Panorama of Dynamics and Geometry of Moduli Spaces and Applications

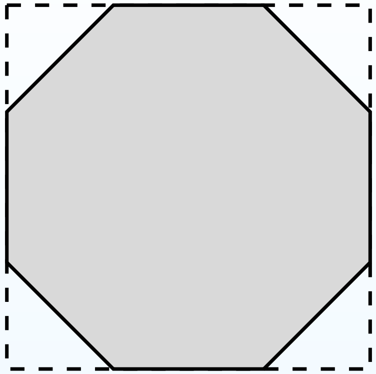
Lecture 5. Square-tiled surfaces

Anton Zorich
University Paris Cité

YMSC, Tsinghua University, April 21, 2022

Masur–Veech volumes of the moduli spaces of Abelian differentials. Square-tiled surfaces

Very flat surface of genus two



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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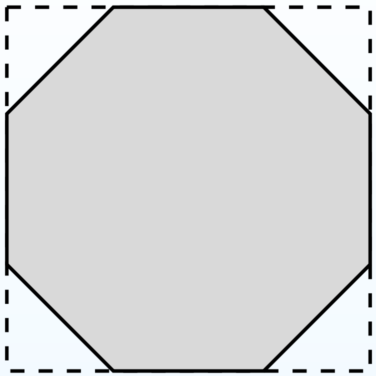

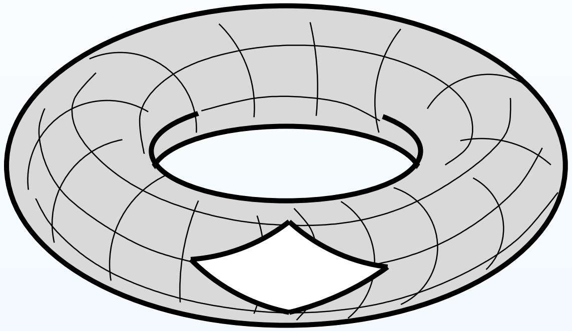

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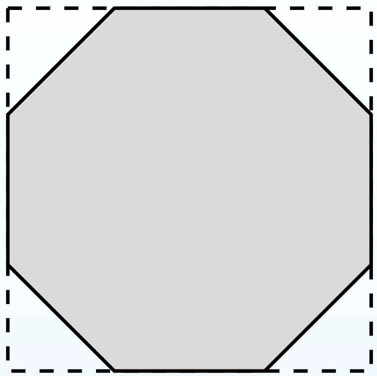

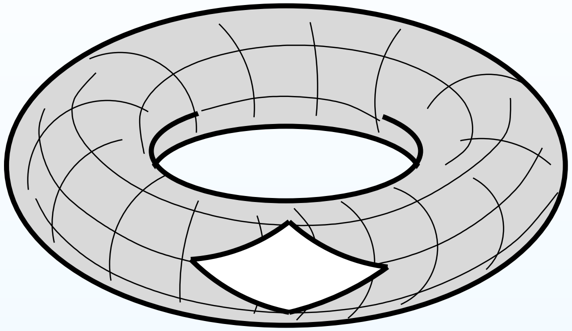

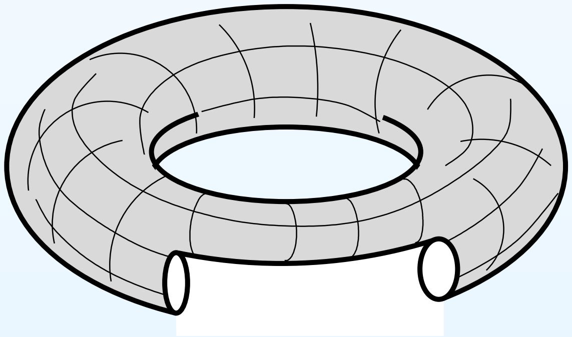

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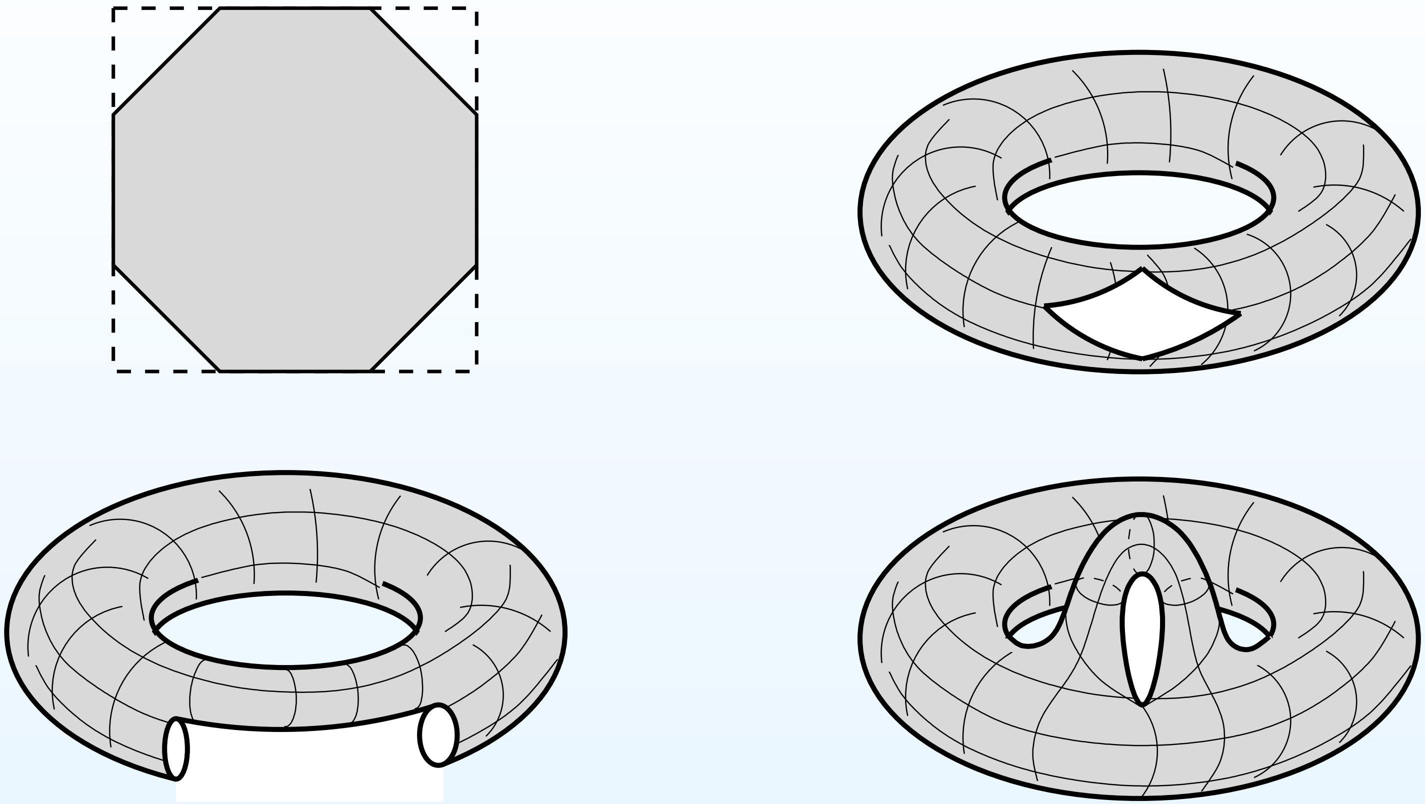

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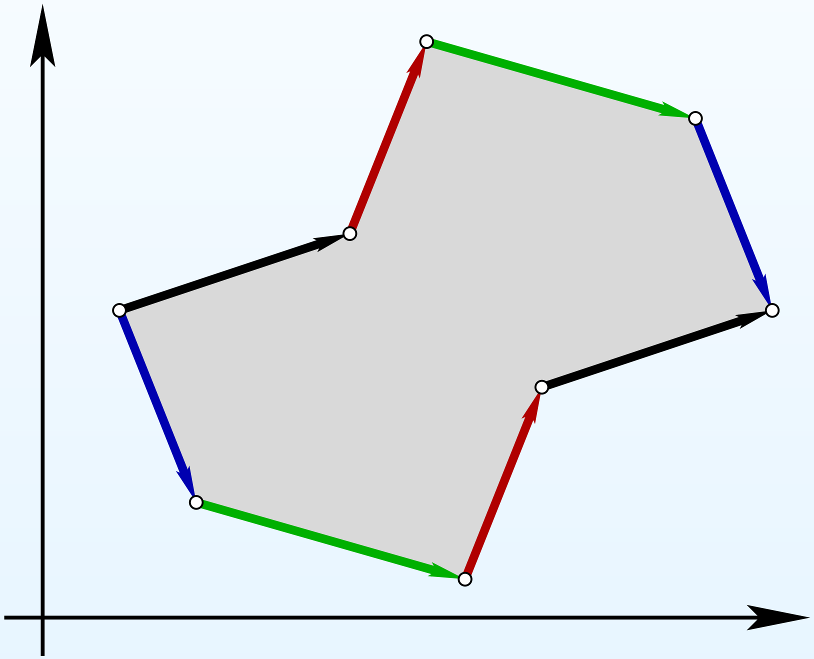
Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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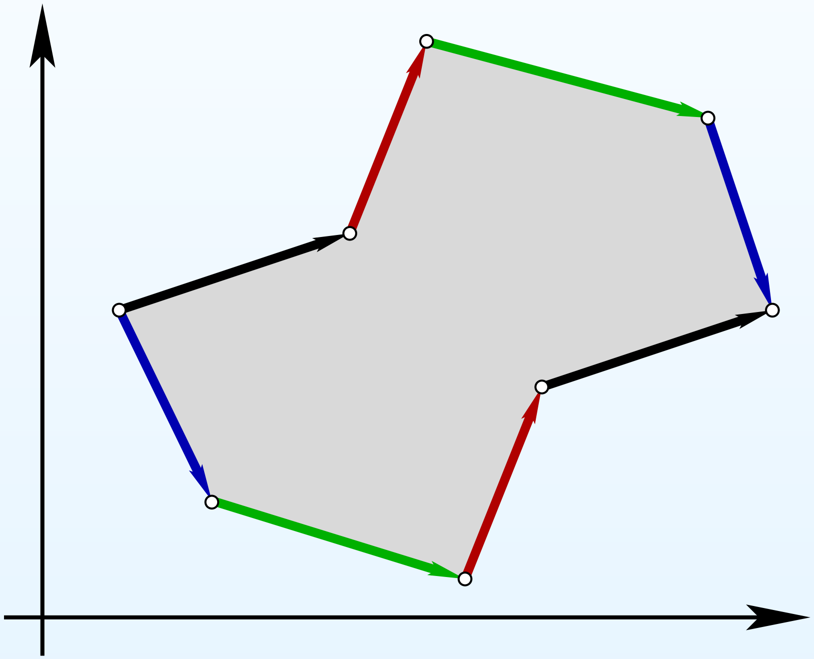

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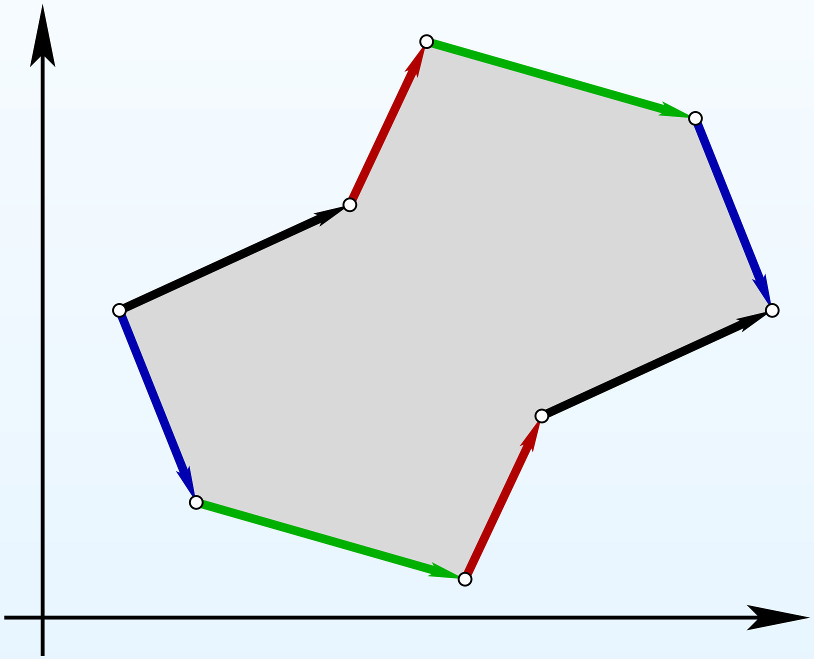

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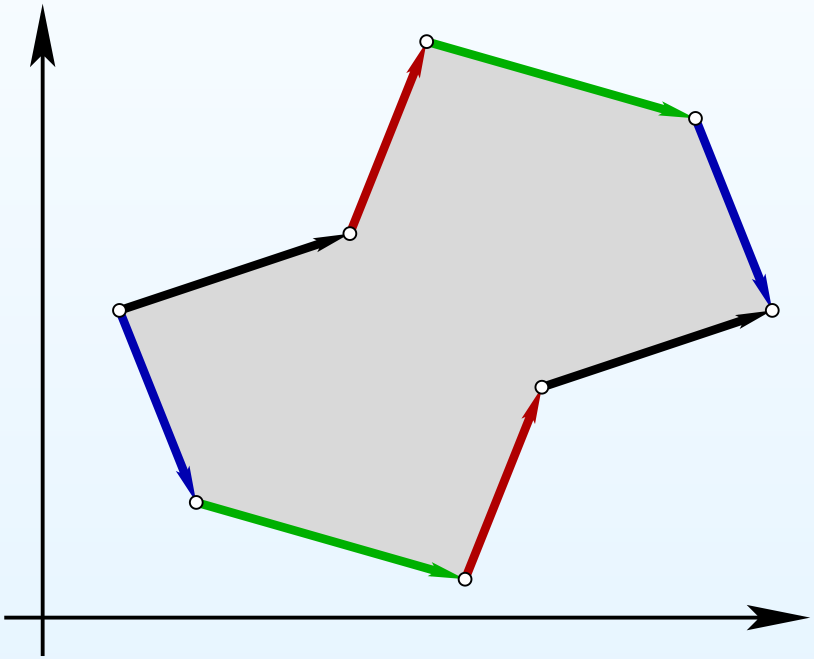
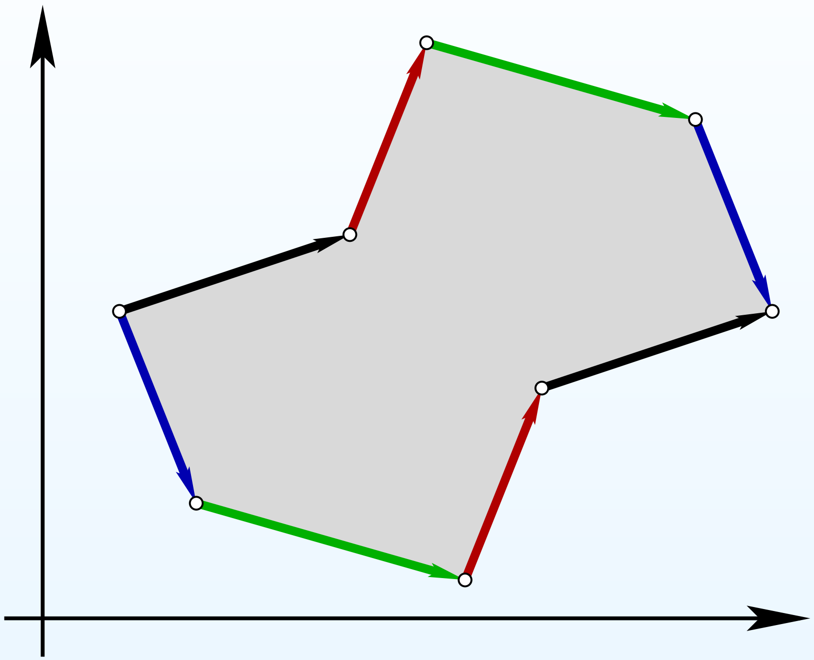


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

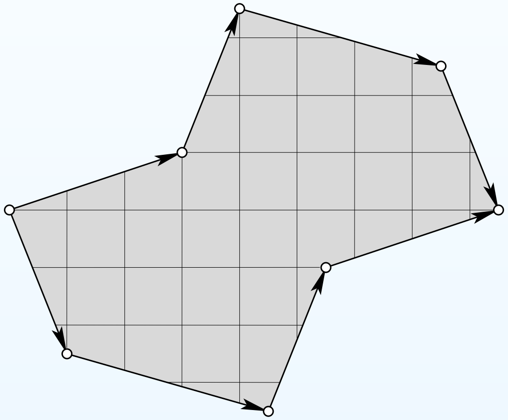
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1, \dots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \dots, m_n$, where $\sum m_i = 2g-2$, is modeled on the vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any moduli space $\mathcal{H}(m_1, \ldots, m_n)$: given a positive integer $r > 0$ we can rescale a flat surface by factor r. The flat area of the surface gets rescaled by the factor r^2.



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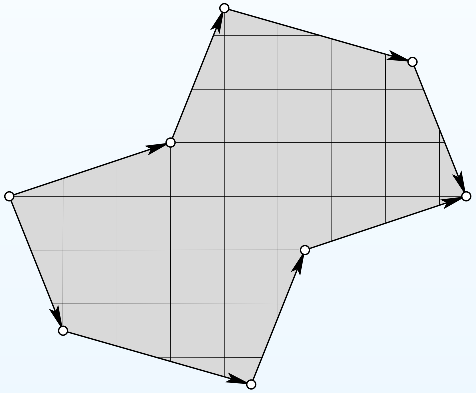

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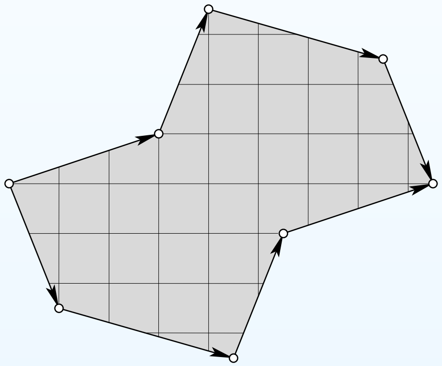

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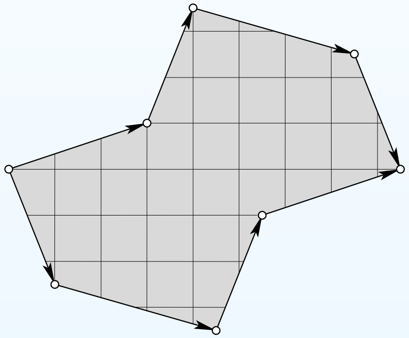

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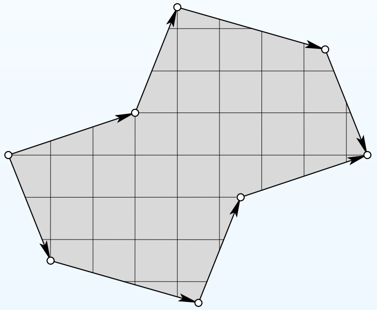

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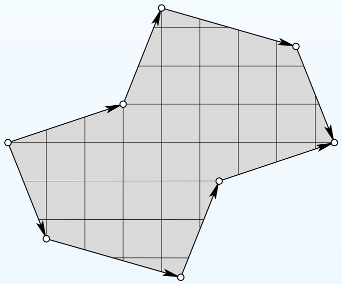

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1, \ldots, m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega \wedge \bar{\omega} = \sum_{i=1}^{g} (A_i \bar{B}_i - \bar{A}_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S = (C, r \cdot \omega)$, where $r > 0$ and $(C, \omega) \in \mathcal{H}_1(m_1, \ldots, m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu = r^{2d-1} dr\, d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \ldots, m_n)$.

Period coordinates and Masur–Veech volume element

The moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g - 2$, is modelled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

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The area function defined on every stratum $\mathcal{H}(m_1,\dots,m_n)$

$$\text{area}(C,\omega)=\frac{i}{2}\int_C \omega\wedge\bar\omega=\frac{i}{2}\sum_{i=1}^{g}(A_i\bar B_i-\bar A_i B_i)\,.$$

allows to define an analog of a “unit ball” $\mathcal{H}_{\le 1}$ in any stratum as a subset of those (C,ω) in $\mathcal{H}(m_1,\dots,m_n)$, where $\operatorname{area}(C,\omega)\le 1$. (Note that in period coordinates the “unit ball” is rather the interior of a “unit hyperboloid”.)

Definition.

$$\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n):=2d\cdot\int_{\mathcal{H}_{\le 1}} d\nu\,,$$

where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$ is just a conventional factor.

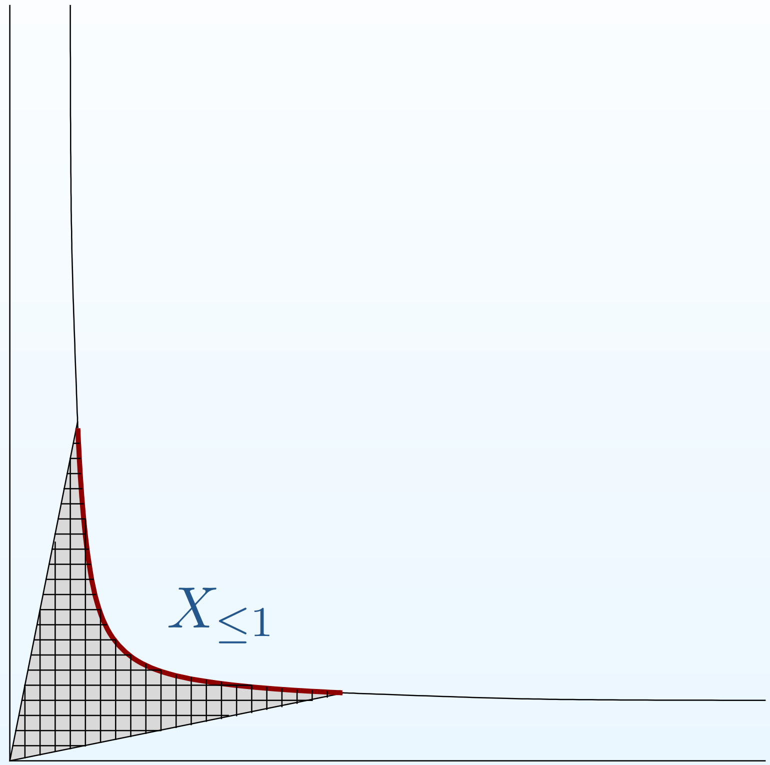
Masur–Veech volume

S0ummary. Every stratum of Abelian differentials admits

- A local structure of a vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$;
- An integer lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

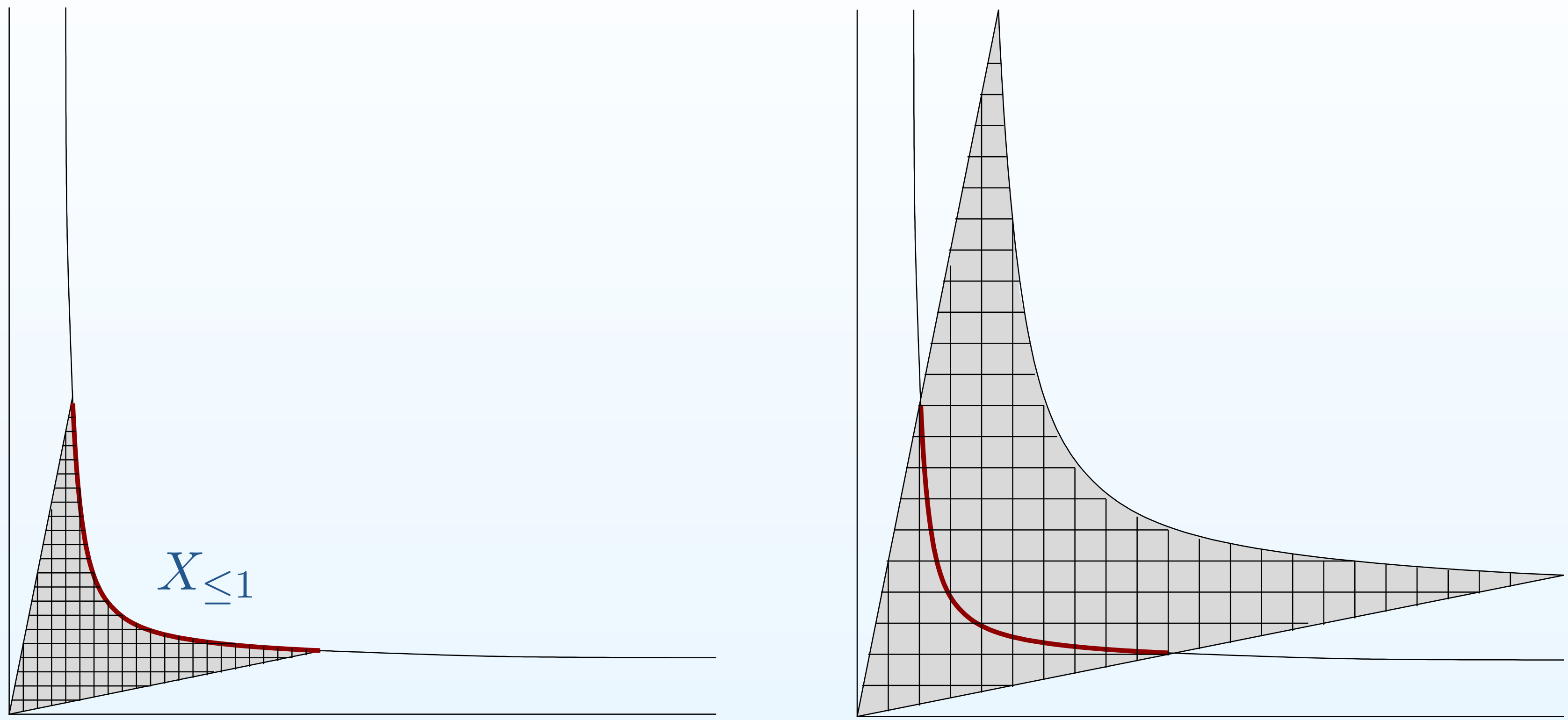
Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum* $\mathcal{H}_1(m_1, \dots, m_n)$ *or* $\mathcal{Q}_1(m_1, \dots, m_n)$ *of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

Counting volume by counting integer points in a large cone



To count volume of the cone $X_{\leq 1}$ one can take an ε-grid and count the number of lattice points inside it.

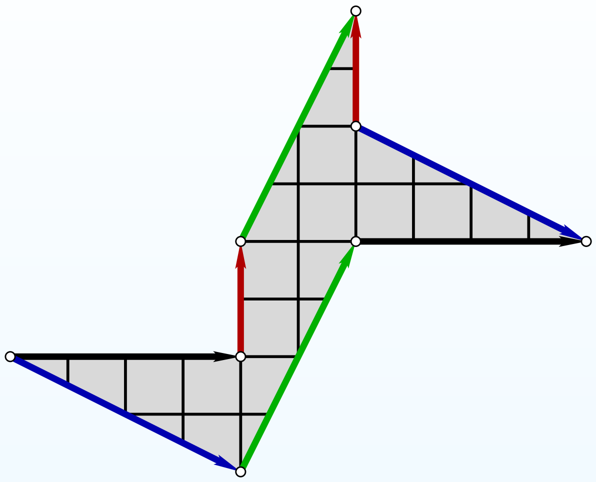
Counting volume by counting integer points in a large cone



To count volume of the cone $X_{\leq 1}$ one can take an ε-grid and count the number of lattice points inside it.

Counting points of the ε-grid in the cone $X_{\leq 1}$ is the same as counting integer points in the proportionally rescaled cone $X_{\leq 1/\varepsilon}$.

Integer points as square-tiled surfaces



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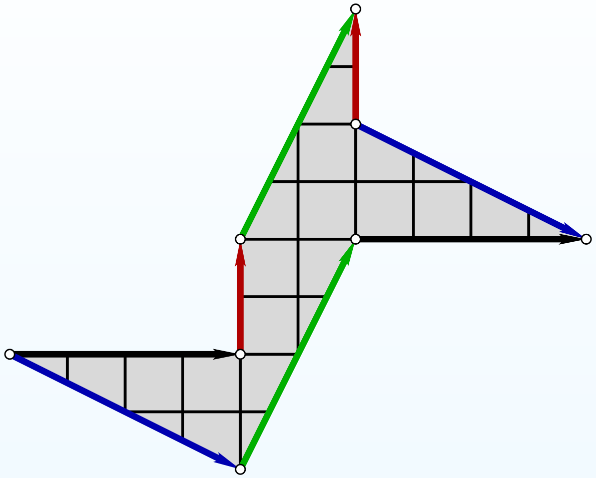

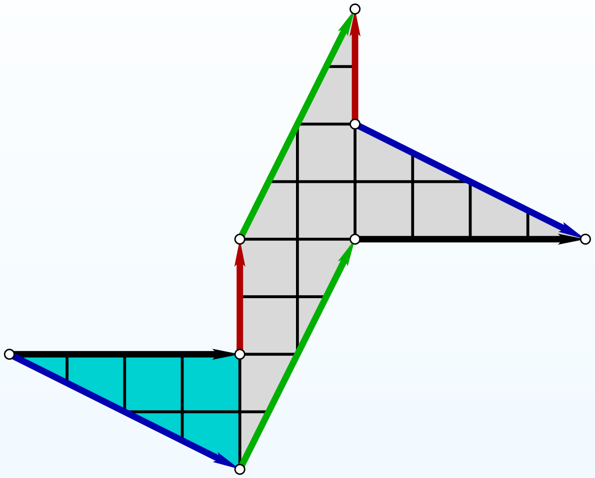

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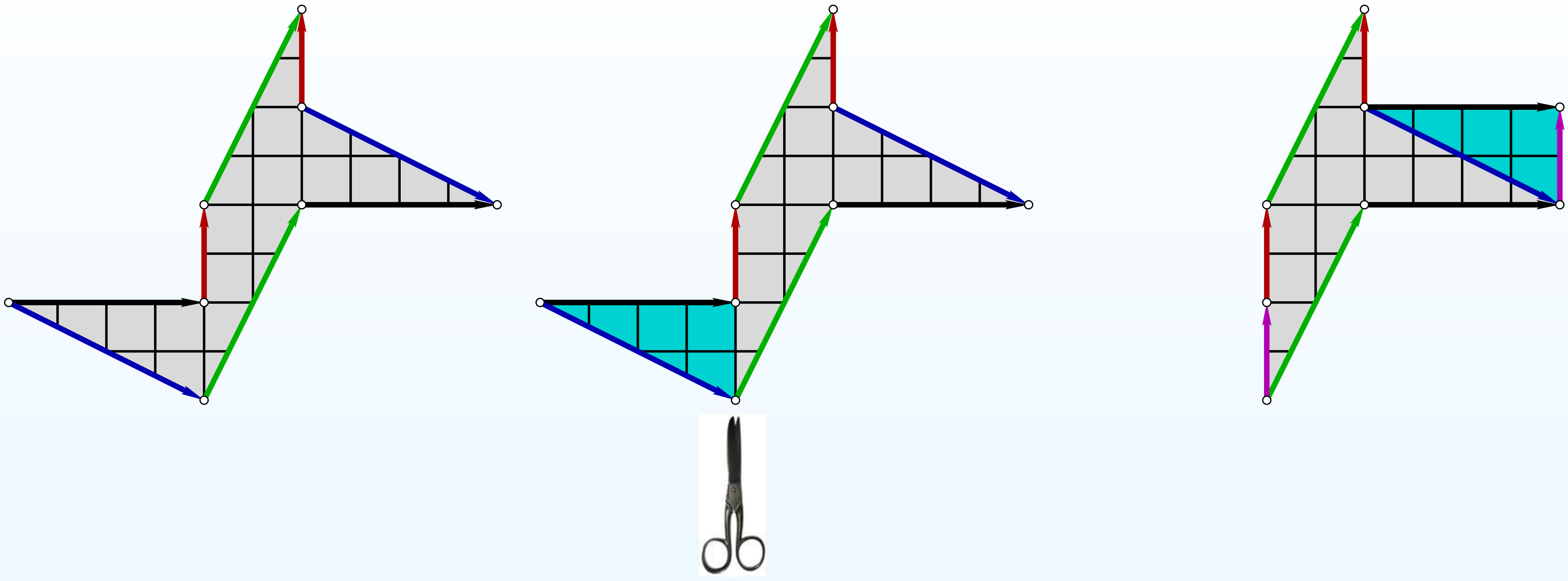

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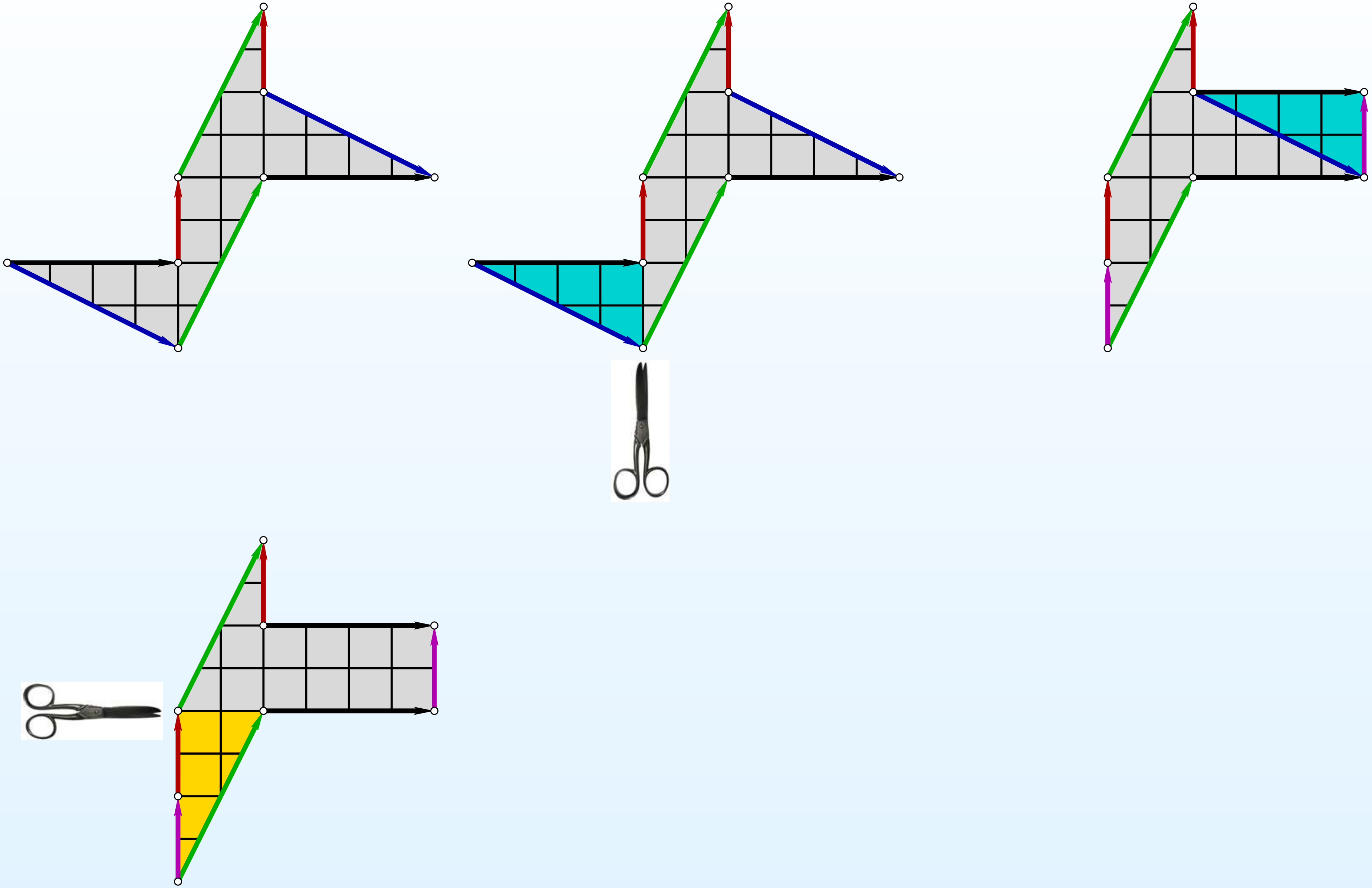

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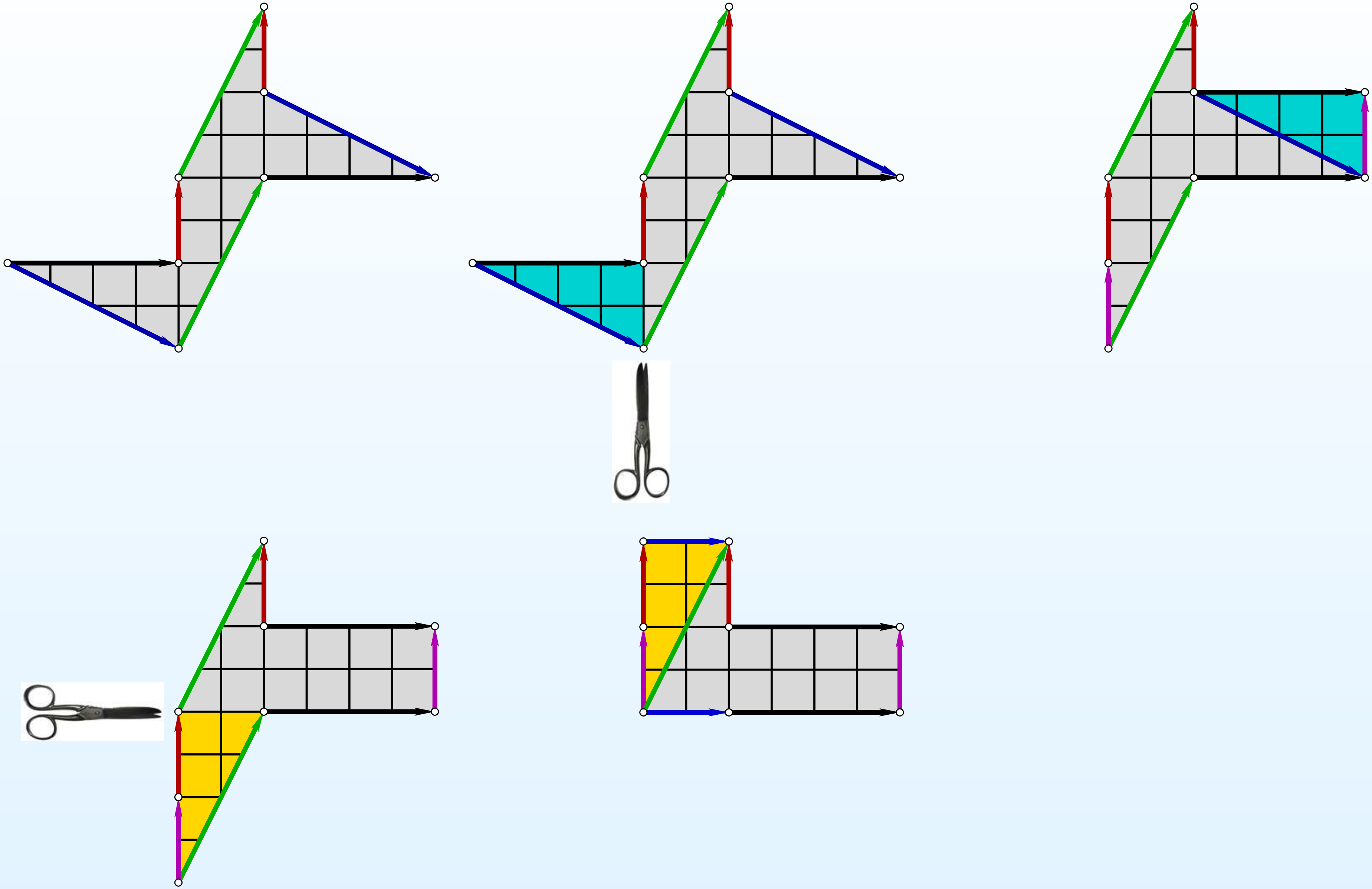

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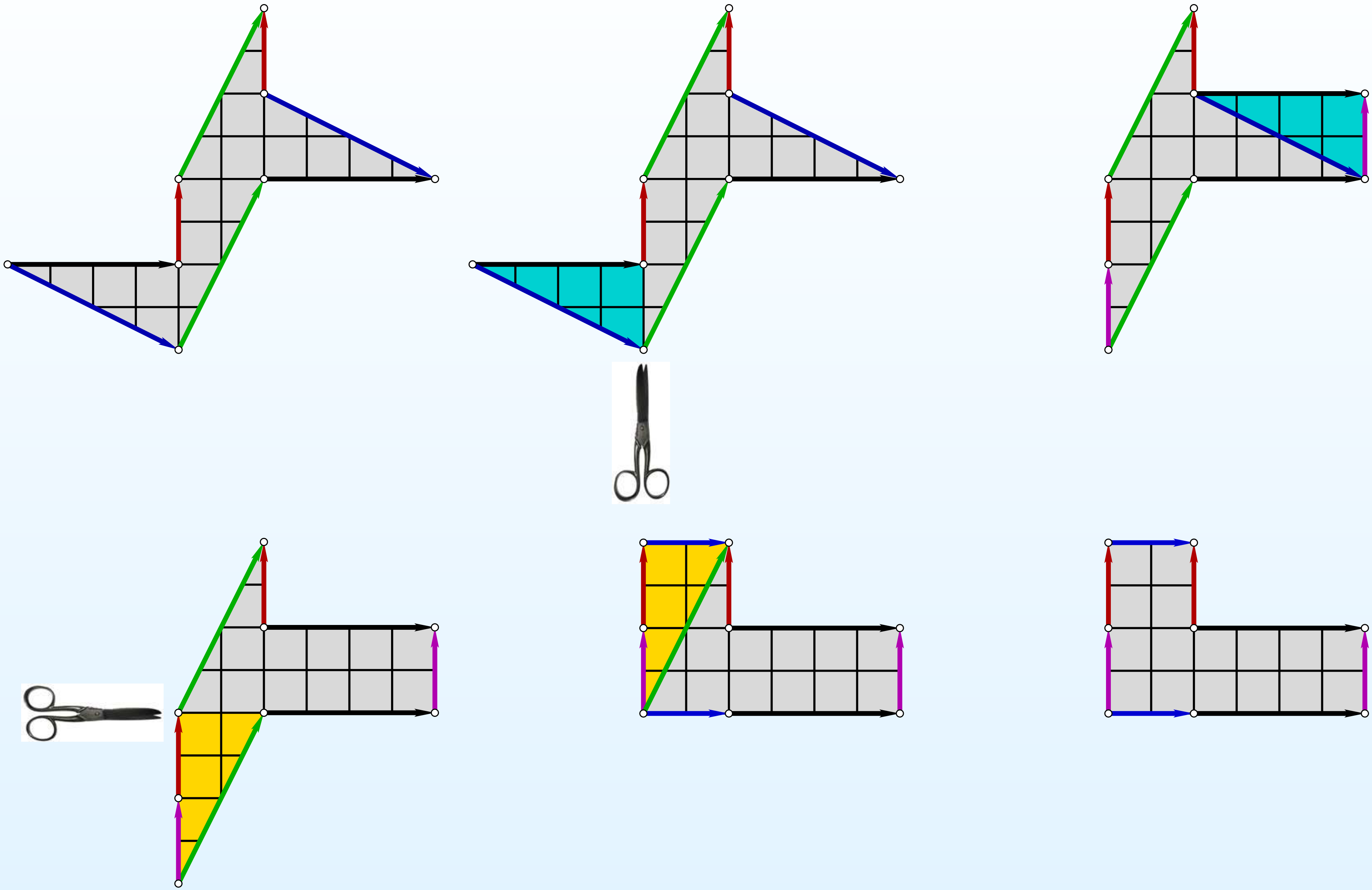

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$$\begin{array}{llll} p: & P & \mapsto & \int_{P_1}^{P} \omega \pmod{\mathbb{Z} \oplus i\mathbb{Z}} \\ p: & C & \to & \mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \end{array}$$

The ramification points of the cover p are exactly the zeroes of ω.

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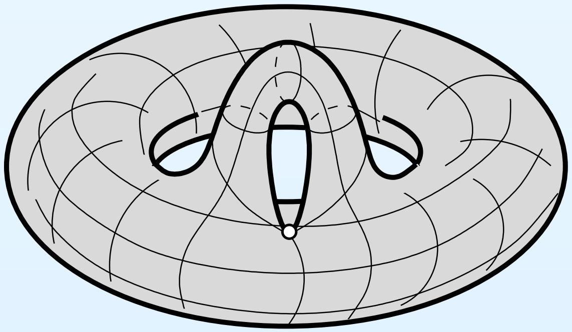
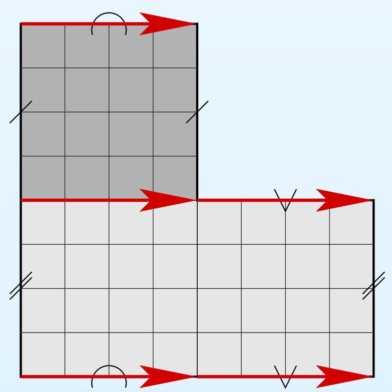
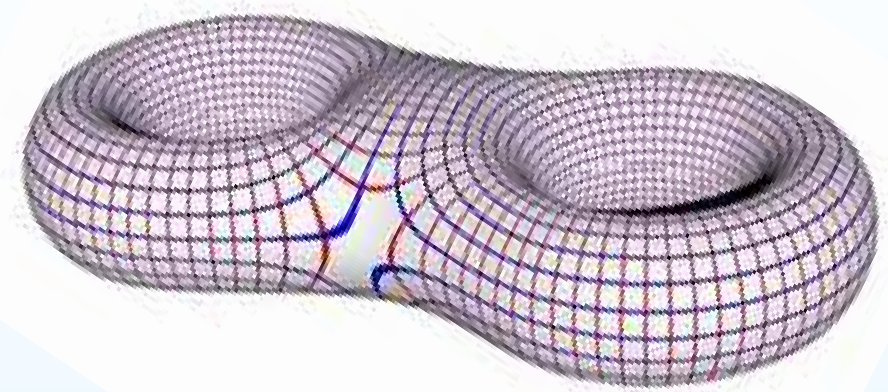
Choosing the standard unit square pattern for $\mathbb{T}$ we get induced tiling of (C, ω) by unit squares which form horizontal and vertical cylinders. The square-tiled surface of genus two in the picture has 2 maximal horizontal cylinders filled with periodic geodesics.





Count of square-tiled surfaces



Picture created by Jian Jiang

We reduced evaluation of the Masur–Veech volumes $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ to a combination of the following two related problems:

- Describe all combinatorial types of square-tiled surfaces in any given stratum $\mathcal{H}(m_1,\dots,m_n)$.
- Count the leading term in the asymptotics of the number of square-tiled surfaces of any given combinatorial type tiled with at most N squares when $N\to+\infty$.

Count of square-tiled surfaces through separatrix diagrams

Multiple zeta-values

Define

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1,\ldots,n_k \geq 1} \frac{1}{n_1^{s_1}(n_1+n_2)^{s_2}\ldots(n_1+\cdots+n_k)^{s_k}}\,.$$

Multiple zeta-values (MZV) are values of $\zeta(s_1, s_2, \ldots, s_k)$ at positive integers $s_j \in \mathbb{N}$, where $s_k \geq 2$. For example

$$\zeta(2) = \frac{\pi^2}{6}\,(Euler); \qquad \zeta(4) = \frac{\pi^4}{90}; \quad \ldots\ \zeta(2n) = \frac{p}{q}\pi^{2n}, \quad \text{where } p, q \in \mathbb{N}\,.$$

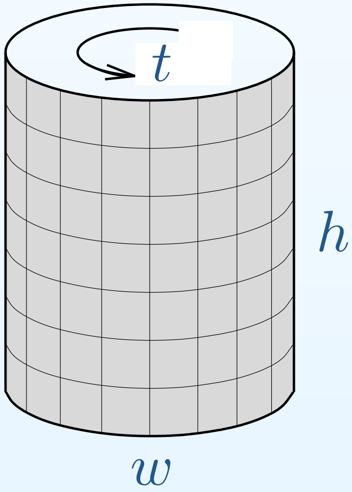
Conjecturally $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

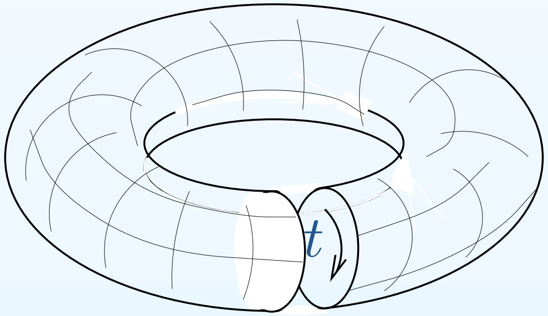
Multiple zeta values satisfy numerous relations. For example

$$\zeta(1,3) = \frac{1}{4}\zeta(4); \qquad \zeta(2,2) = \frac{3}{4}\zeta(4) \quad (Euler)\,.$$

Baby case: decomposition of a square-tiled torus

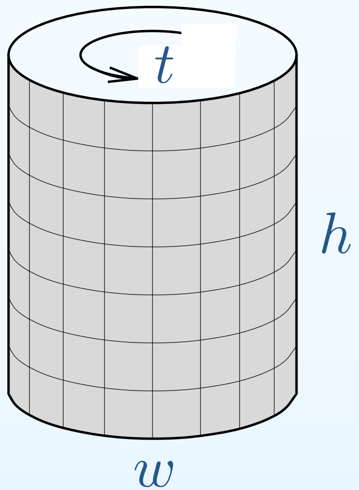
Let us count the number of square-tiled tori tiled by at most $N \gg 1$ squares. Cutting our flat torus by a horizontal waist curve we get a cylinder with a waist curve of length $w \in \mathbb{N}$ and a height $h \in \mathbb{N}$. The number of squares in the tiling equals $w \cdot h$.

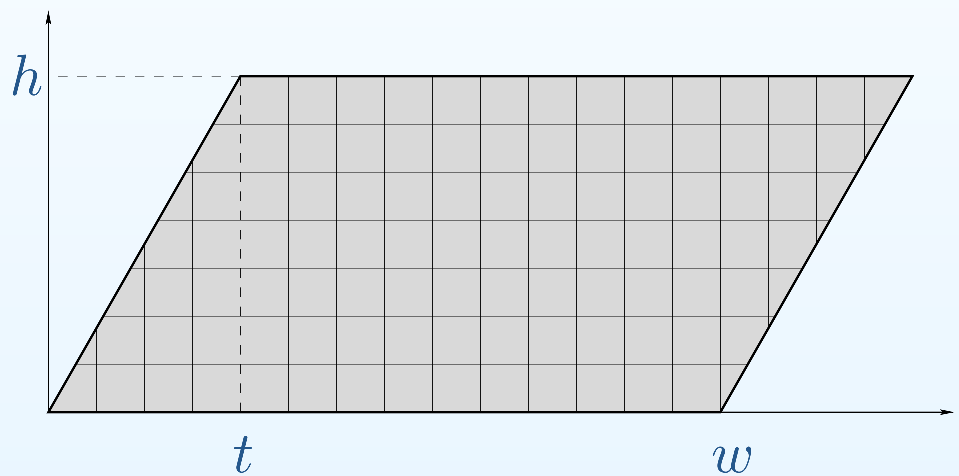




Baby case: counting square-tiled tori

We can glue a torus from a cylinder with some integer twist t. Making an appropriate Dehn twist along the waist curve we can reduce the value of the twist t to one of the values $0, 1, \ldots, w-1$. Fixing the integer perimeter w and height h of a cylinder we get w square-tiled tori.



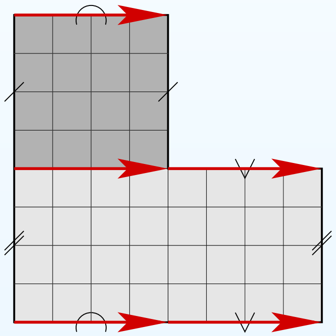


Number of square-tiled tori constructed with at most N squares $=$

$$= \sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w = \sum_{\substack{w,h\in\mathbb{N}\\ w\le \frac{N}{h}}} w \approx \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\zeta(2) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

Critical graphs (separatrix diagrams)

Note that all leaves of the horizontal (vertical) foliation on a square-tiled surface are closed. The *critical graph* Γ (*separatrix diagram*) is the union of all horizontal critical leaves. Vertices of Γ are represented by the conical points; the edges of Γ are formed by horizontal saddle connections.



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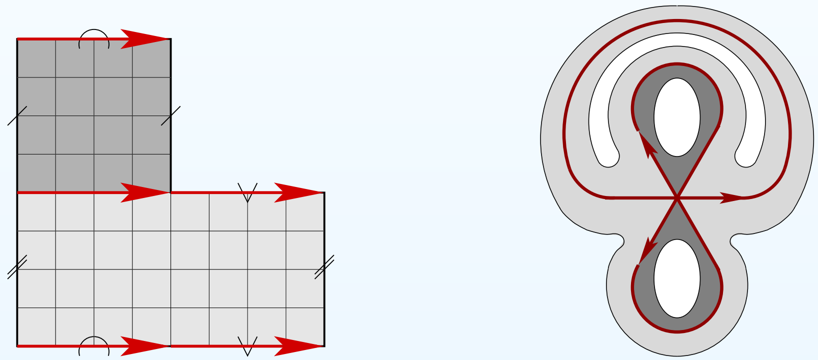
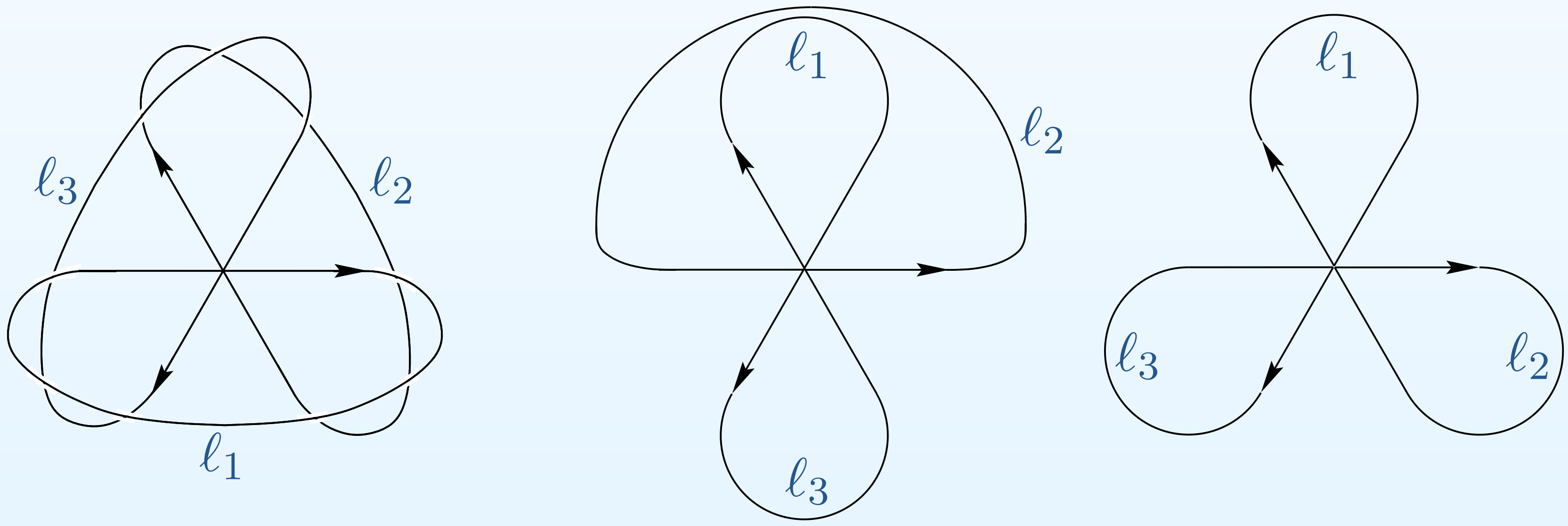


A critical graph Γ is an *oriented ribbon graph* endowed with the following structure:
1. The orientation of edges at any vertex is alternated with respect to the cyclic order of edges at this vertex.
2. The complement $S - \Gamma$ is a finite disjoint union of flat cylinders foliated by oriented circles. Thus, the set of boundary components of the ribbon graph is decomposed into pairs: to each pair of boundary components we glue a cylinder, and there is one positively oriented and one negatively oriented boundary component in each pair.

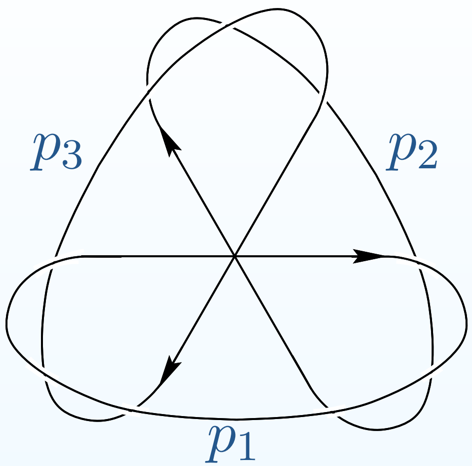
Realizable separatrix diagrams

Note, however, that not all ribbon graphs as above correspond to actual flat surfaces. A flat metric endows saddle connections with positive lengths ℓ_i. The left graph is realizable for any lengths ℓ_1, ℓ_2, ℓ_3. The middle one — only when $\ell_1 = \ell_3$. The rightmost one is never realizable: pairs of boundary components bounding the same cylinder have to have equal length, and we cannot find a pair for the component of length $\ell_1 + \ell_2 + \ell_3$.



Lemma. *The set of all square-tiled surfaces (respectively pillowcase covers) sharing the same realizable separatrix diagram provides a nontrivial contribution to the volume of the corresponding stratum.*

Volume computation for $\mathcal{H}(2)$: the 1-cylinder diagram



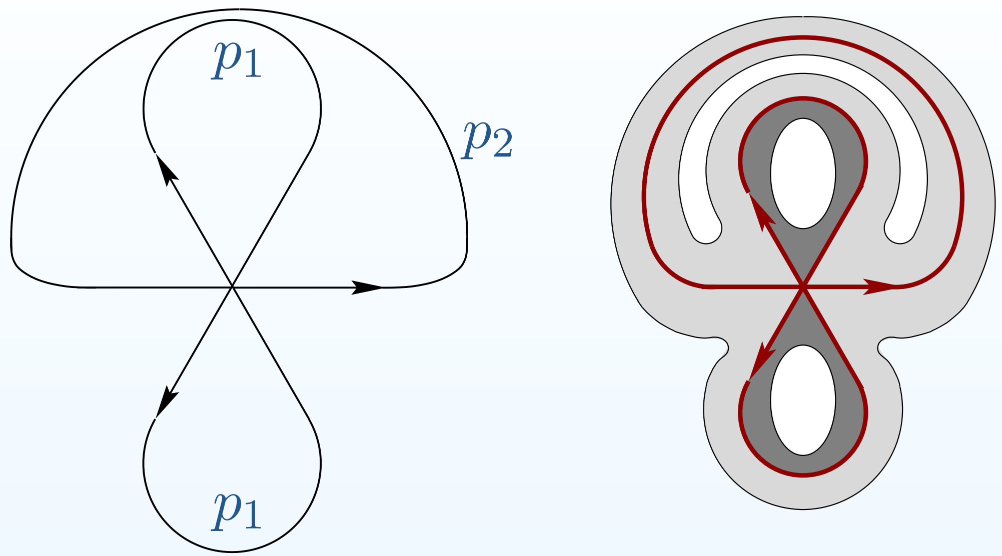
Single cylinder

$$\frac{1}{3} \sum_{\substack{p_1,p_2,p_3,h\in\mathbb{N}\\(p_1+p_2+p_3)h\le N}} (p_1+p_2+p_3) \approx \frac{1}{3}\sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w\cdot\frac{w^2}{2} = \frac{1}{6}\sum_{\substack{w,h\in\mathbb{N}\\ w\le\frac{N}{h}}} w^3$$

$$\approx \frac{1}{6}\sum_{h\in\mathbb{N}}\frac{1}{4}\cdot\left(\frac{N}{h}\right)^4 = \frac{N^4}{24}\cdot\sum_{h\in\mathbb{N}}\frac{1}{h^4}$$

$$= \frac{N^4}{24}\cdot\zeta(4) = \frac{N^4}{24}\cdot\frac{\pi^4}{90}\,.$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram



$$\sum_{\substack{p_1,p_2,h_1,h_2\in\mathbb{N}\\ p_1h_1+(p_1+p_2)h_2\leq N}} p_1(p_1+p_2) = \sum_{\substack{p_1,p_2,h_1,h_2\in\mathbb{N}\\ p_1(h_1+h_2)+p_2h_2\leq N}} (p_1^2+p_1p_2) =$$

$$= \sum_{h_1,h_2\in\mathbb{N}} \sum_{\substack{p_1,p_2\in\mathbb{N}\\ \frac{p_1(h_1+h_2)}{N}+\frac{p_2h_2}{N}\leq 1}} (p_1^2+p_1p_2)\,.$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram

For any fixed h_1, h_2 we can replace the sum with respect to p_1, p_2 by the integral. Let $x_1 := p_1 \cdot \dfrac{h_1+h_2}{N}$ and $x_2 := p_2 \cdot \dfrac{h_2}{N}$ be the new variables, where h_1, h_2 are considered as parameters. After this change of variables our sums with respect to p_1, p_2 become the integral with respect to x_1, x_2, where we integrate over the simplex $\Delta = \{x_1 + x_2 \leq 1 : x_1 \geq 0; x_2 \geq 0\}$:

$$\sum_{\substack{p_1, p_2 \in \mathbb{N} \\ \frac{p_1(h_1+h_2)}{N} + \frac{p_2 h_2}{N} \leq 1}} (p_1^2 + p_1 p_2) \approx$$

$$\approx \int_\Delta \left[\left(\frac{x_1 N}{h_1+h_2}\right)^2 + \left(\frac{x_1 N}{h_1+h_2}\right)\left(\frac{x_2 N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2} dx_1\right)\left(\frac{N}{h_2} dx_2\right) .$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram

$$\sum_{h_1,h_2}\int_\Delta \left[\left(\frac{x_1N}{h_1+h_2}\right)^2+\left(\frac{x_1N}{h_1+h_2}\right)\left(\frac{x_2N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2}dx_1\right)\left(\frac{N}{h_2}dx_2\right)$$

$$= N^4\left[\int_\Delta x_1^2\,dx_1dx_2\cdot\sum_{h_1,h_2\in\mathbb{N}}\frac{1}{h_2(h_1+h_2)^3}\right.$$

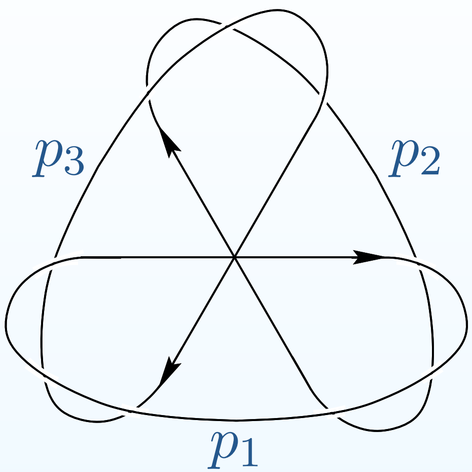
$$\left.+\int_\Delta x_1x_2\,dx_1dx_2\cdot\sum_{h_1,h_2\in\mathbb{N}}\frac{1}{h_2^2(h_1+h_2)^2}\right]$$

$$=\frac{N^4}{24}[2\cdot\zeta(1,3)+\zeta(2,2)]=\frac{N^4}{24}\left[2\cdot\frac{\zeta(4)}{4}+\frac{3\zeta(4)}{4}\right]$$

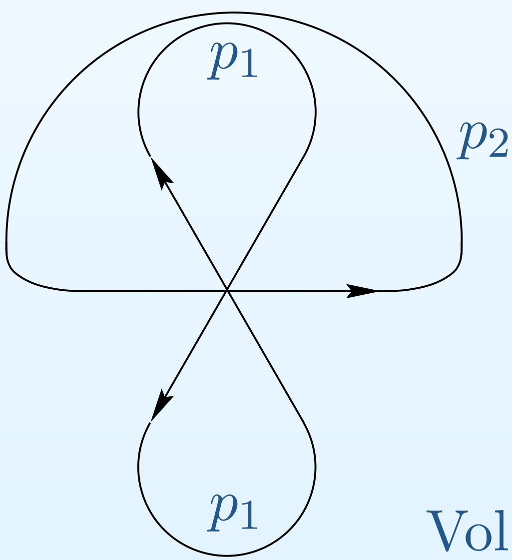
$$=\frac{N^4}{24}\cdot\frac{5}{4}\cdot\frac{\pi^4}{90}.$$

where we used the identities $\zeta(1,3)=\frac{1}{4}\zeta(4),\quad \zeta(2,2)=\frac{3}{4}\zeta(4)$ and the values $\int_\Delta x_1^2\,dx_1dx_2=2\int_\Delta x_1x_2\,dx_1dx_2=2\cdot\frac{1}{4!}$.

Volume computation for $\mathcal{H}(2)$: summary



$$\frac{1}{3} \sum_{\substack{p_1,p_2,p_3,h\in\mathbb{N}\\(p_1+p_2+p_3)h\le N}} (p_1+p_2+p_3) \approx \frac{N^4}{24}\cdot\zeta(4)$$



$$\sum_{\substack{p_1,p_2,h_1,h_2\\p_1h_1+(p_1+p_2)h_2\le N}} p_1(p_1+p_2)$$

$$= \frac{N^4}{24}\big[2\cdot\zeta(1,3)+\zeta(2,2)\big] = \frac{N^4}{24}\cdot\frac{5}{4}\cdot\zeta(4)$$

$$\mathrm{Vol}(\mathcal{H}_1(2)) = \lim_{N\to\infty}\frac{2\cdot 4}{N^4}\cdot\text{(Number of surfaces)} = \frac{\pi^4}{120}$$

Contributions $\operatorname{Vol}_k \mathcal{H}(3,1)$ of k-cylinder surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$\operatorname{Vol}_1\mathcal{H}(3,1) = \frac{\zeta(7)}{15}$$

$$\operatorname{Vol}_2\mathcal{H}(3,1) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{45}$$

$$\begin{aligned}\operatorname{Vol}_3\mathcal{H}(3,1) = \frac{1}{90}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\ &+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\ &- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\ &+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\ &- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\ &- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$\operatorname{Vol}_4\mathcal{H}(3,1) = \frac{2\zeta(2)}{45}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big)\,.$$

After simplification

Multiple zeta values satisfy numerous relations. After simplification (which is now accessible through a SAGE package) we get

$$\operatorname{Vol}_1 \mathcal{H}(3,1) = 1/15 \cdot \zeta(7)$$
$$\operatorname{Vol}_2 \mathcal{H}(3,1) = -7/135 \cdot \zeta(1,6) + 1/135 \cdot \zeta(2,5) + 23/135 \cdot \zeta(7)$$
$$\operatorname{Vol}_3 \mathcal{H}(3,1) = -2/15 \cdot \zeta(1,6) - 2/45 \cdot \zeta(2,5) + 1/5 \cdot \zeta(6) - 4/45 \cdot \zeta(7)$$
$$\operatorname{Vol}_4 (\mathcal{H}(3,1) = 5/27 \cdot \zeta(1,6) + 1/27 \cdot \zeta(2,5) + 7/45 \cdot \zeta(6) - 4/27 \cdot \zeta(7)$$

Conjecturally, multiple zeta values involved in these simplified expressions are linearly independent over rational numbers. However, the total contribution is a rational multiple of π^{2g} in accordance with the general result by A. Eskin and A. Okounkov, 2001:

$$\operatorname{Vol} \mathcal{H}(3,1) = \operatorname{Vol}_1 \mathcal{H}(3,1) + \cdots + \operatorname{Vol}_4 \mathcal{H}(3,1) = \frac{16}{42525}\pi^6$$

Volumes of some low-dimensional strata

$$\mathrm{Vol}(\mathcal{H}_1(\emptyset)) = 2\cdot\zeta(2) = \frac{1}{3}\cdot\pi^2$$

$$\mathrm{Vol}(\mathcal{H}_1(2)) = \frac{2}{3!}\cdot\frac{9}{4}\cdot\zeta(4) = \frac{1}{120}\cdot\pi^4$$

$$\mathrm{Vol}(\mathcal{H}_1(1,1)) = \frac{1}{4!}\cdot 4\cdot\zeta(4) = \frac{1}{135}\cdot\pi^4$$

$$\mathrm{Vol}(\mathcal{H}_1^{hyp}(4)) = \frac{2}{5!}\cdot\frac{135}{16}\cdot\zeta(6) = \frac{1}{6720}\cdot\pi^6$$

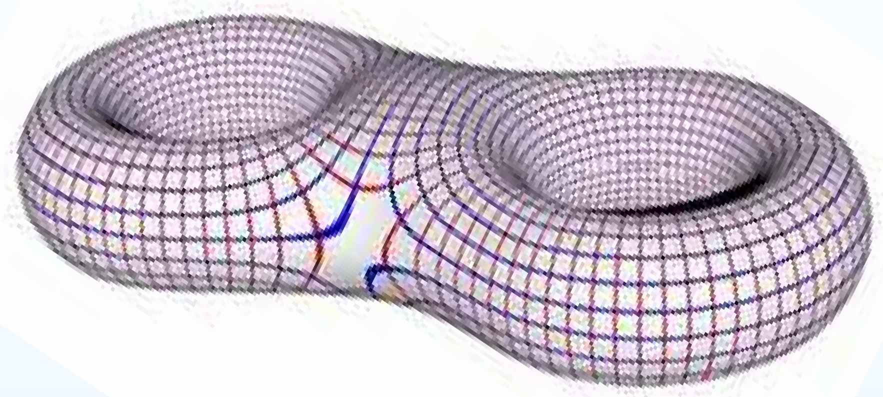
$$\mathrm{Vol}(\mathcal{H}_1^{odd}(4)) = \frac{2}{5!}\cdot\frac{70}{3}\cdot\zeta(6) = \frac{1}{2430}\cdot\pi^6$$

$$\mathrm{Vol}(\mathcal{H}_1(1,3)) = \frac{2}{6!}\cdot 128\cdot\zeta(6) = \frac{16}{42525}\cdot\pi^6$$

$$\mathrm{Vol}(\mathcal{H}_1^{hyp}(6)) = \frac{2}{7!}\cdot\frac{2625}{64}\cdot\zeta(8) = \frac{1}{580608}\cdot\pi^8$$

Homework assignment

Homework assignment



Picture created by Jian Jiang

Questions.

- *To what stratum belongs this square-tiled surface?*
- *Find all realizabe separatrix diagrams for this stratum.*
- *To which of the found diagrams corresponds the square-tiled surface from the picture?*