Panorama of Dynamics and Geometry of Moduli Spaces and Applications

Lecture 4. Right-angled billiards and quadratic differentials

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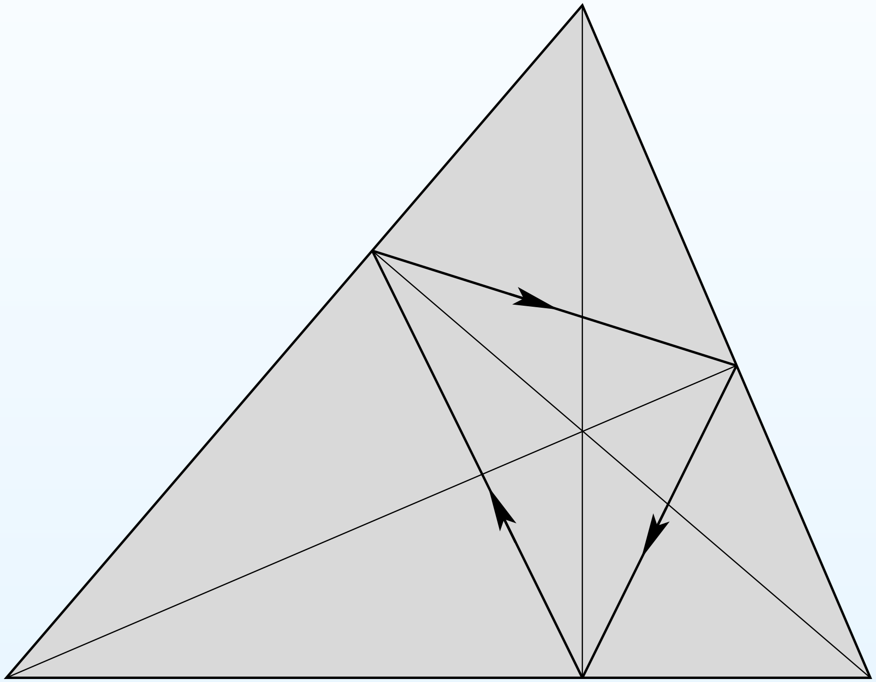
I. Billiards in right-angled polygons

Closed billiard trajectories (reminder)

It is easy to find a periodic billiard trajectory in an acute triangular billiard:



Exercise. Show that the broken line joining the base points of the heights in an acute triangle is a closed billiard trajectory (called *Fagnano trajectory*). Show that it is an inscribed triangle of the minimal possible perimeter.

Challenge

It is difficult to believe, but for an obtuse triangle the problem is open:

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But even if the answer is affirmative, the natural question "And how many?.." is completely and desperately open already for acute triangles.

Open Problem. *Estimate the number $N(\Pi, L)$ of periodic trajectories of length at most L in a polygon Π as $L \to +\infty$.*

Billiards in rational polygons.

Life is better for *rational* polygons with all angles rational multiples of π.

Theorem (H. Masur, 1986). *For any* ***rational*** *polygon Π there exist constants $c(\Pi), C(\Pi)$ such that for L large enough the number $N(\Pi, L)$ of closed trajectories of length at most L in Π satisfies*

$$c(\Pi) \cdot L^2 \le N(\Pi, L) \le C(\Pi) \cdot L^2 \,.$$

For several exceptional rational polygons (namely, for regular polygons; for certain very special triangles; for squares with a vertical barrier; for L-shaped polygons (possibly with a barrier) with ratios of the horizontal and vertical sides in the same quadratic field; and for the finite covers of the above ones) an exact quadratic asymptotics is proved:

$$N(\Pi, L) \sim \mathrm{const}(\Pi) \cdot L^2 \quad \text{as} \quad L \to \infty \,.$$

These polygons correspond to *Veech surfaces*: the translation surfaces for which the $\mathrm{GL}(2, \mathbb{R})$-orbit is closed in the moduli space. The proofs of exact asymptotics and the computation of the values of the constants requires a heavy machinery performed in the papers of Veech, Eskin – Markloff – Morris, Eskin – Masur – Schmoll, Bouw – Möller, Hooper, Bainbridge, and others.

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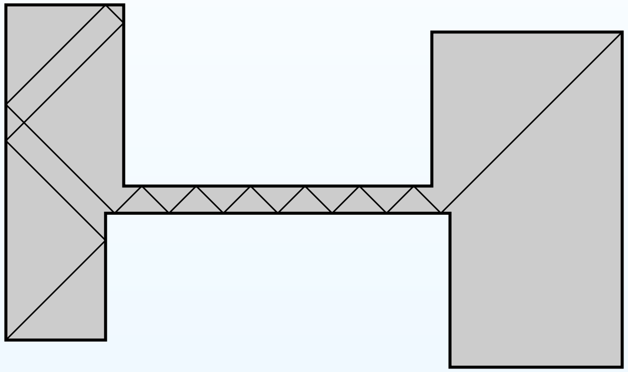
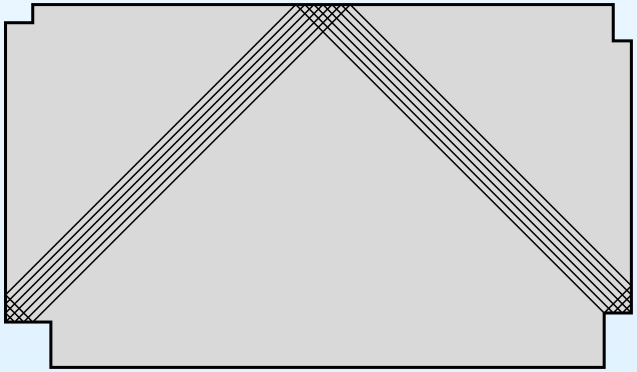
Following Moon Duchin
we shall play on
right-angled billiard tables.

Closed trajectories and generalized diagonals

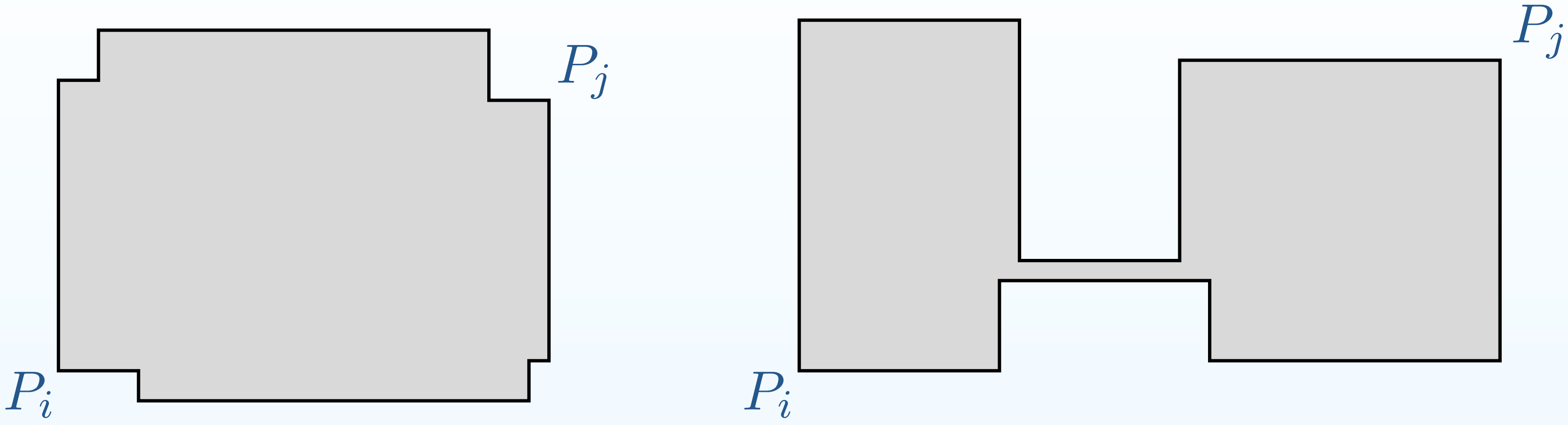
We count the asymptotic number of trajectories of bounded length joining a given pair of corners (“*generalized diagonals*”) as the bound L tends to infinity.



We also want to count the number of periodic trajectories of length at most L, or rather the number of *bands* of periodic trajectories. We might also count the bands with the weight representing the “thickness” of the band.



Number of generalized diagonals

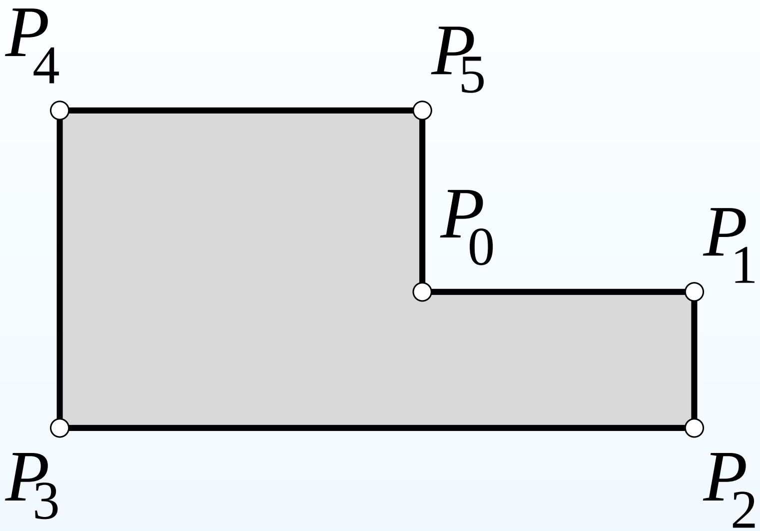


Example of a Theorem. *For almost any right-angled polygon Π in any family $\mathcal{B}(k_1, \dots, k_n)$ of right-angled polygons with angles $k_1\frac{\pi}{2}, \dots k_n\frac{\pi}{2}$, the number $N_{i,j}(\Pi, L)$ of trajectories of length bounded by L joining any two fixed corners with true right angles $\frac{\pi}{2}$ is asymptotically the same as for a rectangle:*

$$N_{i,j}(\Pi, L) \sim \frac{1}{2\pi} \cdot \frac{(\text{bound } L \text{ for the length})^2}{\text{area of the table}} \quad \text{as} \quad L \to \infty$$

and does not depend on the shape of the polygon Π.

Naive intuition does not help...



However, say, for almost any L-shaped polygon Π the number $N_{0,j}(\Pi, L)$ of trajectories joining the corner P_0 with the angle $3\frac{\pi}{2}$ to some other corner P_j has asymptotics

$$N_{0,j}(\Pi, L) \sim \frac{2}{\pi} \cdot \frac{(\text{bound } L \text{ for the length})^2}{\text{area of the table}} \quad \text{as} \quad L \to \infty\,,$$

which is 4 times (and not 3) times bigger than the number of trajectories joining a fixed pair of right corners...

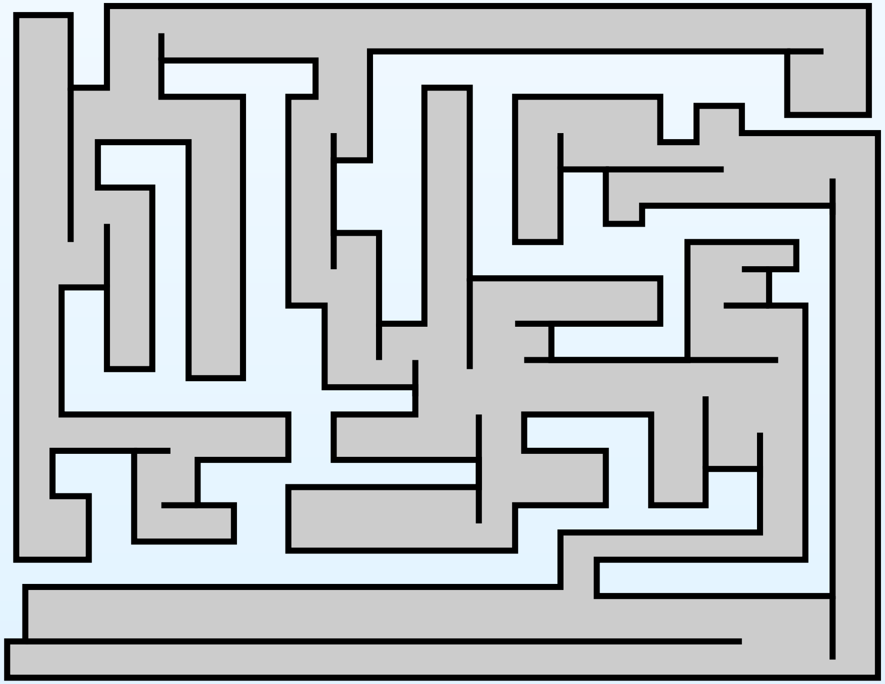
Billiard in a right-angled polygon: general answer

For each family $\mathcal{B}(k_1, \ldots, k_n)$ of right-angled polygons we find all topological types of "admissible" generalized diagonals (closed trajectories). We show that a billiard table Π outside of a zero measure set in $\mathcal{B}(k_1, \ldots, k_n)$ does not contain a single "non-admissible" generalized diagonal (closed trajectory).

Billiard in a right-angled polygon: general answer

Theorem (J. Athreya, A. Eskin, A. Zorich, 2016). *The coefficient in the exact quadratic asymptotics for the number of generalized diagonals (number of closed trajectories) of bounded length L is shared by almost all tables of area one in each billiard family. The coefficient c in the asymptotic number cL^2 of generalized diagonals joining a fixed pair of corners depends only on the angles of the corners. For corners of angles $\frac{\pi}{2}, 3\frac{\pi}{2}, 4\frac{\pi}{2}$ these coefficients are given by the table below:*

$angle$	$\frac{4\pi}{2}$	$\frac{3\pi}{2}$	$\frac{\pi}{2}$
$\frac{4\pi}{2}$	$\frac{9\pi}{10}$	$\frac{45\pi}{64}$	$\frac{9\pi}{32}$
$\frac{3\pi}{2}$	$\frac{45\pi}{64}$	$\frac{16}{3\pi}$	$\frac{2}{\pi}$
$\frac{\pi}{2}$	$\frac{9\pi}{32}$	$\frac{2}{\pi}$	$\frac{1}{2\pi}$



Strategy

- Passing from a right-angled billiard to a flat sphere, obtained by identifying the boundaries of two copies of a billiard, we express count of closed billiard trajectories in terms of count of saddle connections on the associated flat sphere.
- Count of saddle connections in each stratum of “flat spheres” is governed by associated Siegel–Veech constants.
- Siegel–Veech constants are expressed in terms of Masur–Veech volumes of the ambient stratum and of the “principal boundary” strata.
- The *area* Siegel–Veech constant $c_{area}(\mathcal{Q})$ is involved in the formula for the sum of the Lyapunov exponents of the stratum $\mathcal{Q}$. In genus zero the sum vanishes, thus providing an identity for the area Siegel–Veech constant. This identity recursively defines all volumes.
- Guess the correct general expression for the Masur–Veech volume of any stratum and verify that it satisfies the initial data and the general combinatorial identity.

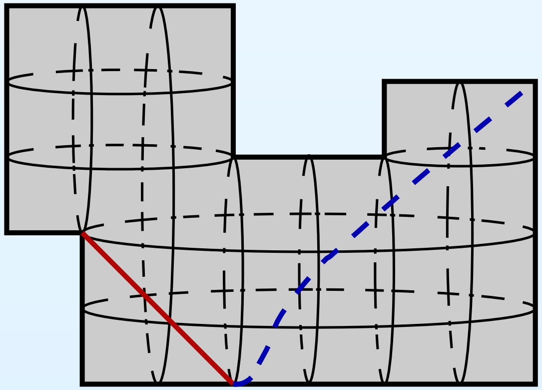
Strategy (continued)

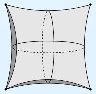
- Using the resulting explicit expression for Masur–Veech volumes in genus zero, we evaluate *all* Siegel–Veech constants (and not only c_{area}) for all configurations of saddle connections in any stratum in genus zero.
- Consider the subset $\mathcal{B}_1 \subset \mathcal{Q}_1$ of flat spheres of unit area obtained by gluing two copies of a billiard table. The tangent space to the ambient stratum $\mathcal{Q}_1$ decomposes into a direct sum of expanding, neutral and contracting subspaces of the Teichmüller flow. Prove that the projection of the tangent space to $\mathcal{B}_1$ to the expanding subspace is onto at any point.
- Using the previous observation we conclude that the closure of the union of $\mathrm{SL}(2,\mathbb{R})$-orbits passing through $\mathcal{B}$ is the entire ambient stratum $\mathcal{Q}(\mathcal{B})$.
- The latter observation already implies certain (weak) count of billiard trajectories for almost all billiards in $\mathcal{B}$ through count of saddle connections for almost all flat spheres in the associated stratum $\mathcal{Q}(\mathcal{B})$.
- An ergodic theorem due to J. Chaika (in the spirit of an analogous theorem due to A. Eskin–G. Margulis–S. Mozes in homogeneous dynamics) allows to obtain exact quadratic asymptotics for the count of billiard trajectories for almost any billiard table in $\mathcal{B}$.

II. Pillowcase covers and Masur–Veech volumes of moduli spaces

Billiards in right-angled polygons versus quadratic differentials on $\mathbb{C}\mathrm{P}^1$

The topological sphere obtained by gluing two copies of the billiard table by the boundary is naturally endowed with a flat metric. This metric has conical singularities at the points coming from vertices of the polygon, otherwise it is nonsingular. In the special case of a *"rectangular polygon"* the flat metric has holonomy in $\mathbb{Z}/(2\mathbb{Z})$, which corresponds to a meromorphic quadratic differential with at most simple poles on $\mathbb{C}\mathrm{P}^1$. As we have seen, billiard trajectories are lifted to geodesics on this flat sphere. Thus, the count closed billiard trajectories is related to count of saddle connections on the associated flat sphere. In other words, we have to count the related Siegel–Veech constants, and hence, the Masur–Veech volumes of the strata in the moduli space of meromorphic quadratic differentials in genus zero.





Integer points in the moduli space of Abelian differentials

When a flat metric on a surface S has trivial holonomy, it corresponds to a holomorphic 1-form. The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of holomorphic 1-forms with zeroes of multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$. The points of this lattice are represented by *square-tiled surfaces*.

Indeed, if a flat surface S is defined by an holomorphic 1-form ω such that $[\omega]\in H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover over the torus $\mathbb{T}=\mathbb{R}^2/(\mathbb{Z}\oplus i\mathbb{Z})$ defined by the map

$$P\mapsto \int_{P_1}^{P}\omega \mod \mathbb{Z}\oplus i\mathbb{Z}\,.$$

Integer points in the strata $\mathcal{Q}(d_1,\dots,d_n)$ of quadratic differentials are represented by “pillowcase covers” over $\mathbb{CP}^1$ branched at four points.

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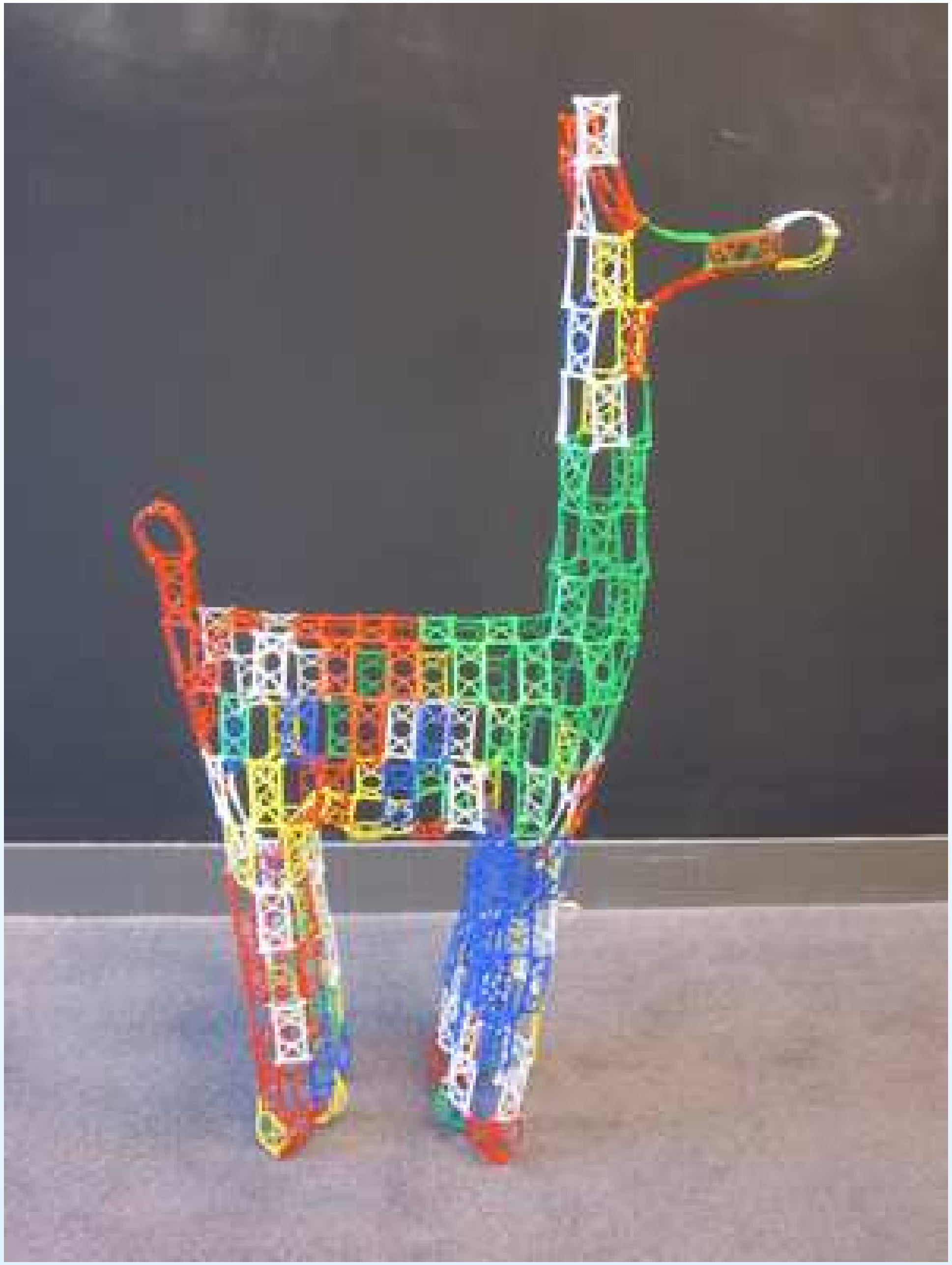
A square-tiled surface of genus zero



Calculation the volume of a “sphere” through counting integer points inside a “ball” of large radius.

The volume of the “unit sphere” $\mathcal{H}_1(m_1,\dots,m_n)$ of flat surfaces of area 1, is a multiple of the volume of the “unit ball” $\mathcal{H}_{\le 1}(m_1,\dots,m_n)$ of flat surfaces of area at most 1 by a dimensional factor:

$$\operatorname{Vol}(\mathcal{H}_1(m_1,\dots,m_n)) = \dim_{\mathbb{R}} \mathcal{H}(m_1,\dots,m_n)\cdot \mu((\mathcal{H}_{\le 1}(m_1,\dots,m_n))\,.$$

The volume of the “unit ball” is equal to the coefficient in the asymptotics of the number of lattice points captured inside the unit ball for the lattice with a grid $1/N$ when $N\to\infty$. The latter number is the same as the number of *integer* points (points represented by square-tiled surfaces) inside a “ball of radius N”.

Thus, to compute the volume of a stratum of flat surfaces, it is sufficient to find the asymptotics for the number $\mathcal{ST}_N(m_1,\dots,m_n)$ of square-tiled surfaces tiled with at most N squares:

$$\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n) = 2\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)\cdot \lim_{N\to+\infty}\frac{\mathcal{ST}_N(m_1,\dots,m_n)}{N^{\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)}}\,.$$

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Canonical double cover defined by a quadratic differential

- A meromorphic quadratic differential with at most simple poles on a Riemann surface S defines a canonical (ramified) double cover $p : \hat{S} \to S$ such that the pullback p^*q to $\hat{S}$ is already a global square of a holomorphic 1-form, $p^*q = (\hat{\omega})^2$.
- The ramification points of the covering are located at the zeroes of odd degrees and at poles of q.
- The topology of the covering is the same for all quadratic differentials in any stratum $\mathcal{Q}(d_1, \ldots, d_m)$.
- The induced flat metric on the double cover already has *trivial* linear holonomy.

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Coordinates in the strata of quadratic differentials

- The natural involution $\tau : \hat{S} \to \hat{S}$ of the canonical double cover induces an involution of relative cohomology of the covering Riemann surface

$$\tau^* : H^1(\hat{S}, \{\text{zeroes}\}; \mathbb{C}) \to H^1(\hat{S}, \{\text{zeroes}\}; \mathbb{C})$$

- The vector space $H^1(\hat{S}, \{\text{zeroes}\}; \mathbb{C})$ decomposes into direct sum of invariant and anti-invariant parts with respect to this involution:

$$H^1(\hat{S}, \{\text{zeroes}\}; \mathbb{C}) = H^1_-(\hat{S}, \{\text{zeroes}\}; \mathbb{C}) \oplus H^1_+(\hat{S}, \{\text{zeroes}\}; \mathbb{C})$$

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Kontsevich conjecture

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

Theorem (J. Athreya, A. Eskin, A. Zorich, 2016). *The volume of any stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ is equal to*

$$\operatorname{Vol} \mathcal{Q}_1(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

Corollary. *The number of pillowcase covers of degree at most N with ramification pattern corresponding to $\mathcal{Q}(d_1, \dots, d_k)$ has the following leading term in the asymptotics as $N \to \infty$*

$$\textit{Number of pillowcase covers} \sim \frac{\pi}{k-2} \prod_{i=1}^{k} v(d_i) \cdot N^{k-2}\,.$$

Masur–Veech volumes of strata of Abeliand differentials: a brief historical retrospective

● Around 1998. Masur–Veech volumes of several low-dimensional strata of Abelian differentials were evaluated by M. Kontsevich and A. Zorich through straightforward count of square-tiled surfaces.

● Around 2001. A. Eskin and A. Okounkov found a much more efficient approach based on quasimodularity of the relevant generating function. A. Eskin wrote a computer code giving volumes of all strata in genera at most 10 and of some strata in genera up to 200.

● 2020. D. Chen, M. Möller, A. Sauvaget and D. Zagier obtained very important advances based on recent BCGGM smooth compactification of the moduli space of Abelian differentials. They developed intersection theory of relevant moduli spaces.

● 2018–2020. D. Chen–M. Möller–A. Sauvaget–D. Zagier and independently A. Aggarwal obtained spectacular results on large genus asymptotics of Masur–Veech volumes uniform for all strata stratum of Abelian differentials proving a conjecture by A. Eskin and of A. Zorich based on their numerical experiments from 2003.

Masur–Veech volumes of strata of quadratic differentials: a brief historical retrospective

The knowledge of Masur–Veech volumes $\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)$ of strata of *quadratic* differentials is still limited.

● Around 1998-2000. Masur–Veech volumes of several low-dimensional strata of quadratic differentials were evaluated by A. Zorich through straightforward count of square-tiled surfaces.

● 2001. A. Eskin and A. Okounkov found a much more efficient approach based on quasimodularity of the generating function counting *pillowcase covers*. However, the resulting expressions contain huge tables of characters of the symmetric group, which makes the computation inefficient. The algorithm is more involved than for Abelian differentials.

● 2016. The algorithm of A. Eskin and A. Okounkov was implemented by E. Goujard. She wrote a code and computed volumes of all strata up to dimension 12.

Masur–Veech volumes of strata of quadratic differentials: a brief historical retrospective

- 2016. J. Athreya–A. Eskin–A. Zorich obtained a close expression (conjectured by M. Kontsevich) for the Masur–Veech volume of any stratum in genus zero through the formula of A. Eskin–M. Kontsevich–A. Zorich for the sum of Lyapunov exponents combined with some combinatorial considerations.
- 2019. V. Delecroix–E. Goujard–P. Zograf–A. Zorich computed volumes of the principal strata (the ones containing only simple zeroes and poles) in terms of Witten–Kontsevich correlators.
- 2019. D. Chen–M. Möller–A. Sauvaget expressed volumes of the principal strata in terms of certain Hodge integrals.
- 2019. J. Andersen–G. Borot–S. Charbonnier–V. Delecroix–A. Giacchetto–D. Lewanski–C. Wheeler used the DGZZ-formula to compute volumes through topological recursion.
- 2020. M. Kazarian and independently Di Yang–D. Zagier–Y. Zhang developed efficient recursion for the Hodge integrals involved in the CMS-formula.
- 2021. A. Aggarwal derived the large genus asymptotics for the volumes of principal strata conjectured by V. Delecroix–E. Goujard–P. Zograf–A. Zorich.

III. Siegel–Veech constants and Lyapunov exponents

Area Siegel—Veech constant

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills a *maximal cylinder* having conical points on each of the boundary components. We have seen that sometimes we might get a *configuration* $\mathcal{C}$ of several cylinders, with homologous waste curves (sharing the same length and direction).

Denote by $N_{area}(S, L)$ the sum of areas of all cylinders spanned by geodesics of length at most L on a translation surface S of area 1.

Theorem [W. Veech; Ya. Vorobets] *For every* $\mathrm{SL}(2,\mathbb{R})$*-invariant finite ergodic measure the following ratio is constant (i.e. does not depend on the value of a positive parameter* L*):*

$$\frac{1}{\pi L^2}\int N_{area}(S,L)\,d\nu_1 = c_{area}(d\nu_1)$$

The constant c_{area} is called the *area Siegel–Veech constant*.

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Evaluation of area Siegel–Veech constant

Theorem (A. Eskin, H. Masur, A. Zorich, 2003.)

$$c_{area} = \frac{1}{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 1} \cdot \sum_q q \cdot \sum_{\substack{Configurations\ \mathcal{C} \\ containing\ q\ cylinders}} c(\mathcal{C})$$

where

$$c(\mathcal{C}) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \frac{\mathrm{Vol}(\text{“}\varepsilon\text{-neighborhood of the cusp } \mathcal{C}\text{ ”})}{\mathrm{Vol}\,\mathcal{H}_1(\beta)} =$$

$$= \big(\text{explicit combinatorial factor}\big) \cdot \frac{\prod_{j=1}^{k} \mathrm{Vol}\,\mathcal{H}_1(\beta'_k)}{\mathrm{Vol}\,\mathcal{H}_1^{comp}(\beta)}$$

Combinatorics is complicated, but explicitly described; volumes are computed by A. Eskin and A. Okounkov, so these constants can be computed exactly. They have the form $\frac{p}{q} \cdot \frac{1}{\pi^2}$. We have tables of rational numbers $\frac{p}{q}$ up to high genera, and good approximation formula for $g \gg 1$.

Sum of the Lyapunov exponents

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014). *Let $d\nu_1$ be an $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measure supported on an invariant suborbifold $\mathcal{M}_1$ in some stratum $\mathcal{H}_1(m_1,\dots,m_n)$. The Lyapunov exponents of the Hodge bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to $\mathcal{M}_1$ satisfy the following relation:*

$$\lambda_1+\lambda_2+\dots+\lambda_g \;=\; \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{m_i(m_i+2)}{m_i+1}+\frac{\pi^2}{3}\cdot c_{area}(d\nu_1)$$

where $c_{\text{area}}(d\nu_1)$ is the area Siegel–Veech constant. The top exponent λ_1 is equal to one, $\lambda_1=1$.

- One can write analogous formulae for quadratic differentials.
- One can write analogous formulae for the sum of exponents of any Hodge $\ast$-invariant covariantly constant subbundle of the Hodge bundle.

In those cases, when the Hodge bundle has maximal possible splitting, one can compute all individual exponents.

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Lyapunov exponents and alternative expression for the Siegel–Veech constant in genus zero

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014). *Let $\mathcal{M}_1$ be an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold in some stratum $\mathcal{Q}_1(d_1,\dots,d_n)$ of meromorphic quadratic differentials with at most simple poles. The Lyapunov exponents of the Hodge bundle H^1_+ along the Teichmüller flow restricted to $\mathcal{M}_1$ satisfy the following relation:*

$$\lambda_1^+ + \lambda_2^+ + \dots + \lambda_g^+ \;=\; \frac{1}{24}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+4)}{d_i+2} + \frac{\pi^2}{3}\cdot c_{area}(\mathcal{M}_1)\,,$$

where $c_{area}(\mathcal{M}_1)$ is the area Siegel–Veech constant of $\mathcal{M}_1$.

Corollary. *For any $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold $\mathcal{M}_1$ in any stratum $\mathcal{Q}_1(d_1,\dots,d_n)$ of quadratic differentials in genus zero:*

$$c_{area}(\mathcal{M}_1) = c_{area}\big(\mathcal{Q}_1(d_1,\dots,d_n)\big) = -\frac{1}{8\pi^2}\sum_{j=1}^{n}\frac{d_j(d_j+4)}{d_j+2}\,.$$

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Combinatorial identities

Combining two expressions for $c_{area}(d_1,\dots,d_n)$ we get series of combinatorial identities recursively defining volumes of all strata:

$$\big(\text{explicit combinatorial factor}\big)\cdot\frac{\prod \operatorname{Vol}\big(\text{adjacent simpler strata}\big)}{\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)}=$$
$$=-\frac{1}{8\pi^2}\sum_{j=1}^{n}\frac{d_j(d_j+4)}{d_j+2}.$$

It remains to verify that the guessed answer satisfy these identities. The verification is reduced to verifying some cute combinatorial identities for multinomial coefficients; it is based on manipulations with appropriate generating functions.

Having proved the formula for the volumes of strata in genus 0, we can plug the values into the formulae for the Siegel–Veech constants, and obtain their numerical values.

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Combinatorial identity to verify

It remains to verify that for any collection $d_1, \ldots, d_m$ of positive integers one has:

$$\left(6+\sum_{i=1}^{m}\frac{d_i(d_i+1)}{d_i+2}\right)\cdot\left(1+\sum_{i=1}^{m}(d_i+1)\right)-\left(4+\sum_{i=1}^{m}d_i\right)\left(3+\sum_{i=1}^{m}d_i\right)$$

$$\stackrel{?}{=} 2\cdot\frac{\left(4+\sum_{i=1}^{m}d_i\right)!}{\left(\sum_{i=1}^{m}(d_i+1)\right)!}\cdot\sum_{1\le i<j\le m}(d_i+1)(d_j+1)$$

$$\times \sum_{\substack{\text{partitions of } \{1,\ldots,m\} \text{ into}\\ \{r_1,\ldots,r_{m_1}\}\sqcup\{s_1,\ldots,s_{m_2}\}\\ \text{such that } i \text{ is in the first subset}\\ \text{and } j \text{ is in the second subset}}} \frac{\left(-1+\sum_{i=1}^{m_1}(d_{r_i}+1)\right)!\cdot\left(-1+\sum_{j=1}^{m_2}(d_{s_j}+1)\right)!}{\left(2+\sum_{i=1}^{m_1}d_{r_i}\right)!\cdot\left(2+\sum_{j=1}^{m_2}d_{s_j}\right)!}.$$

IV. Rigid collections of saddle connections in genus zero.

$\hat{\text{H}}$omologous saddle connections

Recall that a quadratic differential q on a Riemann surface S defines a canonical (ramified) double cover $p : \hat{S} \to S$ such that $p^*q = \omega^2$ is a square of a holomorphic 1-form ω on $\hat{S}$.

Given an oriented saddle connection γ on S let γ', γ'' be its lifts to the double cover. If $[\gamma'] = -[\gamma'']$ as cycles in

$$H_1(\hat{S}, \{\text{preimages of singularities}\}; \mathbb{Z})$$

we let $[\hat{\gamma}] := [\gamma']$, otherwise we define $[\hat{\gamma}]$ as $[\hat{\gamma}] := [\gamma'] - [\gamma'']$.

Definition The saddle connections γ_1, γ_2 on a flat surface S defined by a quadratic differential q are **$\hat{\text{h}}$omologous** if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in

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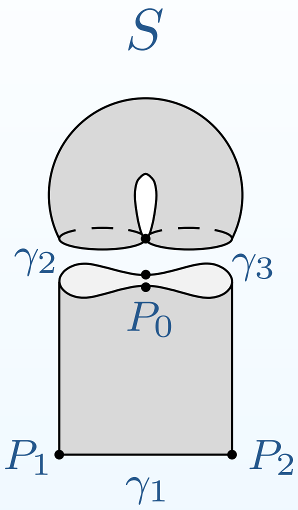
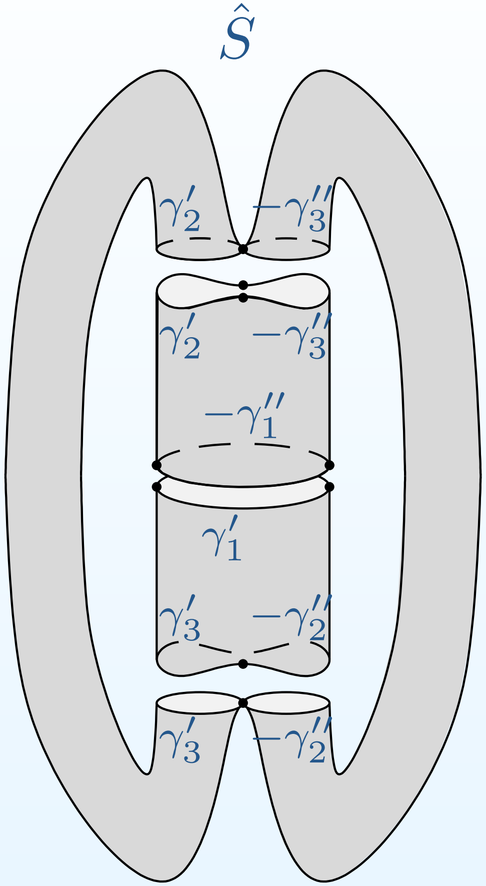
under an appropriate choice of orientations of γ_1, γ_2.

Saddle connection ĥomologous to separatrix loop





Saddle connections $\gamma_1, \gamma_2, \gamma_3$ on the surface S (left picture) are ĥomologous, though γ_1 is a segment joining distinct points P_1, P_2 and γ_2 and γ_3 are closed loops.

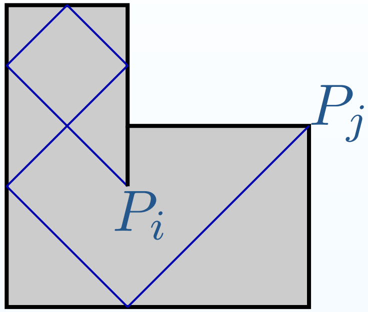
Rigid collections of saddle connections

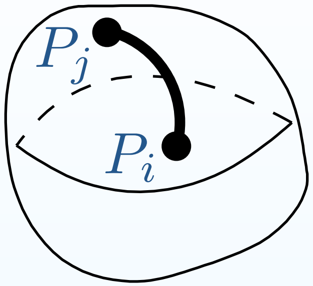
It follows from the definition that $\hat{h}$omologous saddle connections are parallel on S and that their lengths either coincide or differ by a factor of two.

Theorem (H. Masur, A. Zorich, 2008) *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are $\hat{h}$omologous.*

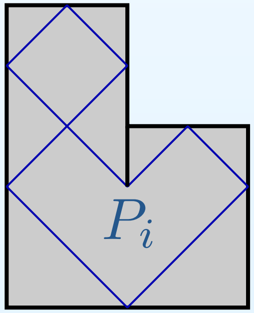
Theorem (H. Masur, A. Zorich, 2008) *Two saddle connections γ_1, γ_2 on S are $\hat{h}$omologous if and only if they have no interior intersections and one of the connected components of the complement $S \setminus (\gamma_1 \cup \gamma_2)$ has trivial linear holonomy. Moreover, if such a component exists, it is unique.*

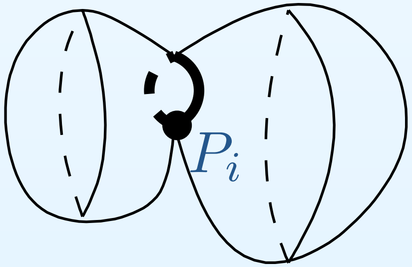
Saddle connections in genus zero (after C. Boissy)





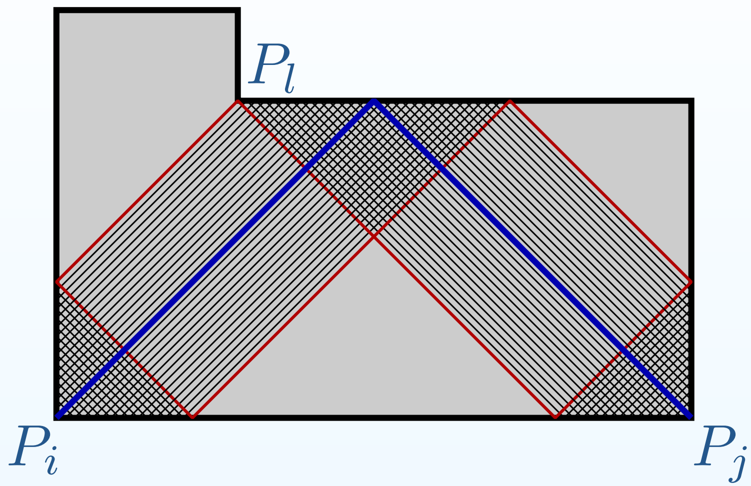
Type I. On the left: a generalized diagonal joining two distinct corners of the billiard, where at least one of the two corners has inner angle at least $\frac{3\pi}{2}$. It does not bound a band of closed trajectories. On the right: a saddle connection on $\mathbb{C}\mathrm{P}^1$ joining a zero to a distinct zero (or to a pole).

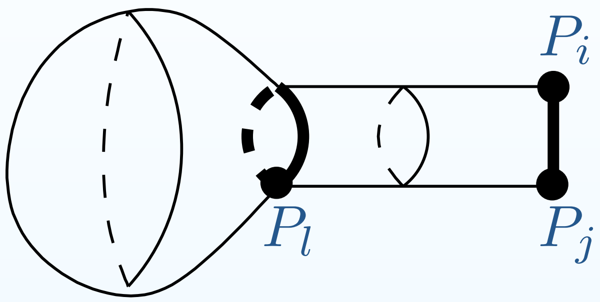




Type II. On the left: a generalized diagonal returning to the same corner. For this type, it does not bound closed trajectories. On the right: the corresponding saddle connection joining a zero (of order at least 2) to itself.

Closed geodesics in genus zero (after C. Boissy)



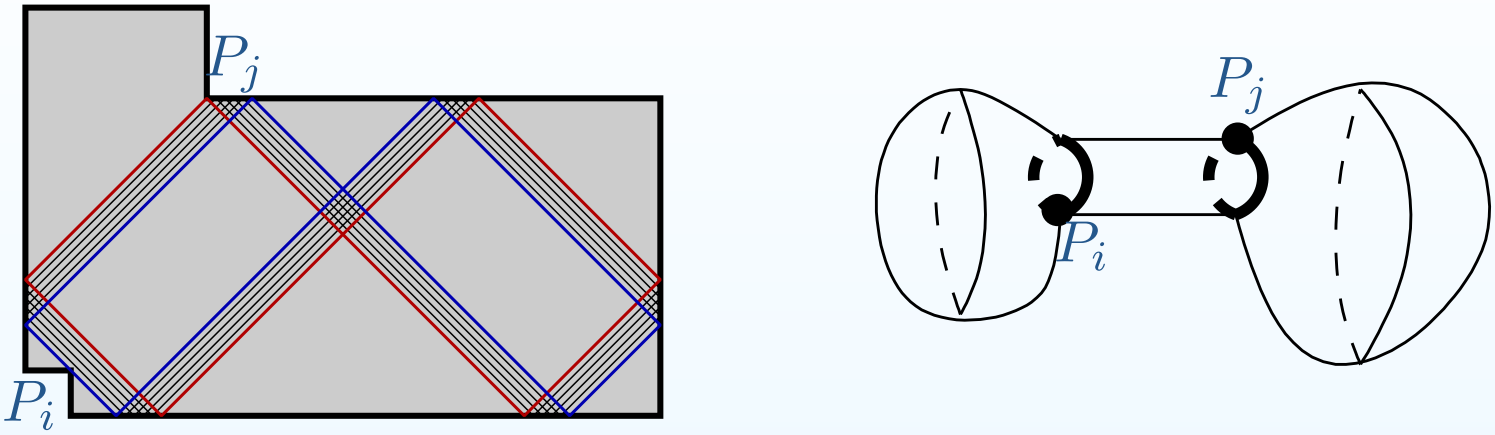


Type III.

On the left: a band of closed trajectories bounded by two generalized diagonals. One of the generalized diagonals joins two distinct corners with angles $\frac{\pi}{2}$; the other diagonal returns to the same corner.

On the right: the corresponding configuration with a cylinder bounded on one side by a saddle connection joining two simple poles, and bounded on the other side by a saddle connection joining a zero to itself.

Closed geodesics in genus zero (after C. Boissy)



Type IV.

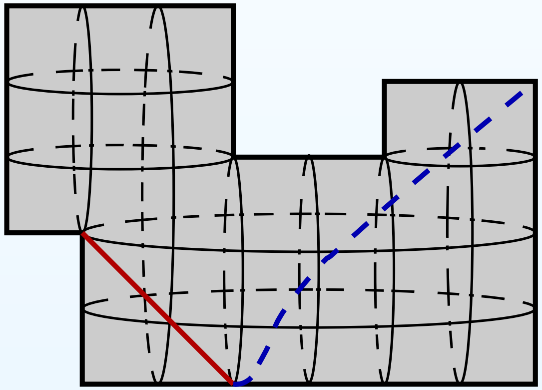
On the left: a band of periodic trajectories, such that each of the two bounding generalized diagonals returns to the same corner.

On the right: a "dumbbell" composed of two flat spheres joined by a cylinder. Each boundary component of the cylinder is a saddle connection joining a zero to itself.

V. Back to billiards in right-angled polygons

Transversality

We have solved the counting problem for almost all flat spheres in any family $\mathcal{Q}_1(d_1, \ldots, d_k)$ in genus zero. The trouble is that the subspace of those flat spheres which correspond to right-angled billiards has large codimension: it is a bit larger than half dimension of the in the ambient family.



Proposition. *Consider the canonical local embedding*

$$\mathcal{B}(k_1, \ldots, k_n) \subset \mathcal{Q}(k_1 - 2, \ldots, k_n - 2).$$

For almost all directional billiards in $\mathcal{B}(k_1, \ldots, k_n)$ the projection of the tangent space $T_*\mathcal{B}(k_1, \ldots, k_n)$ to the unstable subspace of the Teichmüller geodesic flow is a surjective map.

Ergodic Theorem by J. Chaika

J. Chaika used a variation of an argument of G. Margulis to prove equidistribution in the ambient stratum $\mathcal{Q}_1$ of large circles centered at almost all points of the billiard submanifold $\mathcal{B}_1$. This approach is similar in spirit to the approach of A. Eskin–G. Margulis–S. Mozes.

Theorem (J. Chaika). *Let f be any bounded 1-Lipschitz function with a zero mean on a stratum $\mathcal{Q}_1(k_1-2,\dots,k_n-2)$ of quadratic differentials in genus zero. Then for $\mu_{\mathcal{B}}$-almost every right angled billiard Π in $\mathcal{B}_1(k_1,\dots,k_n)$ one has:*

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(g_T r_\theta q_\Pi)\,d\theta=\frac{1}{\mu_1(\mathcal{Q}_1)}\int_{\mathcal{Q}_1} f\,d\mu_1\,.$$

The proof uses, in particular, the result of Athreya on quantitative recurrence of the Teichmüller geodesic flow, and the result of Avila–Resende on exponential mixing of the Teichmüller geodesic flow on $\mathcal{Q}_1$.

Applying these results, one proves the exact quadratic asymptotics for $\mu_{\mathcal{B}}$-almost all quadratic differentials $q(\Pi)$ in the billiard submanifold.

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Ergodic Theorem by J. Chaika

J. Chaika used a variation of an argument of G. Margulis to prove equidistribution in the ambient stratum $\mathcal{Q}_1$ of large circles centered at almost all points of the billiard submanifold $\mathcal{B}_1$. This approach is similar in spirit to the approach of A. Eskin–G. Margulis–S. Mozes.

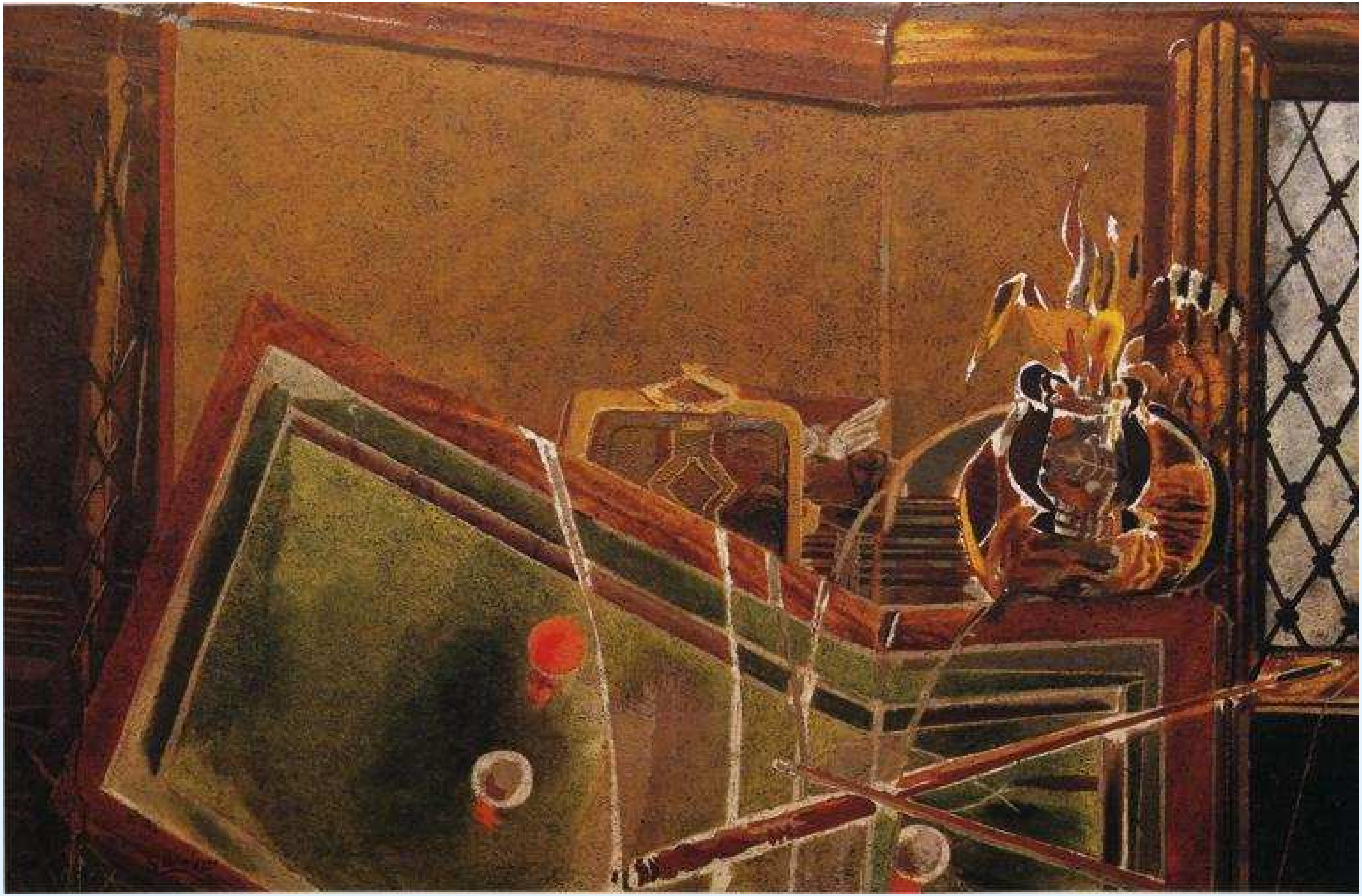
Theorem (J. Chaika). *Let f be any bounded 1-Lipschitz function with a zero mean on a stratum $\mathcal{Q}_1(k_1-2,\dots,k_n-2)$ of quadratic differentials in genus zero. Then for $\mu_{\mathcal{B}}$-almost every right angled billiard Π in $\mathcal{B}_1(k_1,\dots,k_n)$ one has:*

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(g_T r_\theta q_\Pi)\,d\theta = \frac{1}{\mu_1(\mathcal{Q}_1)}\int_{\mathcal{Q}_1} f\,d\mu_1\,.$$

The proof uses, in particular, the result of Athreya on quantitative recurrence of the Teichmüller geodesic flow, and the result of Avila–Resende on exponential mixing of the Teichmüller geodesic flow on $\mathcal{Q}_1$.

Applying these results, one proves the exact quadratic asymptotics for $\mu_{\mathcal{B}}$-almost all quadratic differentials $q(\Pi)$ in the billiard submanifold.

Back to billiards in polygons



Georges Braque, Le Billard (1944). Centre Pompidou, Paris