$\mathbb{C}_{\beta_1} \times \mathbb{C}_{\beta_2} \times \cdots \times \mathbb{C}_{\beta_{m+1}}$ has Calabi–Yau metric (flat in this case) with cone angle $2\pi\beta_i$ along $\{z_i = 0\}$.

Its link S^{2m+1} has a Sasaki-Einstein metric with cone angle

$$2\pi\beta_i \text{ along } S^{2m+1} \cap \{z_i = 0\}.$$

Its Reeb vector field is $\beta_1^{-1} \frac{\partial}{\partial\theta_1} + \cdots + \beta_{m+1}^{-1} \frac{\partial}{\partial\theta_{m+1}}$,

symplectic potential is

$$G = \sum_{a=1}^{m+1} \frac{x_a}{2\beta_a} \log x_a .$$

In Theorem 1-1

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{C}^* \\ \Psi & & \\ \beta = (\beta_1, \cdots, \beta_{m+1}) & \longmapsto & \frac{1}{\beta_1}\frac{\partial}{\partial\theta_1} + \cdots + \frac{1}{\beta_m}\frac{\partial}{\partial\theta_m} \end{array}$$

<u>Prop</u> G is a symplectic potential of a toric Kähler metric with cone angle $2\pi\beta_a$ along $D_a = \mu^{-1}(\{l_a = 0\} \cap C)$

$$\Longleftrightarrow \quad G - \sum \frac{l_a}{2\beta_a} \log l_a \in C^\infty(C).$$

Recall by Guillemin, for $\beta_a = 1$ for all a, $G^{can} := \sum \frac{l_a}{2} \log a$ is obtained by

Delzant construction.

<u>Prop</u> (Abreu formula)

Denote by R_X the scalar curvature of the toric Kähler cone (X, ω). Then, using the symplectic potential G, the Ricci form ρ_X is given by

$$\rho_X = -\frac{1}{2} \sum G^{ij}{}_{ik}\, dy_k \wedge d\theta_j ,$$

the scalar curvature R_X is given by

$$R_X = -\frac{1}{2} \frac{\partial^2 G^{ij}}{\partial y^i \partial y^j} .$$

Further

$$R_X = 0 \iff -\sum_{i,j=1}^{m} \frac{\partial^2 u^{ij}}{\partial \tilde{y}_i \partial \tilde{y}_j} = m(m+1)$$

where $u = G|_{\underbrace{H_3 \cap C}_{P_3}}$, $H_3 = \{ (\xi, x) = \frac{1}{2} \}$,

$\tilde{y}_i$ coord on H_3.

<u>Proof</u>

$$(G_{ij}) = (F_{ij})^{-1} \qquad (G^{ij}) = (G_{ij})^{-1} = (F_{ij})$$

$$(G^{ij}(y) = F_{ij}(x(y)))$$

$$J\frac{\partial}{\partial y_k} = \sum_j G_{jk}\frac{\partial}{\partial\theta^j}, \quad J\frac{\partial}{\partial\theta^k} = -G^{jk}\frac{\partial}{\partial y_j}$$

$$J\,dy_k = -\sum G^{jk}\,d\theta^j$$

$$g = G_{ij}\,dy_i\,dy_j + G^{ij}\,d\theta^i d\theta^j$$

$$= F_{ij}\left(dx^i dx^j + d\theta^i d\theta^j\right)$$

$$d^c f = -df\circ J = \sqrt{-1}(\bar\partial - \partial)f.$$

$$dd^c f = 2i\,\partial\bar\partial f.$$

$$\rho_X = -i\partial\bar\partial\log\det(g_{i\bar j}) = -i\partial\bar\partial\log\det(G^{ij})$$

$$= -\frac{1}{2}dd^c\log\det(G^{ij})$$

$$d^c\log\det(G^{ij}) = -G_{ij}\frac{\partial G^{ij}}{\partial y_k}J\,dy^k$$

$$= +G_{ij}\,G^{ij}{}_{,k}\,G^{kl}\,d\theta^l$$

$$= -\underbrace{G_{ij,k}}_{G_{ikj}}G^{ij}\,G^{kl}\,d\theta^l$$

$$= +G_{ik}\,G^{ij}\,G^{kl}{}_{,j}\,d\theta^l$$

$$= +G^{il}{}_{,j}\,d\theta^l$$

$$R_X = -\frac{1}{2} \sum G^{ji}{}_{,ji} .$$

using Gauss equation for the submanifold $S = \{r=1\} \subset X$ we obtain

$\tilde{R}$: Riemann's scal curv.

$$R_S = \tilde{R}_X + 2m(2m+1)$$

$$\tilde{R}^T = R_S - \underbrace{Ric(\xi,\xi)}_{2m} + 4m \longleftarrow \text{O'Neil formula for submersions}$$

$$= R_S + 2m$$

$$= \tilde{R}_X + 4m(m+1)$$

$$R^T = R_X + m(m+1).$$

$$\therefore R_X = 0 \Longrightarrow R^T = m(m+1).$$

<u>Remark</u> Our $Ric_{i\bar{j}} = g^{k\bar{l}} R_{i}{}^{k}{}_{\bar{j}l}$

Riemannian $Ricci_{ij} = g^{kl} R_{ikjl}$

R = Our Scal $g^{i\bar{j}} R_{i\bar{j}}$

$\tilde{R}$ = Riemannian Scal $g^{ij} R_{ij} = 4R$. $\square$

<u>Prop</u> A toric Kähler cone (X^{m+1}, J, ω) with Reeb vector field ξ and cone angle $2\pi\beta_a$ along D_a is Ricci-flat if and only if the symplectic potential u on P_ξ is a Kähler-Einstein potential, i.e.

$$-\sum_{i=1}^{m} \frac{\partial^2 u^{ij}}{\partial \tilde{y}_i \, \partial \tilde{y}_k} = (m+1)\,\delta_{jk}.$$

Pf Similar

Lemma (Donaldson 2002, JDG.)

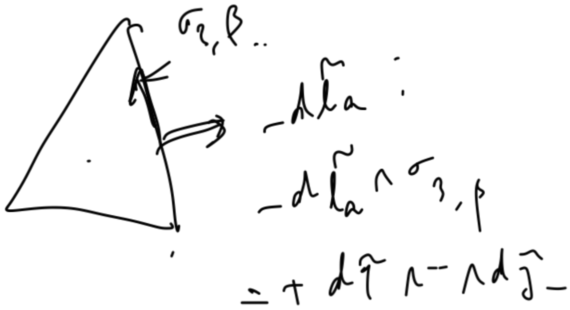
For any symplectic potential on u on P_ξ we have

$$\int_{P_\xi}\left(-\sum_{i,j=1}^{n} u^{ij}{}_{,ij}\right) f(\tilde{y})\, d\tilde{y} = \int_{\partial P_\xi} f(\tilde{y})\, \sigma_{\xi,\beta}$$

for any affine function f, where $\sigma_{\xi,\beta}$ is the measure on ∂P_ξ define by

$$\underbrace{\beta_a^{-1}\, d l_a}_{d\tilde{l}_a} \wedge \sigma_{\xi,\beta} = - d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$$

$\tilde{l}_a := \beta_a^{-1} l_a$



Proof

By integration by parts

normal vector

$$\int_{P_3} u^{ij}{}_{,ij}\, f = -\int_{P_3} u^{ij}{}_{,i}\, f_j - \int_{\partial P_3} u^{ij}{}_{,i}\, f \cdot \nu_j$$

$$= \int_{P_3} u^{ij} f_{ji} - \int_{\partial P_3} u^{ij} f_j \nu_i - \int_{\partial P_3} u^{ij}{}_{,i}\, f \cdot \nu_j$$

$u = \sum x_i \log x_i$, $\nu = (1, 0, \cdots, 0)$, $\partial P_3 = \{x_1 = 0\}$

$u_{11} = \dfrac{1}{x_1}$, $u^{11} = x_1$

$x_1(1+\text{smooth})^0$

$u^{ij} \nu_i = x_1 \to 0$ along ∂P_3

$u^{11}{}_{,1} = 1$

$\therefore u^{ij}{}_{,i}\, \nu_j = 1$ along ∂P_3

Generally $u = \sum x_i \log x_i +$ smooth fn.

Then the same conclusion holds.

$$\therefore \int_{P_3} u^{ij}{}_{,ij}\, f = -\int_{\partial P_3} f$$

$\therefore$

Cor L^2-projection of the transverse scalar curvature $-u^{ij}{}_{,ij}$ to the space of affine functions is independent of the symplectic potential u, therefore indep of the transverse Kähler potential, but only on ξ and $\beta_1^{-1} l_1 \ (= \tilde{l}_1), \ldots, \beta_d^{-1} l_d \ (= \tilde{l}_d)$.

Rem When ξ is regular, $\beta_1, \ldots, \beta_d = 1$, this applies to general compact toric manifold.)
This result is known even for non-toric manifolds.
C.f. Thm 3.3.3 in my Springer Lecture Notes.

Def The L^2-projection of the transverse scalar curvature $-u^{ij}{}_{,ij}$ to the space of affine functions is called the extremal affine function, and the corresponding holomorphic vector field, i.e. its gradient vector field, is called the extremal Kähler vector field.

Write this extremal affine function as

$$A_\beta(\tilde{y}) = A_0 + \sum_{i=1}^{n} A_i \tilde{y}_i .$$

Thus

$$\int_{P_\xi} A_\beta(\tilde{y})\, \tilde{y}_j \, d\tilde{y} = \int_{\partial P_\xi} \tilde{y}_j \, \sigma_{\xi,\beta} \qquad j = 0, 1, \ldots, n$$

where $\tilde{y}_0 = 1$.

In particular, if $-w^{ij}{}_{,ij}$ is constant then A_β is constant and $A_1 = \cdots = A_m = 0$,

$$A_0 \int_{P_\xi} d\tilde{y} = \int_{\partial P_\xi} \sigma_{\xi,\beta}, \quad A_\beta = A_0 = \frac{\int_{\partial P_\xi} \sigma_{\xi,\beta}}{\int_{P_\xi} d\tilde{y}}.$$

<u>Def</u> The (transversal) log Futaki invariant is the linear function

$$L_{\xi,\beta} : \mathfrak{t} \cong \mathrm{Affine}(H_\xi) \longrightarrow \mathbb{R}.$$

$$L_{\xi,\beta}(f) = \int_{\partial P_\xi} f \, \sigma_{\xi,\beta} - \frac{\int_{\partial P_\xi} \sigma_{\xi,\beta}}{\int_{P_\xi} d\tilde{y}} \int_{P_\xi} f \, d\tilde{y}$$

$$= \int_{P_\xi} \left(A_\beta - \frac{\int_{\partial P_\xi} \sigma_{\xi,\beta}}{\int_{P_\xi} d\tilde{y}} \right) f \, d\tilde{y}.$$

<u>Cor.</u> There exists a constant transverse scalar curvature

$\Longrightarrow$ extremal affine function is constant

$\Longleftrightarrow$ $L_{\xi,\beta} = 0$

$\Longleftrightarrow$ The barycenter of P_ξ = the barycenter of ∂P_ξ.

Proof of Theorem 1.1 (i)

$\exists p \in C$ s.t. $\beta_a = l_a(p)$

[Given Reeb vector field $\xi \in C^*$, $\exists_1 \beta \in \mathcal{B}$ cone angle s.t. X has a Calabi-Yau metric with cone angle β_a along D_a, $a = 1, \cdots, d$.]

Def A labelled compact polytope $(P, \tilde{l})$ is monotone $\overset{\text{def}}{\iff}$ $\tilde{l}_1(p) = \cdots = \tilde{l}_d(p) = 1$ for $\exists p \in P_\xi = \{ q \in C \mid \langle q, \xi \rangle = \frac{1}{2} \}$

$\tilde{l}_a = \beta_a^{-1} l_a$

ACB-L used $\frac{1}{m+1}$

Recall the measure $\sigma_{\xi,\beta}$ on ∂P_ξ is defined by
$$\beta_a^{-1} dl_a \wedge \sigma_{\xi,\beta} = -d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$$

Lemma If $(P, \tilde{l})$ is monotone, then
$$\sigma_{\xi,\beta} = \sum_{i=1}^{m} (-1)^{i+1} (\tilde{y}_i - p_i) \, d\tilde{y}_1 \wedge \cdots \wedge \widehat{d\tilde{y}_i} \wedge \cdots \wedge d\tilde{y}_m$$

$\because$
$$\beta_a^{-1} l_a(x) = c_a + \langle \beta_a^{-1} dl_a, x \rangle, \quad c_a = \text{const}$$

$$\beta_a^{-1} dl_a = \sum_{i=1}^{m} s_{a,i} \, d\tilde{y}_i \qquad s_{a,i} = \text{constant}$$

$\beta_a^{-1} dl_a \wedge \sigma_{\xi,\beta}$ For $y \in F_a \cap P_\xi$
$$= \sum s_{a,i} (\tilde{y}_i - p_i) \, d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$$

$$= \langle \beta_a^{-1} d l_a, \tilde{y} - p \rangle \, d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$$

$$= \big(\underbrace{\beta_a^{-1} l_a(\tilde{y})}_{= 0} - \underbrace{\beta_a^{-1} l_a(p)}_{= 1} \big) \, d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$$

$$= - d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$$

suppose $(P, \tilde{e})$ monotone. Then

<u>Lemma</u> p is at the barycenter of $(P_3, d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m)$

$\Longleftrightarrow$ $\mathrm{bar}(P_3, d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m) = \mathrm{bar}(\partial P_3, \sigma_3)$

($\Longleftrightarrow$ Futaki $= 0$.)

$\tilde{\omega} := d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m$ for simplicity.

$$\int_{P_3} \tilde{y}_i \, \tilde{\omega} = \frac{1}{m} \int_{P_3} \tilde{y}_i \, d\sigma_3$$

$$= \frac{1}{m} \int_{\partial P_3} \tilde{y}_i \, \sigma_3 - \frac{1}{m} \int d\tilde{y}_i \wedge \sigma_3$$

$$= \frac{1}{m} \int_{\partial P_3} \tilde{y}_i \, \sigma_3 - \frac{1}{m} \int_{P_3} (\tilde{y}_i - p_i) \, \tilde{\omega}$$

$$\therefore \left(1 + \frac{1}{m}\right) \int_{P_3} \tilde{y}_i \, \tilde{\omega} = \frac{1}{m} \int_{\partial P_3} \tilde{y}_i \, \sigma_3 + \frac{p_i}{m} \int_{P_3} \tilde{\omega} \qquad (*)$$

On the other hand

$$\int_{\partial P_3} \sigma_3 = \int_{P_3} d\sigma_3 = m \int_{P_3} \tilde{\omega}$$

Dividing (*) by $\int_{P_\xi} \tilde{\omega}$ we obtain

$$\left(1+\frac{1}{m}\right)\frac{1}{\int_{P_\xi}\tilde{\omega}}\int_{P_\xi} \tilde{y}_i\, \tilde{\omega} = \frac{1}{\int_{\partial P_\xi}\sigma_\xi}\int_{\partial P_\xi} \tilde{y}_i\, \sigma_\xi + \frac{P_i}{m}$$

$$\therefore \left(\mathrm{bar}\,(P_\xi, \tilde{\omega})\right)_i = P_i$$

$$\iff \mathrm{bar}\,(P_\xi, \tilde{\omega}) = \mathrm{bar}\,(\partial P_\xi, \sigma_\xi).$$

<u>Proof</u> of Theorem 1.1, (1).

Given ξ, choose the barycenter p of P_ξ. Put

$$\beta_a = l_a(p), \quad a = 1, \dots, d$$

Then $\tilde{l}_a(p) = \beta_a^{-1} l_a(p) = 1$, $a = 1, \dots, d$

Thus $(P_\xi, \tilde{l})$ is monotone and p is the barycenter of P_ξ. So, $\mathrm{Fut} = 0$.

Then Wang-Zhu, Donaldson, Legendre have shown $\exists$ K-E symplectic potential with

$$l_a(p) = \beta_a.$$

This corresponds to Calabi-Yau metric on X with cone angle β_a along D_a.

Proof of Thm 1.1, (2)

Suppose $\beta \in B$. We may assume, $\exists\, p_\beta$ s.t.

$\tilde{l}_a(p) = \beta_a^{-1} l_a(p) = 1$. So $(P, \tilde{l})$ monotone

Put $\Xi_\beta = \{ \xi \in C \mid \langle p_\beta, \xi \rangle = \frac{1}{2} \}$, and

let $\mathrm{vol} : \Xi_\beta \longrightarrow \mathbb{R}$ be the volume functional

$$\mathrm{vol}(\xi) = \int_{P_\xi} \tilde{\omega} .$$

Martelli - Sparks - Yau showed

(1) $\mathrm{vol} : \Xi_\beta \to \mathbb{R}$ is a convex proper function. So, $\exists 1$ critical point.

(2) $$\frac{d}{dt} \mathrm{vol}(\xi_t) \Big|_{t=0} = -(m+1) \int_{P_\xi} \langle \dot{\xi}, \tilde{y} \rangle \tilde{\omega} .$$

$\{\xi_t\} \subset \Xi_\beta$, i.e. $\langle p_\beta, \xi_t \rangle = \frac{1}{2}$

$$\therefore \langle p_\beta, \dot{\xi} \rangle = 0 .$$

So, if ξ is the critical point then

$$0 = \frac{d}{dt} \mathrm{vol}(\xi_t) \Big|_{t=0} = -(m+1) \int_{P_\xi} \langle \dot{\xi}, \tilde{y} - p_\beta \rangle \tilde{\omega}$$

$\iff p_\beta$ is the barycenter of P_ξ.

Thus, again by ~~Donaldson~~ Wang-Zhu, Legendre

$\exists$ K-E symplectic potential on P_ξ.

$\therefore \exists$ Calabi-Yau cone metric on X.