Lecture 3. Fundamental groups and knot groups

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May 19, 2022

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Fundamental group

Consider a topological space X and a point $x_0 \in X$. Fix a point a on the circle S^1. Consider the set of continuous mappings $f : S^1 \to X$ such that $f(a) = x_0$. The set of homotopy classes of such mappings admits a group structure. Indeed, the multiplication of two such mappings can be represented by concatenating their paths. The inverse element is obtained by passing over the initial path in the inverse order. Obviously, these operations are well defined up to homotopy.

Definition 1.1

The obtained group is called the fundamental group of the space X; it is denoted by $\pi_1(X, x_0)$.

Lemma 1.2

Show that for the case of connected X the group $\pi_1(X, x_0)$ does not depend on the choice of x_0; i.e., $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

We leave the proof as an exercise.

Remark 1.3

There is no canonical way to define the isomorphism for fundamental groups with different initial points.

Theorem 1.4

The fundamental group is a topological space homotopy invariant.

Let X be a topological space that admits a decomposition $X = X_1 \cup X_2$, where each of the sets X_1, X_2, $X_0 = X_1 \cap X_2$ is open, pathwise–connected and non-empty. Choose a point $A \in X_0$. Suppose that fundamental groups $\pi_1(X_1, A)$ and $\pi_1(X_2, A)$ have presentations $\langle a_1, \dots | f_1 = e, \dots \rangle$ and $\langle b_1, \dots | g_1 = e, \dots \rangle$, respectively. Suppose that the generators $c_1, c_2 \dots$ of $\pi_1(X_0, A)$ (which are elements of both groups $\pi_1(X_1, A)$ and $\pi_1(X_2, A)$) are represented as $c_i = c_i(a_1, \dots)$ and as $c_i = c_i(b_1, \dots)$ in the terms of $\pi_1(X_1, A)$ and $\pi_1(X_2, A)$, respectively.

Then the following theorem holds.

Theorem 1.5 (The van Kampen theorem)

The group $\pi_1(X, A)$ admits a presentation

$$\langle a_i, b_i, | f_i = e, g_i = e, c_i(a) = c_i(b) \rangle.$$

Definition 1.6

Let X be a topological space. X is called simply connected if $\pi_1(X) = \{e\}$.

Corollary 1.7

If both X_1 and X_2 described above are simply connected then X is simply connected as well.

As an example, let us show how to calculate fundamental groups of 2–manifolds.

Theorem 1.8

The fundamental group of the connected oriented 2–surface of genus g ($g > 0$) without boundary has a presentation

$$\langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 {a_1}^{-1} {b_1}^{-1} \ldots a_g b_g {a_g}^{-1} {b_g}^{-1} = e \rangle.$$

The knot group

Now let K be a link in $\mathbb{R}^3$.
Let $M_K = \mathbb{R}^3 \backslash K$ be the complement to K. It is obvious that while performing a smooth isotopy of K in $\mathbb{R}^3$ the complement always says isotopic to itself. Hence, the fundamental group of the complement is an invariant of link isotopy classes.

Definition 2.1

The link (knot) complement fundamental group is also called the link (knot) group.

Remark

In [3] it is shown that the complement to the knot (more precisely, to its small tubular neighbourhood in $\mathbb{R}^3$) is a complete invariant of the knot up to amphicheirality.

There he considered the group together with the element representing an oriented meridian. This was sufficient to distinguish these two trefoils. Later, we will return to this structure (while speaking of the peripheral system of the knot complement).

More generally, one can show that the knot group with a peripheral system can distinguish knots.

However the analogous statement for links is not true. Before constructing a counterexample, let us prove the following lemma.

Lemma 2.2

Let $D^3 \subset \mathbb{R}^3$ be a ball and $T \subset D^3$ be the full torus γ; see Fig. 1. There exists a homeomorphism of $\mathbb{R}^3 \backslash T$ onto itself, mapping the curves AB and CD (as they are shown in Fig. 1.a) to AB and CD (Fig. 1.b) that is constant inside the ball D^3.

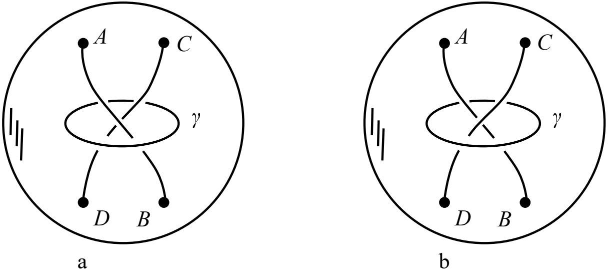


Figure 1: Changing the crossing

Proof

In order to have a more intuitive outlook, let us imagine that the interior diameter of T is very big (so that the “interior” boundary of it represents a high cylinder) in comparison with the exterior one. Thus, we have a deep hole surrounded by the boundary of the full torus; see Fig. 2.

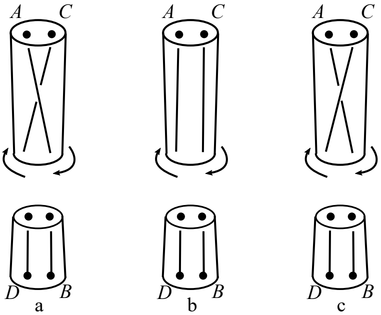


Figure 2: Rotating upper part of the cylinder

Proof : continued

Let us consider the ball D^3 and cut a circle from the plane, as shown in Fig. 2.a. Let us rotate the part of this cut (that is a circle with two marked points) in the direction indicated by arrows. This operation is possible since the full torus T is deleted from D^3. Performing the 180–degree turn, we obtain the configuration shown in Fig. 2.b. Then, let us make one more turnover. Thus we obtain the embedding represented in Fig. 2.c. Thus, each point of the cut returns to the initial position. So, both copies of the cut can be glued together in such a way that the total space remains the same. In this way, we obtain a homeomorphism of the manifold $D^3 \setminus T$ onto itself. This homeomorphism can be extended to a homeomorphism of $\mathbb{R}^3 \setminus T$ onto itself, identical outside D^3. The latter homeomorphism realises the crossing change.

The link complement fundamental group is a very strong invariant. For instance, it recognises trivial links among links with the same number of components. This result follows from Dehn's theorem.

Theorem 2.3 (Dehn)

An m–component link L is trivial if and only if $\pi_1(\mathbb{R}^3 \backslash L)$ is isomorphic to the free group in m generators.

Thus, Dehn's theorem reduces the trivial link recognition problem to the free group recognition problem (for some class of groups). In the general case, the free group recognition problem is undecidable. Dehn's theorem follows from the following lemma.

Lemma 2.4

Let M be a 3–manifold with boundary and let γ be a closed curve on its boundary ∂M. Then if there exists an immersed 2–disc $D \to M$, such that $\partial D = \gamma$ then there exists an embedded disc $D' \subset M$ with the same boundary $\partial D' = \gamma$.

This lemma was first proved by Dehn [4], but this proof contained lacunas. The rigorous proof was first found by Papakyriakopoulos [6]. This proof used the beautiful techniques of towers of 2–folded coverings.

Proof of the Dehn theorem

Having a knot $K \subset \mathbb{R}^3$ with its tubular neighbourhood N(K), let us consider $\pi_1(\mathbb{R}^3 \backslash N(K), A)$ where $A \in T(K) = \partial N(K)$. Obviously, each closed loop which can be isotoped to a loop lying on T(K) then can be represented via the longitude and meridian of T(K), which are non-intersecting closed curves. Obviously, the meridian μ, i.e. the simple curve on T(K) that lies in a small neighbourhood of some point on K and has linking coefficient with K equal to one, cannot be contracted to zero. Let λ be the longitude; i.e. a simple curve in T "parallel to K" and having linking coefficient zero with K. The curves μ and λ generate the fundamental group of the torus T.
Suppose the group of the knot K is isomorphic to $\mathbb{Z}$. Obviously, this group contains $\{\mu\} = \mathbb{Z}$. Besides, no power of λ can be equal to a non-trivial exponent of μ (because of linking coefficients). Thus, the curve λ is isotopic to zero in $\pi_1(\mathbb{R}^3 \backslash N(K))$. Thence, there is a singular disc bounded by λ. By Dehn's lemma, there is a disc embedded in $\mathbb{R}^3 \backslash N(K)$ bounded by λ. Contracting N(K) to K, we obtain a disc bounded by K. Thus, K is the unknot. This completes the proof of Dehn's theorem.

Remark 2.5

The statement of Dehn's theorem shows that the fundamental group is rather a strong invariant. However, it does not allow us to distinguish mirror knots and some other knots. The first example of distinguishing two different mirror knots, namely, the two trefoils, was made by Dehn in [5].

Notice that in the proof of Dehn theorem, we considered not only the knot group, but also a meridian of T(K),which is called the peripheral system of the knot complement. This was sufficient to distinguish these two trefoils and later, we will return to this structure.

The Wirtinger presentation

We present a way of calculating the fundamental group for arbitrary links. Consider a link L given by some planar diagram $\bar{L}$. Consider a point x “hanging” over this plane. Let us classify isotopy classes of loops outgoing from this point. It is easy to see (the proof is left to the reader) that one can choose generators in the following way. All generators are classes of loops outgoing from x and hooking the arcs of $\bar{L}$. Let us decree that the loop corresponding to an oriented edge is a loop turning according to the right–hand screw rule; see Fig. 3.

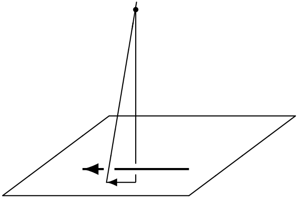


Figure 3: Loops corresponding to edges

The Wirtinger presentation

Now, let us find the system of relations for this group.
It is easy to see the geometrical connection between loops hooking adjacent edges (i.e., edges separated by an overcrossing edge).
Actually, we have $b = cac^{-1}$, where c separates a and b; see Fig. 4.

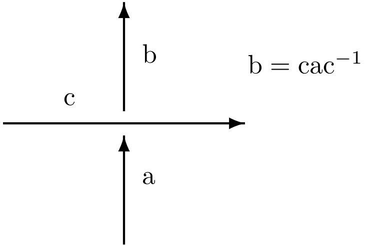


Figure 4: Relation for a crossing

The Wirtinger presentation

Let us show that all relations in the fundamental group of the complement arise from these relations.
Actually, let us consider the projection of a loop on the plane of $\bar{\mathrm{L}}$ and some isotopy of this loop. While transforming the loop, its written form in terms of generators changes only when the projection passes through crossings of the link. Such an isotopy is shown in Fig. 5. During the isotopy process, the arc connecting P and Q passes under the crossing.

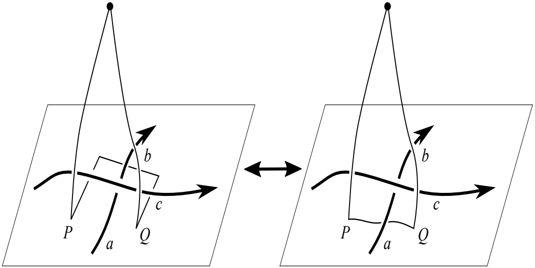


Figure 5: Isotopy generating relation

The Wirtinger presentation

Obviously, the loop shown on the left hand (Fig. 5) generates the element $c^{-1}bc$, that on the right hand is just a.
Thus, our presentation of the fundamental group of the link complement is constructed as follows: arcs correspond to the generators and the generating relations come from crossings: we take $cac^{-1} = b$ for adjacent edges a and b, separated by c, when the edge b lies on the left hand from c with respect to the orientation of c.

Definition 2.6

This presentation of the fundamental group for the knot complement is called the Wirtinger presentation.

The Dehn presentation

We present another way of calculating the fundamental group for arbitrary links, which is called the Dehn presentation. In the Dehn presentation, regions, in which the diagram divides the plane, correspond to the generators and the generating relations come from crossings as described in Fig. 6.

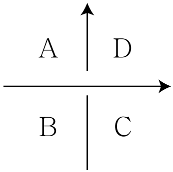


Figure 6: $\mathrm{r(c)} = \mathrm{AB}^{-1}\mathrm{CD}^{-1}$

Then the knot group can be presented by

$$\pi_1(\mathbb{R}^3 \backslash \mathrm{K}) = \langle \{\mathrm{Regions}\} | \mathrm{r(c)} = 1 \rangle.$$

Theorem 2.7

For each knot K, the number $CI(K) + 3$ is equal to the number of homomorphisms of $\pi_1(\mathbb{R}^3 \backslash K)$ to the symmetric group S_3.

Proof

Consider a knot K and an arbitrary planar diagram of it. In order to construct a homeomorphism from $\pi_1(\mathbb{R}^3 \backslash K)$ to S_3, we should first find images of all elements corresponding to arcs of K.
Suppose that there exists at least one such element mapped to an even permutation. Consider the arc s, corresponding to this element. Then any arc s' having a common crossing A with s and separated from s by some overcrossing arc at A, should be mapped to some even permutation. Since K is a knot, we can pass from s to any other arc by means of "passing through undercrossings". Thus, all elements corresponding to arcs of K are mapped to even permutations.
Because the group A_3 is commutative, we conclude that all elements corresponding to arcs are mapped to the same element of A_3 (even symmetric group). There are precisely three such mappings.

Proof:Continued

If at least one element–arc is mapped to an odd permutation then so are all arcs. There are three odd permutations: $(12), (23), (31)$. If we conjugate one of them by means of another one, we get precisely the third one. This operation is well coordinated with the proper colouring rule.

So, all homomorphisms of the group $\pi_1(\mathbb{R}^3 \backslash \mathrm{K})$ to S^3, except three "even" ones, are in one–to–one correspondence with proper colourings of the selected planar diagram of K.

Exercises

1. Show that for the case of connected X the group $\pi_1(X, x_0)$ does not depend on the choice of x_0; i.e., $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.
2. Show that links L_1 and L_2 (Fig. 7.a and 7.b) are not isotopic, but their complements are homeomorphic.

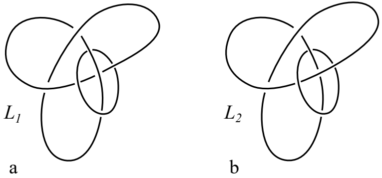


Figure 7: Non-isotopic links with isomorphic complements

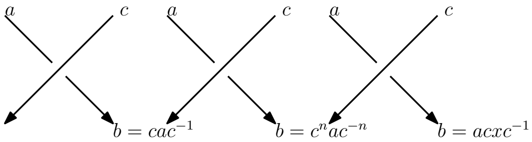
3. (Difficult) Find two non-isotopic (and not mirror) knots with isotopic fundamental groups.

4. Show that the fundamental group of the circle is isomorphic to the fundamental group of the complement to the unknot. Show that they are both isomorphic to $\mathbb{Z}$.
5. Show that the fundamental group of the complement to the trivial n–component link is isomorphic to the free group in n generators.
6. Find a purely algebraic proof that the Wirtinger presentation for two diagrams of isotopic links generates the same group.
7. Find a Wirtinger presentation for the trefoil knot and prove that the two groups presented as $\langle a, b | aba = bab \rangle$ and $\langle c, d | c^3 = d^2 \rangle$ are isomorphic.

8. Show that $\pi_1(A_1 \vee A_2)$ (of the union of spaces A_1 and A_2 with one identified point) is isomorphic to the free product of $\pi_1(A_1)$ and $\pi_1(A_2)$ in the case when both A_1 and A_2 are pathwise connected. Show that for the left and right handed trefoils T_1 and T_2, the fundamental groups of complements for $T_1 \# T_1$ and $T_1 \# T_2$ are isomorphic.
9. Calculate a Wirtinger presentation for the figure eight knot (for the simplest planar diagrams).
10. Calculate a Wirtinger presentation for the Borromean rings

Research problems

We can simply modify the knot groups in the Wirtinger representation by changing the relations from crossings $b = cac^{-1}$ to $b = c^n ac^{-n}$. Apparently, the new group generated by arcs with relations $b = c^n ac^{-n}$ from each crossings is not isomorphic to the knot group, but it is still invariant under Reidemeister moves.



If we consider the relation $b = abxb^{-1}$ (for some fixed element x) from crossings, then it is not invariant under Reidemeister moves. But, it becomes an invariant for framed knots.

Research problems

On the other hands, in [7] Vik. S. Kulikov introduced the notion of C-group. More precisely, C-group is defined by

$$\langle X = \{x_1, \ldots, x_n\} | w_{i,j}^{-1} x_i w_{i,j} = x_j \rangle,$$

where $w_{i,j}$ is a word in $X^{\pm}$. It is proved that every C-group can be realized as the fundamental group of complement of some n-dimensional ($n \geq 2$) compact orientable manifold without boundary embedded in S^{n+2}, that is, the fundamental group of complement of higher dimensional knots.
In [8] V. G. Bardakov, M. V. Neshchadim, and M. Singh constructed a group, which is generated by arcs and relations from crossings. Especially, each crossing gives two relations.

Research problems

- Which groups can be realised by a group presentation generated by arcs with relations from crossings and is an invariant under Reidemeister moves?
- Try to construct "knot group-like" algebraic structure with two different binary operations to distinguish oriented knots with its orientation reversed knots.

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