)$

Schematic picture:



Actual picture:



Attaching 3, 4-handles:

Note $X_3 | X_4 = \natural_{n_3}(S^1 \times D^3)$ $\quad \partial(X_3|X_4) = \#_{n_3}(S^1\times S^2)$

$\partial(X_0|X_1, X_2) \cong Y \# (\#_{n_3} S^1\times S^2)$

Assume $\partial X = \emptyset$ for now, then

$\partial(X_0|X_1|X_2) = \#_{n_3}(S^2 \times S^1)$

need to specify a diffeomorphism

$\phi : \partial(X_0|X_1|X_2) \xrightarrow{\cong} \partial(X_3|X_4)$ to form $X_0|X_1|X_2|X_3|X_4$

However, the choice of ϕ doesn't matter:

Theorem (Laudenbach, Poenaru) Every self-diffeomorphism of $\#^{\ell}(S^1\times S^2)$ extends to a self-diffeomorphism on $\natural^{\ell}(S^1\times D^3)$.

$\boxed{\text{Corollary:}}$ $(X_0|X_1|X_2) \cup_{\phi} (X_3|X_4)$ doesn't depend on ϕ

$\boxed{\text{Corollary}}$: Given any $X_0|X_1|X_2$ with $\partial(X_0|X_1|X_2) = \#^{\ell}(S^1\times S^2)$, there is essentially a unique way to complete it to a closed 4-mfd.

Generalization, given any $X_0|X_1|X_2$ and any decomposition $\partial(X_0|X_1|X_2) \cong Y \# Y'$, where $Y' \cong \#^{\ell}(S^1\times S^2)$. There is a unique way to complete to $X = X_0|X_1|X_2|X_3|X_4$ s.t. $\partial X = Y$.

So we just need to care about how 1-handles, 2-handles are attached to D^4.

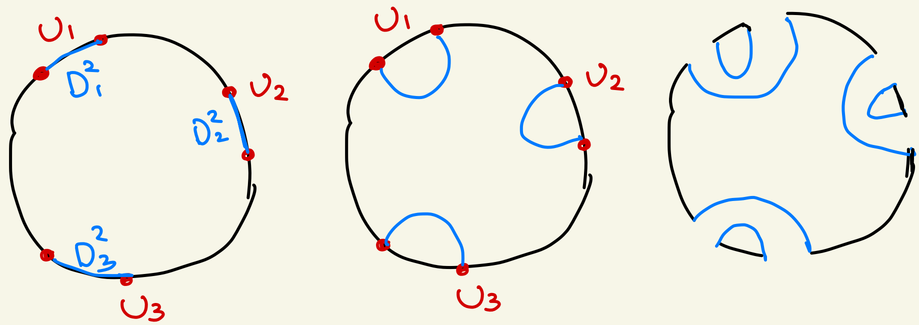
Previous notation: pair of $B^3 \rightsquigarrow$ 1-handles

arcs connecting these balls/knots $\rightsquigarrow$ 2-handles.

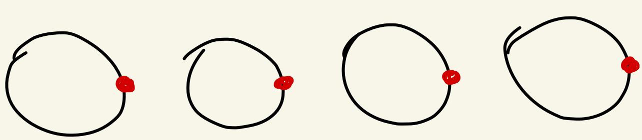
A more convenient way to represent $X_0 \cup X_1 \cup X_2$: dot notation.

Alternative way to get $X_0 \cup X_1 \cong \natural^n (S^1 \times D^3)$:

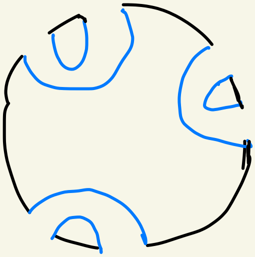
- Take n-component unlink $U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n = U$
- $\sqcup U_i$ bounds $\sqcup D_i^2$ in $S^3 = \partial D^4$
- Push interior of $\sqcup D_i^2$ to the interior of D^4
- Remove tubular neighborhoods of these disks.



So we can represent $X_0 \cup X_1$ using $\sqcup U_i$



We put a dot on each U_i to distinguish them with attaching spheres of 2-handles.



Now we attach 2-handles to $X_0 | X_1$ along a framed link $L' = L_1 \sqcup \cdots \sqcup L_i \subset \partial(X_0|X_1)$

$\partial(X_0|X_1) = (S^3 \setminus \mathring{\nu}(U)) \cup (\sqcup S^1 \times D^2)$

We can isotope L' s.t. $L' \cap (\sqcup S^1 \times D^2) = \phi$.

Since $S^3 \setminus \mathring{\nu}(U) \subset S^3$, L' is a framed link in S^3.

Recall: we can use an integer $a_i = lk(L_i, L_i')$ to record the framing of L_i.

Definition: A Kirby diagram D is a link $L \subset S^3$, with each component decorated with a dot or an integer, such that the dotted components form an unlink.

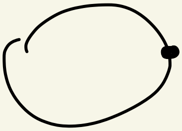
A Kirby diagram represents a 4-mfd $X = X_0 | X_1 | X_2$ with boundary. Given any decomposition $\partial X \cong Y \# Y'$ s.t. $Y' \cong \#^l (S^1 \times S^2)$, there is a unique way to complete it into $\tilde{X} = X_0 | X_1 | X_2 | X_3 | X_4$ with $\partial \tilde{X} = Y$.

In particular, if $\partial X \cong \#^l (S^1 \times S^2)$, then $\exists !$ way to get a closed $\tilde{X}$.

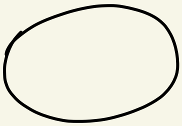
∂X: replace $\bullet$s with 0s and do surgery along L.

Example:

- $L = \emptyset$ Then $X = D^4$ $\tilde{X} = S^4$

- 

  $X = S^1 \times D^3$ $\tilde{X} = S^1 \times S^3$

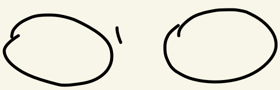
- 

  n

  $X =$ disk bundle over S^2 with euler class n.

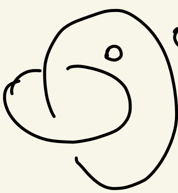
  $\partial X = L(n,1)$

when $n=1$ $\tilde{X} = \mathbb{CP}^2$ when $n=-1$ $\tilde{X} = \overline{\mathbb{CP}}^2$

- 

  1 -1

  $\tilde{X} = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

- 

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  $X = (S^2 \times S^2) \setminus \mathring{D}^4$ $\tilde{X} = S^2 \times S^2$