Toric varieties from fans

First we formally define a fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

<u>Def</u> A (rational polyhedral) <u>cone</u> $\sigma \subseteq N_{\mathbb{R}}$ is generated by a finite subset of N over $\mathbb{R}_{\geq 0}$, that is

$$\sigma = \{ \sum_i \lambda_i n_i \mid \lambda_i \geq 0 \} \text{ for } n_i \in N$$

<u>Note</u> $\{0\}$ is considered a cone.

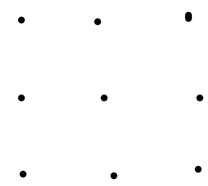
Say σ <u>strongly convex</u> if $\sigma \cap (-\sigma) = \{0\}$

Now for $m \in M$ have

- hyperplane $H_m = \{ n \mid \langle m, n \rangle = 0 \} \subset N_{\mathbb{R}}$
- halfspace $H_m^+ = \{$ same with $\geq 0 \}$

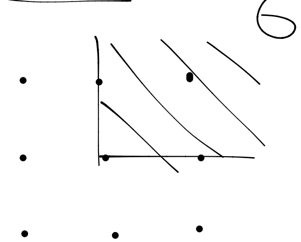
<u>Ex</u>

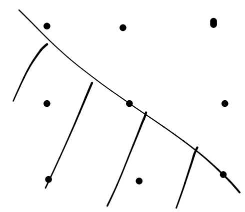
$N \cong \mathbb{Z}^2$



$N_{\mathbb{R}} \cong \mathbb{R}^2$

<u>Ex</u>





ρ not strongly convex

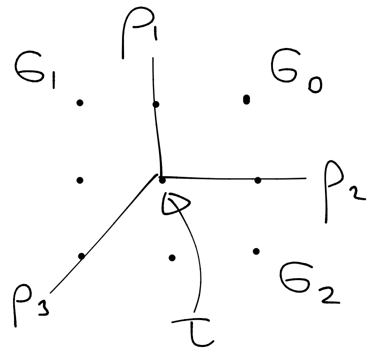
Say H_m <u>supports</u> σ if $\sigma \subset H_m^+$.

In this case, say $H_m \cap \sigma$ is a <u>face</u> of σ.

<u>Def</u> A <u>fan</u> Σ is a finite set of such cones such that

- for $\sigma \in \Sigma$, each face of $\sigma \in \Sigma$
- for $\sigma, \rho \in \Sigma$, $\sigma \cap \rho$ is a face of σ, ρ.

<u>Ex</u> Have a fan as follows in $N_{\mathbb{R}} \cong \mathbb{R}^2$ with

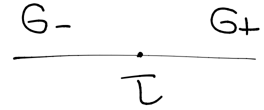


2-dim cones σ

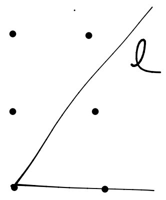
1-dim cones ρ

0-dim cone τ

<u>Ex</u> Have fan in $N_{\mathbb{R}} \cong \mathbb{R}$ with 3 cones:



<u>Ex</u>



This is a cone if and only if the slope of ℓ is <u>rational</u>.

Affine toric varieties

Recall that an affine variety X is determined by its ring of functions R.

<u>Ex</u> $X = \mathbb{C}^2$. Choosing coordinates x, y, have $R = \mathbb{C}[x,y]$.

Write $X = \mathrm{Spec}(R)$, <u>spectrum</u> of a ring.

<u>Ex</u> $X = \mathbb{C}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{\pm 1\}$. Ring of functions spanned by $f = x^2$, $g = xy$, $h = y^2$, with

$$R \cong \mathbb{C}[f,g,h] / fh - g^2.$$

Now cones naturally give rings, as follows. To glue the corresponding affine varieties later, we use:

<u>Def</u> (dual cone) $\sigma^\vee = \{m \mid \langle m,n\rangle \geq 0 \ \forall n \in \sigma\} \subset M_{\mathbb{R}}$

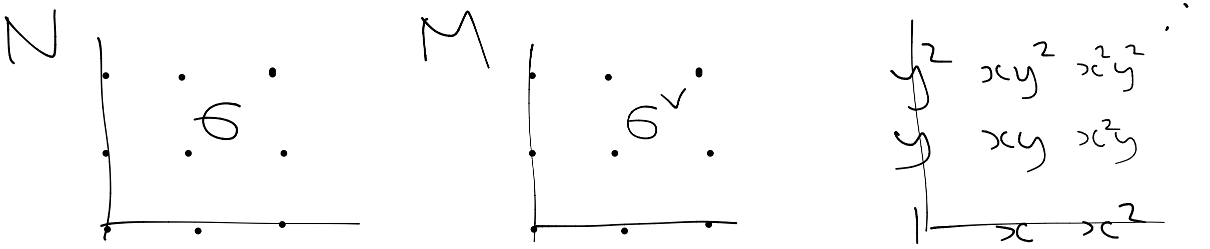
This gives a <u>monoid</u> $M_{\sigma^\vee} := \sigma^\vee \cap M$.

<u>Note</u> a monoid has an associative addition with an identity element, but does not have inverses.

<u>Def</u> ring $\mathbb{C}[\sigma^\vee]$ has generators (as a vector space) z^m for $m \in M_{\sigma^\vee}$ and multiplication $z^{m_1} \cdot z^{m_2} = z^{m_1+m_2}$

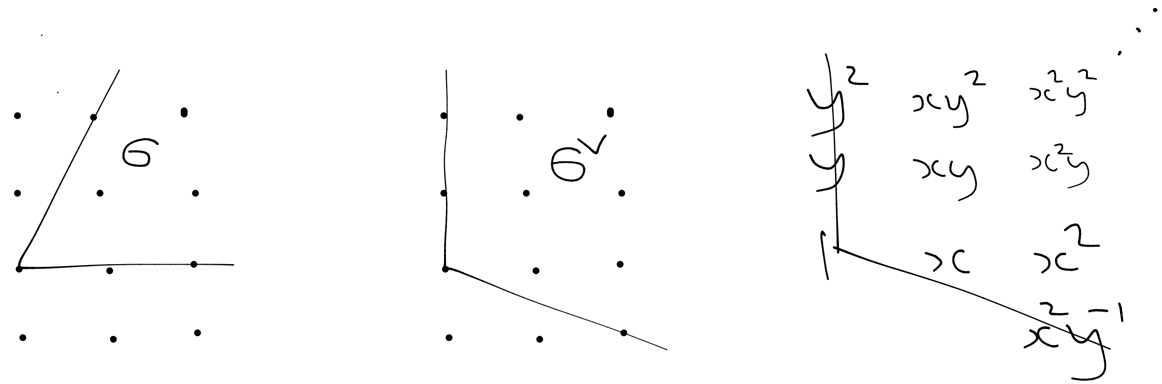
<u>Rem</u> Another way to say this is we associate monomials to elements of dual lattice M, as below.

Ex



here we write $z^{(i,j)} = x^i y^j$, and get $\mathbb{C}[\sigma^\vee] = \mathbb{C}[x,y]$

Ex



here we get $\mathbb{C}[\sigma^\vee] = \mathbb{C}[x,y,x^2y^{-1}] = \mathbb{C}[a,b,c]/a^2-bc$

Note For this example $\mathrm{Spec}\,\mathbb{C}[\sigma^\vee] \cong \mathbb{C}^2/\mathbb{Z}_2$.

Def The affine toric variety for cone σ is

$$U_\sigma := \mathrm{Spec}\,\mathbb{C}[\sigma^\vee]$$

General toric varieties

We now explain how the fan naturally gives a recipe to glue the U_σ together.

Note that

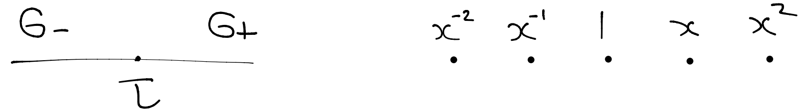
- $\operatorname{Spec}(-)$ is a contraviant functor, that is a morphism $R \to S$ gives a morphism $\operatorname{Spec} S \to \operatorname{Spec} R$.
- The duality $(-)^\vee$ is also contravariant: an inclusion $\tau \hookrightarrow \sigma$ gives an inclusion $\sigma^\vee \hookrightarrow \tau^\vee$.

Combining these, we have that U_σ depends covariantly on σ in the following sense:

$$\boxed{\text{If } \sigma \text{ has face } \tau \hookrightarrow \sigma, \text{ have open embedding } U_\tau \hookrightarrow U_\sigma}$$

The embedding here comes for a localization $\mathbb{C}[\sigma^\vee] \hookrightarrow \mathbb{C}[\tau^\vee]$. For details, see [R, Section 2.2]. We give examples.

<u>Ex</u> $N_\mathbb{R} \cong \mathbb{R}$



As before, $\mathbb{C}[\sigma_+^\vee] = \mathbb{C}[x]$, $\mathbb{C}[\sigma_-^\vee] = \mathbb{C}[x^{-1}]$

For their mutual face $\tau = \{0\} \subset N_\mathbb{R}$

$$\tau^\vee = M_\mathbb{R} \cong \mathbb{R}$$

So $\mathbb{C}[\tau^\vee] = \mathbb{C}[x, x^{-1}]$, with obvious inclusion of $\mathbb{C}[\sigma_\pm^\vee]$.

Now $\operatorname{Spec} \mathbb{C}[x] \cong \mathbb{C}$, $\operatorname{Spec} \mathbb{C}[x, x^{-1}] \cong \mathbb{C}^*$ so we have embeddings as follows:

$$\begin{tikzcd} U_{\sigma_-} \cong \mathbb{C} & U_\tau \cong \mathbb{C}^* \arrow[l, hook', "x^{-1}"'] \arrow[r, hook, "x"] & U_{\sigma_+} \cong \mathbb{C} \end{tikzcd}$$

coord x^{-1} coords x^{-1}, x coord x

Now we may glue the U_σ using these embeddings.

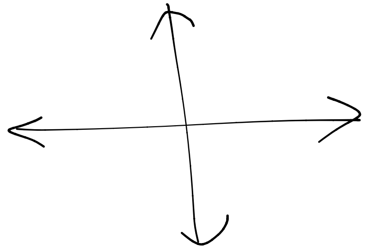
<u>Def</u> The <u>toric variety</u> X_Σ for a fan Σ is a union of U_σ for all $\sigma \in \Sigma$, where if σ_1 and σ_2 share a face τ we glue U_{σ_1} and U_{σ_2} along U_τ.

<u>Ex</u> Continuing the previous example, we get

$$X_\Sigma = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C} = \mathbb{P}^1$$

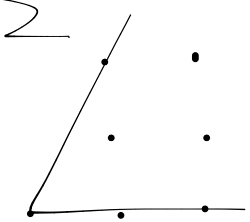
<u>Note</u> By construction have $\mathbb{C}^{*d} \cong U_\tau \hookrightarrow X_\Sigma$ for $\tau = \{0\}$, which allows us to show X_Σ is toric variety according to our first definition (with $\mathbb{C}^{*d}$-action)

Ex Σ

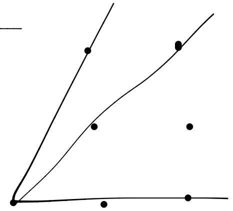


$X_\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$

Ex Σ



Σ'

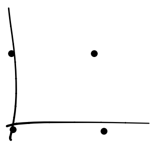


$X_\Sigma = \mathbb{C}^2/\mathbb{Z}_2$ $\quad X_{\Sigma'} = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$

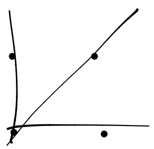
minimal resolution

$\cong$ total space of $\mathcal{O}(-2)$ on $\mathbb{P}^1$

Ex Σ



Σ'



$X_\Sigma = \mathbb{C}^2$ $\quad X_{\Sigma'} = Bl_p\, \mathbb{C}^2 \cong$ total space of $\mathcal{O}(-1)$ on $\mathbb{P}^1$

Structure of toric varieties

Recall that a toric variety X carries an action of a torus $T \cong \mathbb{C}^{*d}$. When X comes from a fan Σ, there is a correspondence

$$\{\text{cones } \sigma \text{ of } \Sigma\} \longleftrightarrow \{T\text{-orbits in } X_\Sigma\}$$

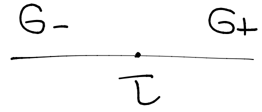
For details, see [R, §5]. In outline:
A cone σ gives an orbit $\mathcal{O}_\sigma \subset U_\sigma$. For the other direction, we take the minimal σ such that some orbit $\mathcal{O}$ satisfies $\mathcal{O} \subset U_\sigma$.

It is inclusion-reversing on closures, that is

$$\sigma \supset \tau \iff \overline{\mathcal{O}_\sigma} \subset \overline{\mathcal{O}_\tau}$$

and $\operatorname{codim} \mathcal{O}_\sigma = \dim \sigma$. We give some examples.

Ex $N_{\mathbb{R}} \cong \mathbb{R}$

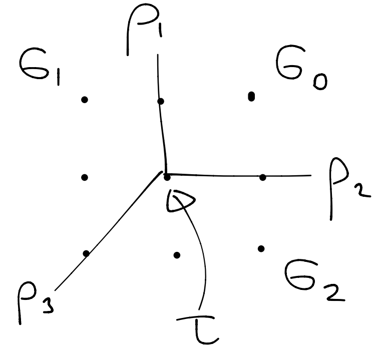


$$X_\Sigma = \mathcal{U}_{\sigma_-} \cup_{\mathcal{U}_\tau} \mathcal{U}_{\sigma_+} = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C} = \mathbb{P}^1$$

$$O_{\sigma_\pm} = \{0\} \subseteq \mathbb{C} = \mathcal{U}_{\sigma_\pm}$$

$$O_\tau = \mathbb{C}^* \subseteq \mathbb{C}^* = \mathcal{U}_\tau$$

Ex $N_{\mathbb{R}} \cong \mathbb{R}^2$



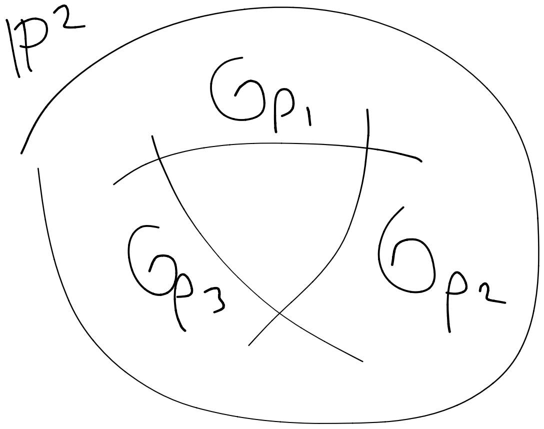
$$X_\Sigma = \mathbb{P}^2 = \{(x_0 : x_1 : x_2)\}$$

$O_{\sigma_i} = \{(0,0)\} \subseteq \mathbb{C}^2 \subset \mathbb{P}^2$ where $\mathbb{C}^2$ is $\{x_i \neq 0\}$

$O_\tau = \{x_j \neq 0 \ \forall j\} = \mathbb{C}^{*2} \subset \mathbb{P}^2$

For ρ_i contained in σ_j and σ_k, $\overline{O_{\rho_i}} \supset O_{\sigma_j}, O_{\sigma_k}$, so $O_{\rho_i} = \mathbb{C}^* \subset \mathbb{P}^1$, the line joining O_{σ_j} and O_{σ_k}.

We thus have a schematic picture $\longrightarrow$



The O_σ and U_σ are beautifully related as follows.

$$\overline{O}_\sigma = \bigcup_{\substack{\sigma \text{ is a} \\ \text{face of } \tau}} O_\tau \qquad U_\sigma = \bigcup_{\substack{\sigma \text{ has a} \\ \text{face } \tau}} O_\tau$$

<u>Note</u>: for this, we consider σ to be a face of σ.

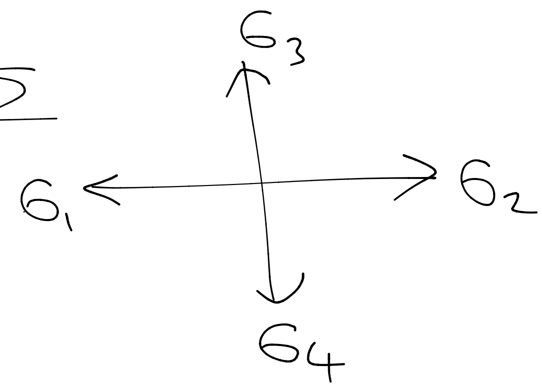
<u>Divisors</u>

In particular 1-dimensional cones $\sigma \subset \Sigma(1)$ correspond to T-invariant divisors $D_\sigma = \overline{O}_\sigma \subset X_\Sigma$.

Here we write $\Sigma(d) = \{d\text{-dimensional cones}\} \subset \Sigma$

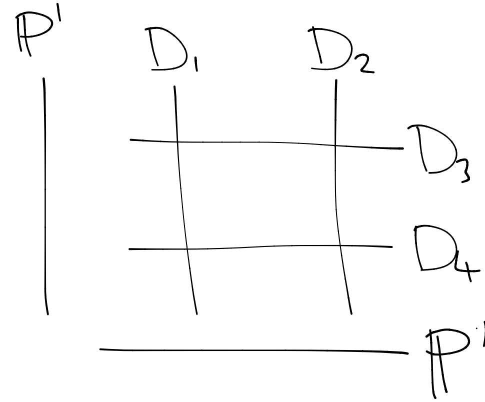
Note By the above D_{σ_1} and D_{σ_2} intersect if the σ_i are faces of some $\tau \in \Sigma$.

Ex Σ



$X_\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$

Let $D_i = D_{\sigma_i}$



Fact The canonical bundle ω_X corresponds to the divisor $-\sum_\sigma D_\sigma$ (see [CK, §3.5])

In other words, the anticanonical divisor $-K_X$ is given by $\sum_\sigma D_\sigma$

Ex $X = \mathbb{P}^2$, $-K_X = \sum_{i=1}^{3} D_{\rho_i}$, corresponding to $\mathcal{O}_{\mathbb{P}^2}(3)$.