More properties of the Seiberg-Witten invariants

- Blow-up formula

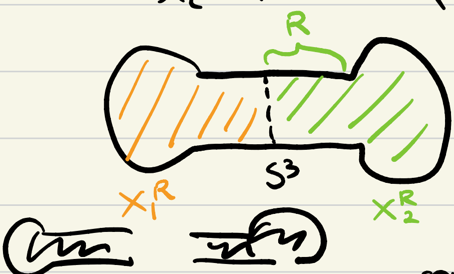
$X = X_1 \# \overline{\mathbb{CP}}^2$ $\qquad$ $H^2(X) \cong H^2(X_1) \oplus \mathbb{Z}\cdot \underset{\text{exceptional divisor}}{P.D.(E)}$

$\mathrm{Char}(X) = \{(k,n) \mid k \in \mathrm{Char}(X_1),\ n \text{ odd.}\}$

Theorem: $SW(k,n) = \begin{cases} SW(k) & n = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$

proof: $X = X_1 \# X_2 = X_1^R \cup_{S^3} X_2^R \qquad X_1 = X_1^R \cup_{S^3} D^4 \qquad X_2 = X_2^R \cup_{S^3} D^4$

$X_i^R = (X_i \setminus \mathring{D}^4) \cup ([0,R] \times S^3)$



Neck stretching: $R \to \infty$

$X \rightsquigarrow X_1^\infty \cup X_2^\infty$

$X_i^\infty = (X_i \setminus D^4) \cup ([0,+\infty) \times S^3)$

So solution on $X \overset{\text{converges}}{\rightsquigarrow}$ (solution on X_1^∞, solution on X_2^∞)

similarly solution on $X_i \overset{\text{converges}}{\rightsquigarrow}$ solution on X_i^∞

Therefore, when $R \gg 0$, we get

$P_X \cong P_{X_1} \times P_{X_2}$ $\qquad$ Here $P_X = \{\text{solutions}\}/\mathcal{G}^0$ $\qquad$ $S^1 \hookrightarrow P_X \to \mathcal{M}_{SW}$

$\mathcal{G}^0$: based gauge group $\{u : X \to S^1 \mid u(b) = 1\}$

Now suppose $X_2 = \overline{\mathbb{CP}}^2$ and $c_1(\mathfrak{s}_2) = \pm P.D.(E)$

Then $P_{X_2} = \{\text{a single reducible solution}\}$

So we get S^1-equivariant diffeomorphism $P_X \cong P_{X_1}$

so $SW(\mathfrak{s}_1 \# \mathfrak{s}_2) = SW(\mathfrak{s}_1)$.

Generalization:

Let $X = X_1 \cup_Y X_2$, $X' = X_1 \cup_Y X_3$.

Suppose 1) $b_2^+(X_2) = b_2^+(X_3) = b_1(X_2) = b_1(X_3) = 0$

2) Y is psc, $b_1(Y) = 0$ (e.g. Y = lens space)

3) $\mathfrak{s}_X|_{X_1} = \mathfrak{s}_{X'}|_{X_1}$

4) $d(\mathfrak{s}_X) = d(\mathfrak{s}_{X'}) = 0$

Then $SW_X(\mathfrak{s}_X) = SW_{X'}(\mathfrak{s}_{X'})$.

Example: Let X' be obtained from X by rational blow-down

Then $SW_{X'}(\mathfrak{s}_{X'}) = SW_X(\mathfrak{s}_X)$.

- Seiberg-Witten invariants for Kähler/algebraic surfaces.

Theorem (Witten) If X is Kähler, then $SW(c_1(TX)) = \pm 1$.

For now, let's assume that X is equipped with an almost complex structure $J: T_*X \to T_*X$. Then J defines a "canonical spin^c structure" $\mathfrak{s}_J$ with $c_1(\mathfrak{s}_J) = c_1(T_*X)$

$$\Lambda^* T^*X \otimes \mathbb{C} = \bigoplus \Lambda^{p,q}(X)$$

$$S^+ = \Lambda^{0,0}(X) \oplus \Lambda^{0,2}(X) = \underline{\mathbb{C}} \oplus \Lambda^{0,2}(X) \qquad S^- = \Lambda^{0,1}(X)$$

$$\rho: \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X) = T^*X \otimes \mathbb{C} \to \mathrm{Hom}(S^+, S^-)$$

$$\rho(v) \cdot \alpha = \sqrt{2}\,(v^{1,0} \wedge \alpha^{0,0} - v^{0,1} \cdot \alpha^{0,2})$$

inner product

$(1, 0)$

So a setion ϕ of S^+ is just a section of $\Lambda^{0,0} \oplus \Lambda^{0,2}$

When X is Kähler, the only point in $\mathcal{M}_{SW}(\mathfrak{s}_J)$ is

constant section of $\Lambda^{0,0} = \mathbb{C}$

$[(A_0, (1,0))]$

unique spin$^{\mathbb{C}}$ connection s.t. $\nabla^{A_0}(1,0) = 0$.

- Similary results holds for a general symplectic manifold.

I.e. If X is a symplectic manifold. then for any compatible almost complex structure J (i.e. $\omega(-, J-)$ is a metric) one has

$SW_X(c_1(T_*X, J)) = \pm 1$. (Taubes)

$SW_X(\mathfrak{s}_J) = \pm 1$.

So combining this with vanishing theorem last time, we get.

Corollary: If X is symplectic, and $X \cong_{diff} X_1 \# X_2$. then $b_2^+(X_1)$ or $b_2^+(X_2)$ must be 0.

Some more works shows the following:

Theorem (Kotschick, Taubes) Let X be simply connected, symplectic manifold. Then X is minimal $\Leftrightarrow$ X is irreducible.

(i.e. $X \neq X' \# \overline{\mathbb{CP}^2}$) $\quad$ ($X \neq_{diff} X_1 \# X_2$ unless X_1, or X_2 is homeomorphic to S^4).

$b^+ > 0 \quad b^+ > 0^{diff}$

$SW(\mathbb{CP}^2 \# \mathbb{CP}^2) = 0$

$b^+(X) > 1$
$\not\ell$

- $\Sigma \hookrightarrow X$ smoothly embedded, essential surface $([\Sigma] \neq 0 \in H_2(X;\mathbb{R}))$
- $\mathfrak{s} \in \mathrm{Spin}^c(X) \quad SW_X(\mathfrak{s}) \neq 0$

Theorem 1 (Kronheimer-Mrowka) If $\Sigma\cdot\Sigma \geq 0$, then

$$2g(\Sigma) - 2 \geq \Sigma\cdot\Sigma + |c_1(\mathfrak{s})\cdot[\Sigma]|$$

2 (Ozsvath-Szabo, Fintushel-Stern) Assume X is simple type, then same results holds even if $\Sigma\cdot\Sigma < 0$.

$$d(\mathfrak{s}) > 0 \Rightarrow SW_X(\mathfrak{s}) = 0$$

This is powerful in the minimal genus problem. We only prove 1, assuming $g(\Sigma) \neq 0$.

Step I:

- replace $\mathfrak{s} \to \bar{\mathfrak{s}}$, then we can assume $c_1(\mathfrak{s})\cdot\Sigma \geq 0$
- replace $X \rightsquigarrow X \# \overline{\mathbb{CP}}^2 \quad \mathfrak{s} \rightsquigarrow \mathfrak{s} \# \mathfrak{s}_0$ with $c_1(\mathfrak{s}_0) = -P.D.(E)$

$$\Sigma \rightsquigarrow \Sigma \# E.$$

Then $SW(\mathfrak{s}\#\mathfrak{s}_0) = SW(\mathfrak{s}) \neq 0$

$$c_1(\mathfrak{s}\#\mathfrak{s}_0)\cdot[\Sigma \# E] = c_1(\mathfrak{s})\cdot[\Sigma] + c_1(\mathfrak{s}_0)\cdot[E]$$
$$= c_1(\mathfrak{s})\cdot[\Sigma] + 1$$

$$(\Sigma\#E)\cdot(\Sigma\#E) = \Sigma\cdot\Sigma - 1$$

So the equality holds for $\Sigma \Leftrightarrow$ it holds for $\Sigma \# E$.

By repeating this $\Sigma\cdot\Sigma$ times, we may assume $\Sigma\cdot\Sigma = 0$ and want to prove $2g(\Sigma) - 2 \geq c_1(\Sigma)\cdot\Sigma$.

Lemma: Let (A,ϕ) be a solution. Then $2\sqrt{2}\,\|F_{A^t}^+\| \le \|s\|$

Here $s: X \to \mathbb{R}$ is the scalar curvature. $\|\cdot\|$ denotes the L^2-norm.

proof: $F_{A^t}^+ = \rho^{-1}(\phi\phi^*)_0$

For $x \in X$, pick an orthonormal basis $\{\frac{\phi(x)}{|\phi(x)|}, \vec{v}\}$ of S_x^+. Then $(\phi\phi^*)_0(x)$ can be expressed as $\begin{pmatrix} \frac{1}{2}|\phi(x)|^2 & 0 \\ 0 & -\frac{1}{2}|\phi(x)|^2 \end{pmatrix}$

so $|(\phi\phi^*)_0(x)| = \frac{1}{2}|\phi(x)|^2$

Note: $\rho: \Lambda_+^2 T_xX \longrightarrow \mathrm{End}(S^+)$ changes the norm by $\sqrt{2}$

so $|F_{A^t}^+(x)| = \frac{1}{2\sqrt{2}}|\phi(x)|^2$

so $\|F_{A^t}^+\|^2 = \frac{1}{8}\int|\phi|^4$

We have proved $\langle \nabla_A^*\nabla_A\phi, \phi\rangle = -\frac{s}{4}|\phi|^2 - \frac{1}{4}|\phi|^4$

so $0 \le \int|\nabla_A\phi|^2 = \int\langle\nabla_A^*\nabla_A\phi,\phi\rangle = \int -\frac{s}{4}|\phi|^2 - \frac{1}{4}|\phi|^4$

so $\int|\phi|^4 \le \int(-s)|\phi|^2 \le \sqrt{\int s^2 \cdot \int|\phi|^4}$

$\Rightarrow \|s\|^2 = \int s^2 \ge \int|\phi|^4 = 8\|F_{A^t}^+\|^2$ $\square$

Lemma: $\|F_{A^t}\|^2 = 2\|F_{A^t}^+\|^2 - 4\pi^2 c_1(\mathfrak{s})^2$

proof: $c_1(\mathfrak{s}) = \frac{i}{2\pi}F_{A^t}$ so

$4\pi^2 c_1(\mathfrak{s})^2 = \int F_{A^t}\wedge F_{A^t} = \int F_{A^t}^+\wedge F_{A^t}^+ + F_{A^t}^-\wedge F_{A^t}^-$

$\|F_{A^t}\|^2 = \int F_{A^t}\wedge * F_{A^t} = \int F_{A^t}^+\wedge F_{A^t}^+ - F_{A^t}^-\wedge F_{A^t}^-$ $\square$

SO $\|F_{A^+}\|^2 = 2\|F_{A^+}^+\|^2 - 4\pi^2 c_1(\mathfrak{s})^2 \leq \frac{1}{4}\|S\|^2 + \text{constant}$

Now decompose

$$X = (X - \nu(\Sigma)) \cup (\underset{\substack{\shortparallel \\ C}}{}\![0,R]\times S^1\times\Sigma) \cup (\Sigma\times D^2)$$

Fix a metric on $X-\nu(\Sigma)$, $\Sigma\times D^2$

Let g_C = product metric $[0,R]\times S^1\times g_\Sigma$

where $\quad \mathrm{vol}(S^1) = \mathrm{vol}(g_\Sigma) = 1$, $K(g_\Sigma)$ = constant K_0

Gauss-Bonnet $\int K = 2\pi(2-2g)$ so $K_0 = 2\pi(2-2g)$

On C, we have $S = 2K_0 = 4\pi(2-2g)$

so $\frac{1}{4}\|S\|^2 = \frac{1}{4}\int_C |S|^2 + \int_{X\setminus C} S = R(2\pi(2-2g))^2 + \text{constant}$

$$\|F_{A^+}\|^2 = \int_C \|F_{A^+}\|^2 + \int_{X\setminus C} \|F_{A^+}\|^2$$

$$= \int_{S^1}\int_{[0,R]}\int_{a\times t\times\Sigma} \|F_{A^+}\|^2 + \text{constant}$$

Note $\int_{a\times t\times\Sigma} \|F_{A^+}\|^2 = \int_{a\times t\times\Sigma} \|F_{A^+}\|^2 \cdot \int_{a\times t\times\Sigma} 1$

$$\geq \Big(\int_{a\times t\times\Sigma} |F_{A^+}|\Big)^2 \geq \Big|\int_{a\times t\times\Sigma} F_{A^+}\Big|^2 = \big(2\pi(c_1(\mathfrak{s})\cdot[\Sigma])\big)^2$$

so $\|F_{A^+}\|^2 \geq 4\pi^2 R\,(c_1(\mathfrak{s})\cdot[\Sigma])^2 + \text{constant}$

$\frac{1}{4}\|S\|^2 = 4\pi^2 R\,(2-2g)^2 + \text{constant}$

$R\to+\infty$, we get $\quad c_1(\mathfrak{s})\cdot[\Sigma] \leq 2g-2 \qquad \square$

Why do we call $2g-2 \le \Sigma\cdot\Sigma + |c_1(TX)[\Sigma]|$ the adjunction inequality?

Lemma: Let J be an almost complex structure on X.
Suppose $\Sigma \hookrightarrow X$ is a psedo holomorphic curve (i.e. $J(T_x\Sigma) \subset T_x\Sigma \ \forall x \in \Sigma$) then we have the adjunction formula

$$2g(\Sigma)-2 = \Sigma\cdot\Sigma - c_1(TX)\cdot[\Sigma]$$

proof: $TX|_\Sigma = T\Sigma \oplus N\Sigma$

since $J(T\Sigma) \subset T\Sigma$, this is actually a complex decomposition

so $c_1(TX)\cdot[\Sigma] = c_1(T\Sigma)\cdot[\Sigma] + c_1(N\Sigma)\cdot[\Sigma]$

$= e(T\Sigma)\cdot[\Sigma] + e(N\Sigma)\cdot[\Sigma]$

$= (2-2g) + \Sigma\cdot\Sigma.$ $\square$.

Theorem (The symplectic Thom conjecture, Ozsvath-Szabo 1998)
Let (X,ω) be a symplectic 4-manifold. Let $\Sigma \xhookrightarrow{i} X$ be a symplectic surface. then Σ is genus minimizing in it's homology class.
(i.e. $i^*(\omega)$ is nowhere vanishing)

Proof: Since Σ is symplectic, $\exists J$ s.t. Σ is psedoholomorphic

so $2g(\Sigma)-2 = \Sigma\cdot\Sigma - c_1(TX)\cdot\Sigma$

Note $\int_\Sigma \omega > 0$ so $[\Sigma] \neq 0 \in H_2(X;\mathbb{R})$

Let $S \hookrightarrow X$ be another surface with $[S]=[\Sigma]$.

Taubes's theorem : $SW_X(c_1(T^*X)) = \pm 1 \neq 0$

X is of simple type.

so S satisfies

$$2g(S)-2 \geqslant S\cdot S - c_1(TX)\cdot S = \Sigma\cdot\Sigma - c_1(TX)\cdot\Sigma = 2g(\Sigma)-2 . \square$$