PERFECTOIDS AND GALOIS COHOMOLOGY: A PEDAGOGICAL INTRODUCTION TO p-ADIC HODGE THEORY

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Abstract. Since Tate proposed the famous Hodge-Tate decomposition conjecture in the 1960s, p-adic Hodge theory has undergone profound and continuous development over the subsequent sixty years, with new ideas and tools constantly emerging. Among these, the theory of perfectoid rings is one of the most striking breakthroughs and has gradually become a foundational language for understanding modern p-adic geometry.

These lecture notes start from a historical perspective to explain the role and status of perfectoids in p-adic geometry, and uses this as a main thread to introduce the basic framework and core ideas of p-adic Hodge theory. We will present the deep and beautiful techniques of p-adic geometry to graduate students and advanced undergraduates in a friendly and detailed manner.

More specifically, we begin with how Tate used local class field theory to compute Galois cohomology in the discretely valued case, then introduce the notion of perfectoids and prove several key results, including the tilting correspondence, cohomological descent in the arc topology, and the almost purity theorem. Using these tools, we compute the cohomology of the fundamental group of smooth algebraic varieties, which has been a central topic of p-adic Hodge theory over the past sixty years. Finally, we discuss the extension of these methods to general (non-discrete) valuation rings and look ahead to the future development of p-adic Hodge theory.

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1. A Glance at p-adic Hodge Theory

1.a. **Hodge decomposition.** One of the most fundamental theorem in complex geometry concerning about the singular cohomology of complex manifolds is the following so-called *Hodge decomposition.*

Theorem 1.1. *Let X be a projective smooth variety over $\mathbb{C}$. Then, there is a canonical decomposition*

$$H^n_{\mathrm{sing}}(X,\mathbb{C}) \cong \bigoplus_{i+j=n} H^j(X,\Omega^i_{X/\mathbb{C}}). \tag{1.1.1}$$

The standard proof used essentially techniques in analysis: the n-th de Rham cohomology classes are represented uniquely by n-th harmonic forms ([Voi02, 5.23]), and the latter can be decomposed into direct sums of (i,j)-type harmonic forms ([Voi02, 6.10]), the space of which is canonically isomorphic to $H^j(X,\Omega^i_{X/\mathbb{C}})$ ([Voi02, 6.18]).

Let's take a view from the p-adic geometry.

1.b. $\mathbb{C}$ **and** $\mathbb{C}_p$**.** If we complete the field of rational numbers $\mathbb{Q}$ with respect to the archimedean norm, we obtain the field of real number $\mathbb{R}$; if we complete $\mathbb{Q}$ with respect to a non-archimedean norm, we obtain the field of p-adic numbers $\mathbb{Q}_p$. Recall that $\mathbb{R}$ and $\mathbb{Q}_p$ are the only two types of completions that $\mathbb{Q}$ has by a theorem of Ostrowski.

Recall that the non-archimedean norm on $\mathbb{Q}_p$ corresponds to the discrete valuation ring

$$\mathbb{Z}_p = \lim_{n\to\infty} \mathbb{Z}/p^n\mathbb{Z} \tag{1.1.2}$$

where the valuation map is

$$\begin{aligned} v_p : \mathbb{Z}_p &\longrightarrow \mathbb{N}\cup\{\infty\}, \\ p^n u &\longmapsto n,\ \forall n\in\mathbb{N} \text{ and } u\in\mathbb{Z}_p^\times, \\ 0 &\longmapsto \infty. \end{aligned} \tag{1.1.3}$$

The discrete valuation field $\mathbb{Q}_p$ is the fraction field of $\mathbb{Z}_p$ given by inverting p: $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. We refer to [Bou06a] for basic theory on valuation rings.

Taking an algebraic closure of $\mathbb{R}$, we obtain the field of complex numbers $\mathbb{C}$ which has degree 2 over $\mathbb{R}$; taking an algebraic closure of $\mathbb{Q}_p$, we obtain an infinite Galois extension $\overline{\mathbb{Q}}_p$. Notice that $\overline{\mathbb{Q}}_p$ is still a valuation field (but not discrete) with respect to the valuation ring $\overline{\mathbb{Z}}_p$, where the latter is the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$ (see [Bou06a, VI, §8.6, Proposition 6] and [Sta26, 04GH]). The extended valuation map is

$$\begin{aligned} v_p : \overline{\mathbb{Z}}_p &\longrightarrow \mathbb{Q}_{\geq 0}\cup\{\infty\}, \\ x &\longmapsto v_p(\mathrm{N}_{\mathbb{Q}_p(x)/\mathbb{Q}_p}(x))/[\mathbb{Q}_p(x):\mathbb{Q}_p]. \end{aligned} \tag{1.1.4}$$

But $\overline{\mathbb{Q}}_p$ is not complete with respect to its valuation (i.e., $\overline{\mathbb{Z}}_p$ is not p-adically complete, $\overline{\mathbb{Z}}_p \neq \widehat{\overline{\mathbb{Z}}}_p = \lim_{n\to\infty} \overline{\mathbb{Z}}_p/p^n\overline{\mathbb{Z}}_p$ as $1+p^{1+1/p}+p^{2+1/p^2}+\cdots \in \widehat{\overline{\mathbb{Z}}}_p$ is transcendental over $\mathbb{Q}_p$). We put $\mathbb{C}_p = \widehat{\overline{\mathbb{Z}}}_p[1/p]$, which is a complete algebraically closed valuation field by Krasner's lemma.

Notice that $\mathbb{C}$ and $\mathbb{C}_p$ have the same cardinalities as that of $\mathbb{R}$. Hence, they have the same transcendental degree $\mathfrak{a}$ over $\mathbb{Q}$ and thus they are both algebraic closures of the purely transcendental extension $\mathbb{Q}(T_i|i\in\mathfrak{a})$ of $\mathbb{Q}$ (see [Sta26, 030D, 09GV]). In conclusion, there is a field isomorphism

$$\mathbb{C} \cong \mathbb{C}_p. \tag{1.1.5}$$

Although these two fields are isomorphic, the way they are defined actually endows them totally different topology. It is clear that the Euclidean topology on $\mathbb{C}$ is connected, while the non-archimedean topology on $\mathbb{C}_p$ is totally disconnected.

1.c. **Hodge-Tate decomposition.** The same thing happens to a projective smooth variety X over $\mathbb{C}_p$. When fixing a field isomorphism $\mathbb{C} \cong \mathbb{C}_p$, we have an isomorphism of schemes $X_{\mathbb{C}} \cong X$. However, the Euclidean topology on $X_{\mathbb{C}}$ as a manifold is totally different from the étale or Zariski topology on X as a scheme.

But a surprising fact that these two different topology actually give the same cohomological invariants (which thus reflects the geometric nature of X) as long as we fix $\mathbb{C} \cong \mathbb{C}_p$:

$$H^n_{\mathrm{sing}}(X_{\mathbb{C}}, \mathbb{C}) \cong H^n_{\text{ét}}(X, \mathbb{C}_p), \tag{1.1.6}$$

where the latter is defined as $\mathbb{C}_p \otimes_{\mathbb{Z}_p} (\lim_{r\to\infty} H^n_{\text{ét}}(X, \mathbb{Z}/p^r\mathbb{Z}))$. This is Artin's comparison theorem, see [SGA 4_{III}, XI.4.4].

Therefore, the terms involved in the Hodge decomposition (1.1.1) actually come from algebraic geometry and Theorem 1.1 implies that

$$H^n_{\text{ét}}(X, \mathbb{C}_p) \cong \bigoplus_{i+j=n} H^j(X, \Omega^i_{X/\mathbb{C}}). \tag{1.1.7}$$

A priori, this isomorphism depends on the arbitrary choice of the field isomorphism $\mathbb{C} \cong \mathbb{C}_p$. But both sides are algebraic, we naturally ask

Question 1.2. *Is there a purely algebraic proof or a canonical construction of* (1.1.7)*? If so, how is the valuation ring structure $\mathbb{Q}_p \supseteq \mathbb{Z}_p$ involved here?*

This question is the central theme of p-adic Hodge theory. It started by Tate [Tat67], where he explained what does the "canonical construction" should mean and solve the question for abelian varieties over a finite extension K of $\mathbb{Q}_p$ with good reductions. Although it looks like a very special case, his strategy is generalized greatly by Faltings [Fal88] to solve the question for proper smooth varieties over K. Thus, the canonical decomposition (1.1.7) is also called the *Hodge-Tate decomposition*. While Tate's proof specializes only to abelian varieties, Faltings invented a bunch of new techniques to realize Tate's strategy over general smooth varieties, including *almost purity theorem* and *Galois cohomology computation*. Nowadays, Faltings' techniques have been developed and subsumed within *perfectoid theory* after Scholze [Sch12, Sch13a], which we are going to explain to graduate and undergraduate students in a friendly and detailed manner in this lecture series.

It would be too technical and difficult if we start directly with these deep techniques. Instead, we begin with Tate's groundbreaking work [Tat67] to trace the origins of these modern techniques.

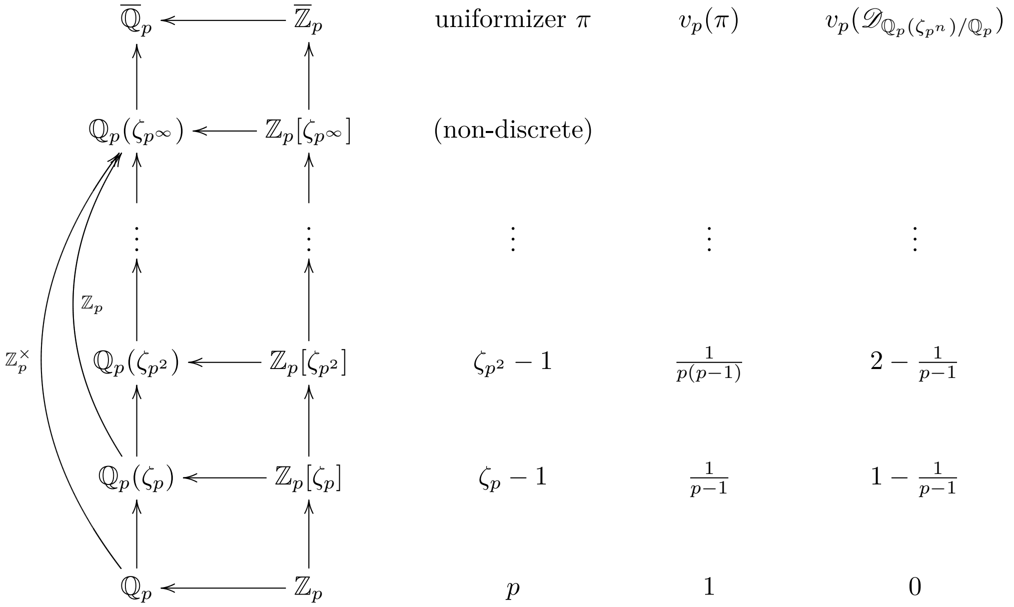
Question 1.2 essentially requires a good understanding of the p-adic cohomology. The key making p-adic cohomology distinguished is the valuation ring structure $\mathbb{Q}_p \supseteq \mathbb{Z}_p$ and the ramification above it. Before we try to understand ramification above X following Faltings, let's simply understand ramification above the single point $\mathbb{Q}_p$ following Tate.

1.d. **Ramification of $\overline{\mathbb{Q}}_p$ over $\mathbb{Q}_p$.**

Example 1.3. Consider a compatible system of primitive p^n-th roots of unity $(\zeta_{p^n})_{n\in\mathbb{N}}$, i.e., $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ and $\zeta_p \neq \zeta_1 = 1$. Using basics in algebraic number theory, we can prove that $\mathbb{Q}_p(\zeta_{p^n})$ is a totally ramified extension of $\mathbb{Q}_p$ with valuation ring $\mathbb{Z}_p[\zeta_{p^n}]$ (see [Ser79] or [He25a, 5.4]). This explicit expression of valuation rings (or integral closures) enables us to compute every invariant about the ramification behavior. For instance, $\zeta_{p^n} - 1$ is a uniformizer of $\mathbb{Q}_p(\zeta_{p^n})$ with valuation

$v_p(\zeta_{p^n} - 1) = \frac{1}{p^{n-1}(p-1)}$, and the valuation of the different ideal $\mathscr{D}_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}$ is $n - \frac{1}{p-1}$ for $n \geq 1$.

(1.3.1)



But how to understand ramification above $\mathbb{Q}_p(\zeta_{p^\infty})$? Tate answers this question by the following theorem.

Theorem 1.4 ([Tat67, §3])**.** *Let K be a complete discrete valuation field extension of $\mathbb{Q}_p$, K_∞ a totally ramified $\mathbb{Z}_p$-extension of K. Let K_n be the subfield of K_∞ corresponding to the closed subgroup $p^n\mathbb{Z}_p$ of $\mathrm{Gal}(K_\infty/K) = \mathbb{Z}_p$ for any $n \in \mathbb{N}$.*

(1) *(Regular ramification) There is a constant c and a bounded sequence $(a_n)_{n\in\mathbb{N}}$ of integers such that for any $n \in \mathbb{N}$, the valuation of the different ideal*

$$v_p(\mathscr{D}_{K_n/K}) = n + c + p^{-n}a_n. \tag{1.4.1}$$

(2) *(Almost unramification) For any finite field extension L of K, if we denote by L_n the composite of L with K_n for any $n \in \mathbb{N} \cup \{\infty\}$. Then,*

$$v_p(\mathscr{D}_{L_n/K_n}) \to 0 \text{ when } n \to \infty. \tag{1.4.2}$$

In other words, $\mathscr{D}_{L_\infty/K_\infty}$ ([He25a, 4.1.2]) *is equal to $\mathfrak{m}_{L_\infty}$ or $\mathcal{O}_{L_\infty}$ (we call L_∞ is* almost unramified *over K_∞).*

Remark 1.5.
(1) Tate's proof to these results essentially used higher ramification groups and local class field theory.
(2) Tate used these results to compute the p-adic cohomology for $\mathbb{Q}_p$, i.e., $H^n_{\text{ét}}(\mathrm{Spec}(\mathbb{Q}_p), \mathbb{C}_p)$, see [Tat67, §3.3].
(3) Faltings adopted the same strategy to understand the ramification above a smooth variety X. Roughly speaking, for a small smooth algebra R over $\mathbb{C}_p$, he constructed a "regularly ramified" tower $R \to R_\infty$ such that there is no more ramification beyond R_∞ in the almost sense. In fact, this R_∞ is "pre-perfectoid" and we will show the almost purity theorem for perfectoid rings and Galois cohomology computation for this specific tower $R \to R_\infty$.

2. Definition of Perfectoids

2.a. **Review of deformation theory.** We refer to [Ill71] and [Ill72] for a systematic development of deformation theory and suggest to read Grothendieck's definitions of smoothness [EGA IV$_4$, §17] or Illusie's expository notes [Ill96, §1,2] at first before jumping into the most general theory.

Recall that a *thickening* of affine schemes is a closed immersion $\mathrm{Spec}(R_0) \to \mathrm{Spec}(R)$ such that $R_0 = R/I$ with $I^2 = 0$. For example, each closed immersion in $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}/p^3\mathbb{Z}) \to \cdots$ is a thickening.

Question 2.1. *Given a flat R_0-algebra A_0, is there a flat R-algebra A with $A_0 = A \otimes_R R_0$?*

(2.1.1)
$$\begin{array}{ccc} \mathrm{Spec}(A) & \dashleftarrow & \mathrm{Spec}(A_0) \\ \downarrow & \llcorner & \downarrow \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R_0) \end{array}$$

Example 2.2. Consider the baby case $A_0 = R_0[T]$. Then, there is an obvious lifting $A = R[T]$.

(2.2.1)
$$\begin{array}{ccc} R[T] & \longrightarrow & R_0[T] \\ \uparrow & & \uparrow \\ R & \longrightarrow & R_0 \end{array}$$

In fact, any flat lifting of $R_0[T]$ is isomorphic to $R[T]$: let A' be a flat R-algebra with $A'/IA' = R_0[T]$. Then, we consider the R-algebra homomorphism $R[T] \to A'$ sending T to $T' \in A'$ a lifting of $T \in R_0[T]$. It is an isomorphism by the exact sequence $0 \to IA' \to A' \to A'/IA' \to 0$ and the identity $IA' = I \otimes_R A' = I \otimes_{R_0} A'/IA'$. Moreover, the automorphism group of the flat lifting $R[T]$ is isomorphic to $IA = I \otimes_{R_0} A_0$, where each $a \in IA$ corresponds to the automorphism sending T to $T + a$.

In general, there is a standard simplicial resolution of A_0 by free algebras over R_0 ([Ill71, I.1.5.5.6], see also [Sta26, 08N8])

(2.2.2)
$$\cdots \rightrightarrows P_1 = R_0[R_0[A_0]] \rightrightarrows P_0 = R_0[A_0] \longrightarrow A_0.$$

The *cotangent complex* of A_0 over R_0 is the associated complex of A_0-modules ([Ill71, II.1.2.3], see also [Sta26, 08PL])

(2.2.3)
$$\mathbb{L}_{A_0/R_0} = (\cdots \to \Omega^1_{P_1/R_0} \otimes_{P_1} A_0 \to \Omega^1_{P_0/R_0} \otimes_{P_0} A_0).$$

Theorem 2.3 ([Ill71, III.2.1.2.3])**.** *For the lifting problem* 2.1*, we have:*

(1) *There is an element $\omega \in \mathrm{Ext}^2_{A_0}(\mathbb{L}_{A_0/R_0}, A_0 \otimes_{R_0} I)$, which vanishes if and only if there exists a flat lifting A.*
(2) *When $\omega = 0$, the set of isomorphism classes of all the flat liftings A is a torsor under* $\mathrm{Ext}^1_{A_0}(\mathbb{L}_{A_0/R_0}, A_0 \otimes_{R_0} I)$.
(3) *The automorphism group of a flat lifting A is canonical isomorphic to* $\mathrm{Ext}^0_{A_0}(\mathbb{L}_{A_0/R_0}, A_0 \otimes_{R_0} I)$.

In particular, when $A_0 = R_0[T]$, we see that $\mathbb{L}_{A_0/R_0} = \Omega^1_{A_0/R_0} \cong A_0$ is a free A_0-module of rank 1. Thus, we can deduce 2.2 from 2.3.

Question 2.4. *Given a morphism of flat R_0-algebras $f_0 : A_0 \to A'_0$ together with fixed flat R-algebras A and A' with $A_0 = A \otimes_R R_0$ and $A'_0 = A' \otimes_R R_0$, is there a morphism $f : A \to A'$ with $f_0 = f \otimes_R R_0$?*

(2.4.1)
$$\begin{array}{ccc} \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(A'_0) \\ \downarrow{\scriptstyle f} & \llcorner & \downarrow{\scriptstyle f_0} \\ \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(A_0) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R_0) \end{array}$$

Theorem 2.5 ([Ill71, III.2.2.2])**.** *For the lifting problem* 2.4*, we have:*

(1) *There is an element $\omega \in \mathrm{Ext}^1_{A_0}(\mathbb{L}_{A_0/R_0}, A'_0 \otimes_{R_0} I)$, which vanishes if and only if there exists a lifting f.*
(2) *When $\omega = 0$, the set of isomorphism classes of all the liftings f is a torsor under* $\mathrm{Ext}^0_{A_0}(\mathbb{L}_{A_0/R_0}, A'_0 \otimes_{R_0} I)$.

2.b. **Universal p-deformation: Witt rings.** We fix a *perfect* $\mathbb{F}_p$-algebra R in this subsection, i.e., the Frobenius map $\mathrm{Frob} : R \to R$ sending x to x^p is an isomorphism.

Lemma 2.6 ([GR03, 6.5.13.(i)])**.** *The cotangent complex $\mathbb{L}_{R/\mathbb{F}_p} = 0$ in the derived category of R-modules.*

Proof. The Frobenius induces an endomorphism of the standard resolution

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle \mathrm{Frob}} & & \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0. \end{array} \tag{2.6.1}$$

Since $\mathrm{Frob} : R \xrightarrow{\sim} R$ is an isomorphism, by the functoriality of cotangent complexes ([Ill71, II.1.2.3.2]), we see that $\mathrm{Frob} : \mathbb{L}_{R/\mathbb{F}_p} \to \mathbb{L}_{R/\mathbb{F}_p}$ is an isomorphism of complexes (this morphism does not coincides with (2.6.2) in the level of complexes). On the other hand, it coincides with the following morphism in the derived category of R-modules ([Ill71, II.1.2.6.2])

$$\begin{array}{ccccc} \cdots & \longrightarrow & \Omega^1_{P_1/\mathbb{F}_p} \otimes_{P_1} R & \longrightarrow & \Omega^1_{P_0/\mathbb{F}_p} \otimes_{P_1} R \\ & & \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle \mathrm{Frob}} \\ \cdots & \longrightarrow & \Omega^1_{P_1/\mathbb{F}_p} \otimes_{P_1} R & \longrightarrow & \Omega^1_{P_0/\mathbb{F}_p} \otimes_{P_1} R. \end{array} \tag{2.6.2}$$

Since $\mathrm{Frob}(\mathrm{d}x) = \mathrm{d}x^p = px^{p-1}\mathrm{d}x = 0$ for any $\mathrm{d}x \in \Omega^1_{P_n/\mathbb{F}_p}$ and $n \in \mathbb{N}$. We see that the isomorphism $\mathrm{Frob} : \mathbb{L}_{R/\mathbb{F}_p} \to \mathbb{L}_{R/\mathbb{F}_p}$ is the zero map in the derived category of R-module and thus $\mathbb{L}_{R/\mathbb{F}_p} = 0$. □

Proposition 2.7. *There exists a p-adically complete and flat $\mathbb{Z}_p$-algebra W with $W/pW = R$. Moreover, it is unique up to a unique isomorphism.*

Proof. By deformation theory (2.3 and 2.6), there is a unique flat $\mathbb{Z}/p^2\mathbb{Z}$-algebra R_2 with $R_2/pR_2 = R$. Consider the derived tensor product of $\mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})}$ with the exact sequence of R_2-modules $0 \to pR_2 \to R_2 \to R \to 0$, we obtain a distinguished triangle (where we used the fact that $R \otimes^{\mathrm{L}}_{R_2} \mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})} = \mathbb{L}_{R/\mathbb{F}_p}$ by [Ill71, II.2.2.1])

$$pR_2 \otimes^{\mathrm{L}}_{R} \mathbb{L}_{R/\mathbb{F}_p} \longrightarrow \mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})} \longrightarrow \mathbb{L}_{R/\mathbb{F}_p} \longrightarrow \tag{2.7.1}$$

which implies that $\mathbb{L}_{R_2/(\mathbb{Z}/p^2\mathbb{Z})} = 0$ by 2.6. Repeating this argument, we obtain unique (up to a unique isomorphism) flat liftings

$$\begin{array}{ccccccc} \cdots & \twoheadrightarrow & R_3 & \twoheadrightarrow & R_2 & \twoheadrightarrow & R_1 = R \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \twoheadrightarrow & \mathbb{Z}/p^3\mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p^2\mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \end{array} \tag{2.7.2}$$

with $\mathbb{L}_{R_n/(\mathbb{Z}/p^n\mathbb{Z})} = 0$ in the derived category.

Then, we take $W = \lim_{n\to\infty} R_n$. As $R_{n+1}/p^n R_{n+1} = R_n$ by construction, we have $W/p^nW = R_n$ for any $n \geq 1$ ([Sta26, 09B8]) and thus W is p-adically complete.

Consider the injection $\mathbb{Z}/p^{n-1}\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^n\mathbb{Z}$. Tensoring with the flat $\mathbb{Z}/p^n\mathbb{Z}$-module R_n, we obtain an injection $R_{n-1} \xrightarrow{\cdot p} R_n$. Taking inverse limit over $n \in \mathbb{N}$, we get an injection $W \xrightarrow{\cdot p} W$, in other words, W is p-torsion free (hence flat over $\mathbb{Z}_p$).

The uniqueness of W follows from that of the diagram (2.7.2). □

Definition 2.8. We denote by $W(R)$ the unique p-adically complete flat $\mathbb{Z}_p$-algebra with $W(R)/pW(R) = R$. We call it the *Witt ring* of the perfect $\mathbb{F}_p$-algebra R.

Remark 2.9. By deformation theory (2.5 and 2.6), any morphism of perfect $\mathbb{F}_p$-algebras $R \to R'$ lifts uniquely to a ring homomorphism $W(R) \to W(R')$. In other words, there is an equivalence of categories

$$\begin{aligned} \{p\text{-complete flat } \mathbb{Z}_p\text{-algebras } A \text{ with } A/pA \text{ perfect}\} &\cong \{\text{perfect } \mathbb{F}_p\text{-algebras } R\} \\ A &\mapsto A/pA \\ W(R) &\leftarrow R. \end{aligned} \tag{2.9.1}$$

Lemma 2.10. *There is a unique multiplicative section $[\ ] : R \to W(R)$ of the canonical surjection $W(R) \to R$.*

Proof. For any $x \in R$ and $n \in \mathbb{N}$, we take a lifting $y_n \in W(R)$ of $x^{1/p^n} \in R$. Since $(y_n + pz)^{p^n} \equiv y_n^{p^n}$ mod $p^nW(R)$ for any $z \in W(R)$, we see that the image of $y_n^{p^n}$ in $W(R)/p^nW(R)$ is a lifting of $x \in R$ independent of the choice of y_n. We take

$$[x] = \lim_{n\to\infty} y_n^{p^n} = (\dots, y_2^{p^2}, y_1^p, y_0) \in W(R) = \lim_{n\to\infty} W(R)/p^nW(R). \tag{2.10.1}$$

It is clear that $[\ ] : R \to W(R)$ is a well-defined multiplicative section of $W(R) \to R$. This verifies the existence.

For the uniqueness, let $[\ ]' : R \to W(R)$ be another multiplicative section. For any $x \in R$, we have $[x]' = [x] + py$ for some $y \in W(R)$. Taking p^n-th power, we get $[x^{p^n}]' \equiv [x^{p^n}] \mod p^n W(R)$. Since R is perfect, any element of R is of the form x^{p^n} for some $x \in R$. Thus, $[x]' \equiv [x] \mod p^n W(R)$ for any $x \in R$ and $n \in \mathbb{N}$. Taking inverse limit over $n \in \mathbb{N}$, we get $[x]' = [x]$ in $W(R)$. □

Proposition 2.11 (Teichmüler expansion). *For any $x \in W(R)$, there is a unique sequence $x_0, x_1, x_2, \dots$ in R such that*

$$x = [x_0] + p[x_1] + p^2[x_2] + \cdots . \tag{2.11.1}$$

In particular, $x \in W(R)^\times$ if and only if $x_0 \in R^\times$.

Proof. For any $x \in W(R)$, let x_0 be its image in R. Then, $x = [x_0] + px'$ for a unique $x' \in W(R)$ by the flatness of $W(R)$ over $\mathbb{Z}_p$. Inductively repeating the construction, we obtain the existence and uniqueness of the sequence $x_0, x_1, x_2, \dots$.

If $x \in W(R)^\times$, then its image $x_0 \in R$ is also a unit. The converse is also true since $W(R)$ is p-adically complete. □

Remark 2.12.
(1) (Frobenius) By 2.9, there is a unique ring isomorphism $F : W(R) \to W(R)$ lifting the Frobenius on R. In particular, for any $x = [x_0] + p[x_1] + p^2[x_2] + \cdots \in W(R)$, we have $F(x) = [x_0^p] + p[x_1^p] + p^2[x_2^p] + \cdots$.
(2) (Verschiebung) There is a canonical additive map $V = pF^{-1} : W(R) \to W(R)$ sending $x = [x_0] + p[x_1] + p^2[x_2] + \cdots \in W(R)$ to $V(x) = p[x_0^{1/p}] + p^2[x_1^{1/p}] + p^3[x_2^{1/p}] + \cdots \in W(R)$.
(3) (Witt vectors) There is a canonical bijection

$$W(R) \xrightarrow{\sim} \prod_{n=0}^{\infty} R \tag{2.12.1}$$

$$\sum_{n=0}^{\infty} p^n[x_n] \longmapsto (a_n^{p^n})_{n\in\mathbb{N}}.$$

The latter is the usual presentation of the elements in Witt rings, see [Bou06b, IX.§1] or [Ser79, II.§6].
(4) (Addition and multiplication formulas in Teichmüler expansions) For any $x, y \in W(R)$, we put $x = [x_0]+p[x_1]+p^2[x_2]+\cdots$ and $y = [y_0]+p[y_1]+p^2[y_2]+\cdots$. We hope to write explicitly the Teichmüler expansions of $x + y$ and xy in terms of $x_0, x_1, \dots, y_0, y_1, \dots$. Unwinding the construction 2.10 of Teichmüler liftings, we can compute out by hand that

$$(x + y)_0 = x_0 + y_0, \tag{2.12.2}$$

$$(x + y)_1 = x_1 + y_1 + \frac{(x_0^{1/p})^p + (y_0^{1/p})^p - (x_0^{1/p} + y_0^{1/p})^p}{p} = x_1 + y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_0^{\frac{i}{p}} y_0^{\frac{p-i}{p}}, \tag{2.12.3}$$

$$(xy)_0 = x_0y_0, \tag{2.12.4}$$

$$(xy)_1 = x_0y_1 + x_1y_0, \tag{2.12.5}$$

$$(xy)_2 = x_0y_2 + x_2y_0 + x_1y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} (x_1y_0)^{\frac{i}{p}} (x_0y_1)^{\frac{p-i}{p}}. \tag{2.12.6}$$

In general, after passing to the form of Witt vectors in (3), then the explicit addition and multiplication formulas are inductively computed out in [Bou06b, IX.§1.3, (12) and (13)] or [Ser79, II.§6, Theorem 6], which can be translated back into the following theorem.

Theorem 2.13 ([Bou06b, IX.§1.3, (a) and (b)]). *For any $x, y \in W(R)$, we put $x = [x_0] + p[x_1] + p^2[x_2] + \cdots$ and $y = [y_0] + p[y_1] + p^2[y_2] + \cdots$.*

(1) *There is a homogeneous polynomial $S_n \in \mathbb{Z}[X_0^{1/p^n}, X_1^{1/p^{n-1}}, \dots, X_n, Y_0^{1/p^n}, Y_1^{1/p^{n-1}}, \dots, Y_n]$ of degree 1 for any $n \in \mathbb{N}$ such that for any $x, y \in W(R)$,*

$$x + y = [S_0(x, y)] + p[S_1(x, y)] + p^2[S_2(x, y)] + \cdots \in W(R), \tag{2.13.1}$$

where $S_n(x, y) \in R$ is the value of the polynomial S_n at $X_i = x_i$ and $Y_i = y_i$ for any $0 \le i \le n$.
(2) *There is a homogeneous polynomial $P_n \in \mathbb{Z}[X_0^{1/p^n}, X_1^{1/p^{n-1}}, \dots, X_n, Y_0^{1/p^n}, Y_1^{1/p^{n-1}}, \dots, Y_n]$ of degree 2 that is homogeneous of degree 1 with respect to the variables $(X_0^{1/p^n}, X_1^{1/p^{n-1}}, \dots, X_n)$*

and also homogeneous of degree 1 *with respect to the variables* $(Y_0^{1/p^n}, Y_1^{1/p^{n-1}}, \dots, Y_n)$ *for any* $n \in \mathbb{N}$ *such that for any* $x, y \in W(R)$,

$$xy = [P_0(x,y)] + p[P_1(x,y)] + p^2[P_2(x,y)] + \cdots \in W(R), \tag{2.13.2}$$

where $P_n(x,y) \in R$ *is the value of the polynomial* P_n *at* $X_i = x_i$ *and* $Y_i = y_i$ *for any* $0 \le i \le n$.

2.c. **Universal ξ-deformation: perfect prisms.** Since $W(R)$ is the "universal p-deformation" of a perfect $\mathbb{F}_p$-algebra R, in order to define the "mixed-characteristic analogue of perfect algebras", we would like to realize $W(R)$ as a "universal ξ-deformation". We firstly need to define what ξ is.

Definition 2.14. A *perfect prism* is a pair (A, I) consisting of a ring A and an ideal I of A such that

(1) A is a p-complete flat $\mathbb{Z}_p$-algebra with $R = A/pA$ perfect (i.e., $A = W(R)$).
(2) $I = (\xi)$ for some $\xi = [\xi_0] + p[\xi_1] + p^2[\xi_2] + \cdots \in W(R)$ such that R is ξ_0-complete and $\xi_1 \in R^\times$ (we call such an element of $W(R)$ *distinguished*).

Remark 2.15. Since we want to realize $A = W(R)$ as a "universal ξ-deformation", it is natural to require that it is ξ-complete and ξ-torsion free. We will see that they are guaranteed by the second condition 2.14.(2) in the following lemmas.

Lemma 2.16. *Let* R *be a perfect* $\mathbb{F}_p$*-algebra,* $d \in R$*. Then, any element of* R *that is killed by a power of* d *is also killed by a* p*-power root of* d*, i.e.,* $R[d^\infty] = R[d^{1/p^\infty}]$*. In particular,* R *is* d*-torsion-bounded.*

Proof. If $dx = 0$, then $(dx)^{1/p^n} = 0$ by perfectness, i.e., $d^{1/p^n} x^{1/p^n} = 0$. Hence, $d^{1/p^n} x = d^{1/p^n} x^{1/p^n} \cdot x^{1-1/p^n} = 0$. □

Lemma 2.17 (completeness). *Any perfect prism* $(A, (\xi))$ *is* (p, ξ)*-complete.*

Proof. Firstly, we take induction on $n \ge 1$ to see that $W(R)/p^n$ is ξ-complete (where $R = A/pA$). By $\mathbb{Z}_p$-flatness of $W(R)$, there is an exact sequence $0 \to W(R)/p^{n-1} \xrightarrow{\cdot p} W(R)/p^n \to W(R)/p = R \to 0$. Since R is ξ-torsion bounded by 2.16, taking ξ-completion still produces an exact sequence $0 \to (W(R)/p^{n-1})^\wedge \to (W(R)/p^n)^\wedge \to \widehat{R} = R \to 0$ ([He25a, 8.8]), where R is ξ-complete by definition 2.14.(2). By induction, we see that $(W(R)/p^n)^\wedge = W(R)/p^n$.

Then, as $W(R)$ is p-adically complete by definition, we have $W(R) = \lim_{n\to\infty} W(R)/p^n = \lim_{n\to\infty} \lim_{m\to\infty} W(R)/(p^n, \xi^m) = \lim_{r\to\infty} W(R)/(p^r, \xi^r) = \lim_{r\to\infty} W(R)/(p,\xi)^r$, where the last equality follows from $(p^{2r}, \xi^{2r}) \subseteq (p,\xi)^{2r} \subseteq (p^r, \xi^r)$. In other words, $W(R)$ is (p,ξ)-complete. □

Lemma 2.18. *Let* A *be a* p*-complete* $\mathbb{Z}_p$*-flat algebra with* A/pA *perfect,* ξ *a distinguished element of* A*,* $x \in A$*. Then,* $\xi \cdot x$ *is distinguished if and only if* $x \in A^\times$*.*

Proof. We write $A = W(R)$ and $\xi = [\xi_0] + p\xi'$, $x = [x_0] + px' \in W(R)$.

If $x \in W(R)^\times$, i.e., $x_0 \in R^\times$ by 2.11. Then, $\xi \cdot x = [\xi_0 x_0] + p(\xi'[x_0] + [\xi_0]x' + p\xi'x')$. On the one hand, R is $\xi_0 x_0$-complete as it is ξ_0-complete. On the other hand, since $\xi'[x_0] + [\xi_0]x' + p\xi'x' \equiv \xi' x_0 \mod (p,\xi)$ is a unit, it is also a unit in $W(R)$ by (p,ξ)-completeness (2.17). Therefore, $\xi \cdot x$ is distinguished.

Conversely, if $\xi \cdot x$ is distinguished, then $\xi'[x_0] + [\xi_0]x' + p\xi'x'$ is a unit in $W(R)$. Modulo (p,ξ), we see from the previous discussion that $\xi' x_0$ is a unit in $R/\xi_0 R$. This implies that $x_0 \in R^\times$ (as R is ξ_0-complete) and thus $x \in W(R)^\times$. □

Lemma 2.19 (nonzero divisor)**.** *Let* (A, I) *be a perfect prism. Then, any generator* ξ *of* I *is a distinguished nonzero divisor of* A*.*

Proof. By definition, there exists a distinguished generator $\xi = [\xi_0] + p\xi'$ of I, i.e., $\xi' \in W(R)^\times$ by 2.11. Then, any generator of I is still distinguished by 2.18.

To see any generator of I is a nonzero divisor, consider $x = [x_0] + p[x_1] + p^2[x_2] + \cdots \in W(R)$ and suppose that $\xi \cdot x = 0$. Then, we have $([\xi_0] + p\xi')x = 0$. For any positive odd number n, we have $([\xi_0]^n + p^n \xi'^n)x = 0$ and thus $p^n x \in [\xi_0^n]W(R)$. By the uniqueness of the Teichmüler expansion, we see that each $x_i \in \xi_0^n R$ for any odd number n. But since R is ξ_0-complete, we must have $x_i = 0$, i.e., $x = 0$. □

Now we start to technically realize A as a "universal ξ-deformation" of $A/\xi A$.

Lemma 2.20. *Let* R *be a ring,* I *an ideal of* R *such that* R/I *is of characteristic* p *and that* R *is* I*-adically complete. Then, the canonical map*

$$\varprojlim_{\mathrm{Frob}} R \longrightarrow \varprojlim_{\mathrm{Frob}} R/IR \tag{2.20.1}$$

is a bijection, where $\varprojlim_{\mathrm{Frob}} R := \lim(\cdots \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R)$ *as a multiplicative monoid.*

Proof. For $(\cdots, x_2, x_1, x_0) \in \varprojlim_{\mathrm{Frob}} R/IR$, we take liftings $\cdots, y_2, y_1, y_0$ of these coordinates in R. Notice that for any $n, m \in \mathbb{N}$ and $z \in I$, $(y_{n+m} + z)^{p^n} \equiv y_{n+m}^{p^n} \mod I^{n+1}R$ as $p \in I$. Thus, $y_{n+m}^{p^n} \in R/I^{n+1}R$ does not depend on the choice of y_{n+m}. Then, we see that $\lim_{n\to\infty} y_{n+m}^{p^n}$ is a well-defined element in $R = \lim_{n\to\infty} R/I^{n+1}R$. We put

$$y = (\cdots, \lim_{n\to\infty} y_{n+2}^{p^n}, \lim_{n\to\infty} y_{n+1}^{p^n}, \lim_{n\to\infty} y_n^{p^n}) \in \varprojlim_{\mathrm{Frob}} R. \tag{2.20.2}$$

It is clearly that y is well-defined and the assignment $x \mapsto y$ gives an inverse to the canonical map (2.20.1). ☐

Proposition 2.21. *The following functor from the category of of perfect prisms to the category of rings*

$$\begin{aligned} \{\textit{perfect prisms}\} &\longrightarrow \{\textit{rings}\} \\ (A, I) &\longmapsto A/I, \end{aligned} \tag{2.21.1}$$

is fully faithful.

Proof. Let S be a ring lying in the essential image of (2.21.1). We take a perfect prism (A, I) with $A/I \cong S$. Then, we have

$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S \cong A/I & \twoheadrightarrow & S/p \cong R/\xi_0. \end{array} \tag{2.21.2}$$

Since R is a ξ_0-complete perfect $\mathbb{F}_p$-algebra, we deduce from 2.20 that

$$R \xleftarrow{\sim} \varprojlim_{\mathrm{Frob}} R \xrightarrow{\sim} \varprojlim_{\mathrm{Frob}} R/\xi_0 R \cong S^\flat := \varprojlim_{\mathrm{Frob}} S/pS, \tag{2.21.3}$$

where the first isomorphism is the projection onto the first component. In particular, the canonical map $S^\flat \to S/pS$ is surjective. By deformation theory (2.5 and 2.6), the canonical surjection $S^\flat \to S/pS$ lifts uniquely to a morphism $\theta : W(S^\flat) \to S$ (which remains surjective by déviassage). By deformation theory again, we see that the isomorphism $A/I \cong S$ lifts uniquely to an isomorphism $A \cong W(S^\flat)$. All in all, the functor from the essential image of (2.21.1) to the category of perfect prisms sending S to $(W(S^\flat), \ker(\theta))$ is well-defined and forms a quasi-inverse to (2.21.1). ☐

Definition 2.22. A *perfectoid ring* is a ring S such that $S \cong A/I$ for some perfect prism (A, I).

Our presentation of perfectoids is different from the original reference [BMS18] but follows closely Bhatt's latest lecture notes [Bha25] in the spirits of prismatic cohomology. We suggest the readers to read [Sch12, §3-5], which is the very beginning resource of perfectoids (in the almost sense), then move to the original reference for perfectoid rings [BMS18, §3] and some complements [CS24, §2], and finally to change the perspective to prisms via [BS22, §2-3] together with some helpful lecture notes [Bha18b, §2-4] and [Bha25, §3].

Remark 2.23. Note that possibly many perfectoid rings S could have the same perfect $\mathbb{F}_p$-algebra $S^\flat$, since the choice of an distinguished principal ideal (ξ) on $W(S^\flat)$ could be many (even if we fix $\xi_0 \in R = S^\flat$, it seems that different choices of $\xi_1 \in R^\times$ could lead to different ideals $I = (\xi) \subseteq W(R)$). But I don't have an explicit example in hand.

However, this issue does not exists when we work over a fixed perfect prism, see the tilting correspondence in the following.

2.d. **Tilting correspondence of perfectoids.** Our definition for perfectoids immediately implies the tilting correspondence as long as we have the following rigidity lemma:

Lemma 2.24 (rigidity)**.** *Let $(A, I) \to (B, J)$ be a morphism of perfect prisms. Then, $J = IB$.*

Proof. We only need to show that for generator ξ of J, if $\xi \cdot x$ is distinguished then $x \in B^\times$. This is proved in 2.18. ☐

Theorem 2.25. *Given a perfect prism (A, I), we put*

$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S = A/\xi & \twoheadrightarrow & S/p = R/\xi_0. \end{array} \tag{2.25.1}$$

Then, the base change induces equivalences of categories

(2.25.2)

$$\begin{array}{ccc}\{\textit{perfect } (A,I)\textit{-prisms } (A',IA')\} & \xrightarrow[\beta]{\sim} & \{\xi_0\textit{-complete perfect } R\textit{-algebras } R'\} \\ \alpha\big\downarrow\wr & & \gamma\big\downarrow\wr \\ \{\textit{perfectoid } S\textit{-algebras } S'\} & \xrightarrow[\delta]{\sim} & \{\textit{relatively perfect } (S/p)=(R/\xi_0)\textit{-algebras } T \textit{ with } T=T^\flat/\xi_0^\flat T^\flat\}\end{array}$$

where $\xi_0^\flat=(\cdots,\xi_0^{1/p^2},\xi_0^{1/p},\xi_0)\in(R/\xi_0R)^\flat$ *and an* (R/ξ_0)*-algebra* T *is called* relatively perfect *if the relative Frobenius* $T\otimes_{R/\xi_0,\mathrm{Frob}}R/\xi_0\to T$ *is an isomorphism, i.e.,* $T/\xi_0^{1/p}\xrightarrow{\sim}T$.

Proof. By the rigidity lemma 2.24, the category of perfect (A,I)-prisms (A',IA') is the category of perfect prisms (B,J) with a morphism $(A,I)\to(B,J)$. Hence, it is equivalent to the category of perfectoid rings S' with a morphism $S=A/I\to S'$ by 2.21, i.e., α is an equivalence.

Unwinding the definition 2.14, the category of perfect (A,I)-prisms (A',IA') is the category of p-complete $\mathbb{Z}_p$-algebras A' with A'/pA' perfect ξ_0-complete and a morphism $A\to A'$. Hence, it is equivalent to the category of ξ_0-complete perfect $\mathbb{F}_p$-algebras R' with a morphism $R=A/pA\to R'$ by 2.9, i.e., β is an equivalence.

Recall that $R\xleftarrow{\sim}R^\flat\xrightarrow{\sim}(R/\xi_0R)^\flat$ identifying ξ_0 with $\xi_0^\flat$ by 2.20 and that the Frobenius induces an isomorphism $R/\xi_0^{1/p}\xrightarrow{\sim}R/\xi_0$. The same holds true for any ξ_0-complete perfect R-algebra R'. In particular, γ is a well-defined functor. To see that it is an equivalence, we only need to show that for any relatively perfect (R/ξ_0)-algebra T with $T=T^\flat/\xi_0^\flat T^\flat$, $T^\flat$ is a ξ_0-complete perfect R-algebra. As $R=(R/\xi_0)^\flat$, we see $T^\flat$ is naturally a perfect R-algebra. Moreover, $T^\flat=\lim(\cdots\xrightarrow{\mathrm{Frob}}T^\flat/\xi_0T^\flat\xrightarrow{\mathrm{Frob}}T^\flat/\xi_0T^\flat\xrightarrow{\mathrm{Frob}}T^\flat/\xi_0T^\flat)=\lim(\cdots\to T^\flat/\xi_0^{p^2}T^\flat\to T^\flat/\xi_0^pT^\flat\to T^\flat/\xi_0T^\flat)$, where we applied the identification $\mathrm{Frob}^n:T^\flat/\xi_0T^\flat\xrightarrow{\sim}T^\flat/\xi_0^{p^n}T^\flat$. This shows that $T^\flat$ is ξ_0-complete. Hence, γ is an equivalence.

The proof of 2.21 shows that δ is a well-defined functor making the diagram (2.25.2) commutative. Hence, δ is an equivalence. □

Remark 2.26. We couldn't simply apply deformation theory in the setting of 2.25 because the relatively perfect $(S/p)=(R/\xi_0)$-algebra T may not be flat. To resolve this issue, one may consider instead relatively perfect *animated* $(S/p)=(R/\xi_0)$-algebra T, i.e., animated algebra T such that the relative Frobenius $T\otimes^{\mathrm{L}}_{R/\xi_0,\mathrm{Frob}}R/\xi_0\to T$ is an isomorphism, and then apply deformation theory for animated algebras, see [Bha25, 3.2.6]. In another way, one can impose flatness assumptions in order to use the classical deformation theory as follows.

Theorem 2.27. *Given a perfect prism* (A,I)*, we put*

(2.27.1)
$$\begin{array}{ccc}A=W(R) & \twoheadrightarrow & A/p=R \\ \big\downarrow & & \big\downarrow \\ S=A/\xi & \twoheadrightarrow & S/p=R/\xi_0.\end{array}$$

Then, the base change induces equivalences of categories

(2.27.2)

$$\begin{array}{ccc}\{(p,\xi)\textit{-completely flat perfect } (A,I)\textit{-prisms } (A',IA')\} & \xrightarrow[\beta]{\sim} & \{\xi_0\textit{-completely flat } \xi_0\textit{-complete perfect } R\textit{-algebras } R'\} \\ \alpha\big\downarrow\wr & & \gamma\big\downarrow\wr \\ \{p\textit{-completely flat perfectoid } S\textit{-algebras } S'\} & \xrightarrow[\delta]{\sim} & \{\textit{flat relatively perfect } (S/p)=(R/\xi_0)\textit{-algebras } T\}\end{array}$$

where "I-completely flat" means "flat after modulo I^n for any $n\in\mathbb{N}$" here.

Proof. Let T be a flat relatively perfect $(S/p)=(R/\xi_0)$-algebra. By deformation over $R\to R/\xi_0$ and $\mathbb{L}_{T/(R/\xi_0)}=0$ ([GR03, 6.5.13.(i)]), there exists a unique ξ_0-completely flat ξ_0-complete R-algebra R' with $R'/\xi_0R'=T$ (see the proof of 2.7 and 2.9). To see that γ is an equivalence, it remains to check that R' is perfect. As T is relatively perfect, we have $R'/\xi_0R'\otimes_{R/\xi_0R,\mathrm{Frob}^{p^n}}R/\xi_0R=R'/\xi_0R'$ for any $n\in\mathbb{N}$. Since $\mathrm{Frob}^{p^n}:R/\xi_0R\to R/\xi_0R$ factors as $R/\xi_0R\xrightarrow{\sim}R/\xi_0^{p^n}R\to R/\xi_0R$, the uniqueness of the liftings implies that the Frobenius induces isomorphism $\mathrm{Frob}^{p^n}:R'/\xi_0R'\xrightarrow{\sim}R'/\xi_0^{p^n}R'$. Thus, $R'=\lim_{n\to\infty}R'/\xi_0^{p^n}R'=\varprojlim_{\mathrm{Frob}}R'/\xi_0R'=R'^\flat$ is perfect.

Similarly, by deformation over $S \to S/p$ and $\mathbb{L}_{T/(S/p)} = 0$, there exists a unique p-completely flat p-complete S-algebra S' with $S'/pS' = T$. To see that δ is an equivalence, it remains to check that S' is perfectoid. It suffices to check that S' lies in the essential image of α. As the diagram (2.27.2) commutes, we only need to prove that β is an equivalence.

We claim that a perfect (A, I)-prism (A', IA') is (p, ξ)-completely flat if and only if $A'/(p, \xi)A'$ is flat over $A/(p, \xi)A$. Since A' is p-torsion-free, we have $A' \otimes^{\mathrm{L}}_A A/pA = A'/pA'$. Thus, $A' \otimes^{\mathrm{L}}_A A/(p, \xi) = A'/pA' \otimes^{\mathrm{L}}_{A/pA} A/(p, \xi) = R' \otimes^{\mathrm{L}}_R R/\xi_0$, where $R' = A'/pA'$ is a perfect $\mathbb{F}_p$-algebra. In particular, $\mathrm{Tor}^A_1(A', A/(p, \xi)) = \mathrm{Tor}^R_1(R', R/\xi_0) = 0$ by 2.28. Then, the claim follows directly from [Sta26, 051C].

The claim implies that the category of (p, ξ)-completely flat perfect (A, I)-prisms (A', IA') is equivalent to the category of p-complete $\mathbb{Z}_p$-algebras A' with A'/pA' perfect ξ_0-complete ξ_0-completely flat and a morphism $A \to A'$. Hence, it is equivalent to the category of ξ_0-completely flat ξ_0-complete perfect $\mathbb{F}_p$-algebras R' with a morphism $R = A/pA \to R'$ by 2.9, i.e., β is an equivalence. □

Lemma 2.28. *Let $R \to R'$ be a morphism of perfect $\mathbb{F}_p$-algebras, $d \in R$. Consider the following conditions:*

(1) *$R/dR \to R'/dR'$ is flat.*
(2) *$R/d^nR \to R'/d^nR'$ is flat for any $n \in \mathbb{N}$.*
(3) *$R[d] \otimes_R R' \to R'[d]$ is an isomorphism.*
(4) *$R[d] \otimes_R R' \to R'[d]$ is surjective.*
(5) *$\mathrm{Tor}^R_1(R', R/d) = 0$.*
(6) *$R/dR \otimes^{\mathrm{L}}_R R' \to R'/dR'$ is an isomorphism.*

Then, we have (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6).

Proof. (1) ⇒ (2): As R and R' are perfect, the Frobenius induces isomorphism between $R/dR \to R'/dR'$ with $R/d^{p^n}R \to R'/d^{p^n}R'$. Thus, the latter is also flat.

(2) ⇒ (3): Note that $R[d] \cap dR = 0$ by 2.16. Thus, the sequence of R/d^2R-modules $0 \to R[d] \to R/dR \xrightarrow{\cdot d} R/d^2R$ is exact. Tensoring with R'/d^2R', the flatness implies that $R'[d] = R[d] \otimes_R R'$.

(3) ⇒ (4): This is clear.

(4) ⇒ (5): Consider the exact sequence $0 \to R[d] \to R \xrightarrow{\cdot d} R \to R/dR \to 0$. Then, $R/dR \otimes^{\mathrm{L}}_R R'$ is represented by the total complex of $R[d] \otimes^{\mathrm{L}}_R R' \to R' \xrightarrow{\cdot d} R'$. In particular, $\mathrm{Tor}^R_1(R', R/d) = H_1(R/dR \otimes^{\mathrm{L}}_R R') = \mathrm{Coker}(R[d] \otimes_R R' \to R'[d]) = 0$.

(5) ⇒ (6): As R is perfect, $R[d] = R[d^{1/p^\infty}]$ by 2.16 so that $R/R[d]$ is a perfect $\mathbb{F}_p$-algebra. Recall that $R/R[d] \otimes^{\mathrm{L}}_R R' = R/R[d] \otimes_R R'$ by [BS17, 11.6]. We deduce from the exact sequence $0 \to R[d] \to R \to R/R[d] \to 0$ that $R[d] \otimes^{\mathrm{L}}_R R' = R[d] \otimes_R R'$. In particular, $R/dR \otimes^{\mathrm{L}}_R R'$ is concentrated in degree $[-1, 0]$ by previous discussion. Thus, condition (5) implies that $R/dR \otimes^{\mathrm{L}}_R R' = R/dR \otimes_R R' = R'/dR'$. □

2.e. **Properties of perfectoids.** We fix a perfectoid ring S in this section. Recall that it is associated with a commutative diagram

$$\begin{array}{ccc} A = W(R) & \twoheadrightarrow & A/p = R \\ \downarrow & & \downarrow \\ S = A/\xi & \twoheadrightarrow & S/p = R/\xi_0. \end{array} \tag{2.28.1}$$

where $R = S^\flat$ by 2.21 and its proof. We put

$$(-)^\sharp : S^\flat \xrightarrow{[\ ]} W(S^\flat) \xrightarrow{\theta} S. \tag{2.28.2}$$

Definition 2.29. A *strict pseudo-uniformizer* of a perfectoid ring S is an element ϖ of S equipped with a compatible system of p-power roots $(\varpi^{1/p^n})_{n\in\mathbb{N}}$ such that there exists a distinguished generator ξ of $\mathrm{Ker}(\theta : W(S^\flat) \to S)$ with $\varpi^{1/p^n} = \theta([\xi_0^{1/p^n}]) = (\xi_0^{1/p^n})^\sharp$, where ξ_0 is the image of ξ in $S^\flat$.

Remark 2.30. If we write $\xi_0 = (\cdots, x_2, x_1, x_0) \in S^\flat = \varprojlim_{\mathrm{Frob}} S/pS$, then $\xi_0^{1/p^n} = (\cdots, x_{n+2}, x_{n+1}, x_n) \in S^\flat$ so that $\varpi^{1/p^n} = \theta([\xi_0^{1/p^n}]) \equiv x_n \mod pS$. In other words, $\xi_0 = (\cdots, \varpi^{1/p^2}, \varpi^{1/p}, \varpi) \in S^\flat$ is determined by the data of a strict pseudo-uniformizer ϖ. We denote it simply by $\varpi^\flat \in S^\flat$.

Lemma 2.31. *Any strict pseudo-uniformizer $\varpi \in pS^\times$.*

Proof. We write $\xi = [\xi_0] + p\xi' \in W(R)$ with $\xi' \in W(R)^\times$. Modulo ξ, we see that $\varpi = -p\xi'$ in S so that $\varpi \in pS^\times$. □

Lemma 2.32 (Frobenius isomorphism)**.** *The Frobenius induces an isomorphism* $S/\varpi^{1/p}S \xrightarrow{\sim} S/\varpi S$.

Proof. Since R is perfect, the Frobenius on $R/\xi_0 R$ is surjective with kernel generated by $\xi_0^{1/p}$. Since $R/\xi_0 = S/p$ identifying $\xi_0^{1/p}$ with $\varpi^{1/p}$ via the commutative diagram (2.28.1), the conclusion follows immediately. ☐

Lemma 2.33 (almost torsion-free)**.** $S[\varpi^\infty] = S[\varpi^{1/p^\infty}] = R[\xi_0^{1/p^\infty}] = R[\xi_0^\infty]$. *In particular, S is p-torsion bounded.*

Proof. Since p, ξ are both nonzero divisors on A by 2.19, there are canonical isomorphisms

$$(A/\xi)[p] \xleftarrow{\sim} \frac{\{(x,y) \in A^2 \mid \xi x = py\}}{\{(pz, \xi z) \mid z \in A\}} \xrightarrow{\sim} (A/p)[\xi] \qquad (2.33.1)$$

$$y \longleftarrow (x,y) \longmapsto x.$$

Thus, we have $S[\varpi] = R[\xi_0]$ as $(S/p) = (R/\xi_0)$-modules. Then, for any $n \geq 1$, we have $S[\varpi^{1/p^n}] = (S[\varpi])[\varpi^{1/p^n}] = (R[\xi_0])[\xi_0^{1/p^n}] = R[\xi_0^{1/p^n}]$.

Since R is perfect, we have $R[\xi_0^{1/p^\infty}] = R[\xi_0]$ by 2.16. The above discussion implies that $S[\varpi^{1/p^\infty}] = S[\varpi]$. This implies furthermore that $S[\varpi^{1/p^\infty}] = S[\varpi^\infty]$. ☐

Lemma 2.34 (completeness)**.** *S is p-complete.*

Proof. As ξ is a nonzero divisor on A (2.19), there is an exact sequence $0 \to A \xrightarrow{\cdot\xi} A \to S \to 0$. Since S is p-torsion bounded by 2.33, taking p-completion still produces an exact sequence $0 \to A \to A \to \widehat{S} \to 0$ ([He25a, 8.8]), where $\widehat{A} = A$ by 2.17. Hence, we get $\widehat{S} = S$, i.e., S is p-complete. ☐

Proposition 2.35 (perfectoidness criterion)**.** *A p-torsion-free ring S is perfectoid if and only if the following conditions hold:*

1. *S is p-complete.*
2. *There exists $\pi \in S$ such that $\pi^p \in pS^\times$.*
3. *The Frobenius induces an isomorphism $S/\pi S \xrightarrow{\sim} S/pS$.*

Proof. These conditions are necessary by 2.34, 2.31 and 2.32. To see they are also sufficient, consider $S^\flat = \lim_{\longleftarrow \mathrm{Frob}} S/pS$. The surjectivity part of condition (3) implies that the canonical projection $S^\flat \to S/pS$, $(\cdots, x_2, x_1, x_0) \mapsto x_0$ is surjective. Hence, we can take $\xi_0 = (\cdots, \pi_2, \pi_1 = \pi, \pi_0 = \pi^p) \in S^\flat$. Since S is p-torsion-free, the injectivity part of condition (3) implies that the kernel of $S^\flat \to S/pS$ is generated by ξ_0. By deformation theory and a dévissage argument (2.5 and 2.6, see also 2.9), the exact sequence $S^\flat \xrightarrow{\cdot\xi_0} S^\flat \to S/pS \to 0$ lifts uniquely to an exact sequence $0 \to W(S^\flat) \to W(S^\flat) \to S \to 0$, where we used the fact that S is p-complete and p-torsion-free. Let ξ be the image of 1 under the map $W(S^\flat) \to W(S^\flat)$ and denote the surjection $W(S^\flat) \to S$ by θ.

To see that S is perfectoid, it remains to show that ξ is distinguished. By condition (2) we write $\pi_0 = \pi^p = pu$. As $\theta([\xi_0^{1/p}]) \equiv \pi_1 \mod pS$, we have $\theta([\xi_0]) \equiv \pi_1^p \equiv pu \mod p^2 S$. This shows that $\theta([\xi_0]) = pv$ for some $v \in S^\times$ as S is p-complete and p-torsion-free. Let $w \in W(S^\flat)^\times$ be a lifting of v. Then, $[\xi_0] - pw \in \ker(\theta) = (\xi)$ and is distinguished by construction. This implies that ξ is also distinguished by a similar argument of 2.18. ☐

Remark 2.36. See [Bha18b, §4, 2.10] for a general criterion removing the "p-torsion-free" assumption on S.

2.f. **Examples of perfectoids.**

Lemma 2.37 (adding p^∞-roots)**.** *Let S be a perfectoid ring. Then, the p-adic completion $S\langle X^{1/p^\infty}\rangle$ of $S[X^{1/p^\infty}]$ is also perfectoid.*

Proof. Since $(S/pS)[X^{1/p^\infty}]$ is a flat relative perfect (S/pS)-algebra with the unique flat lifting $S\langle X^{1/p^\infty}\rangle$. Thus, the proof of 2.27 shows that $S\langle X^{1/p^\infty}\rangle$ is perfectoid. ☐

Lemma 2.38 (perfect=perfectoid over $\mathbb{F}_p$)**.** *Let S be an $\mathbb{F}_p$-algebra. Then, S is perfectoid if and only if S is perfect.*

Proof. If S is perfect, then $S = W(S)/pW(S)$ with perfect prism $(W(S), (p))$ (i.e., p is distinguished).

If S is perfectoid, then $\mathbb{F}_p \to S$ is a morphism of perfectoid rings, which corresponds to a morphism of perfect prisms $(\mathbb{Z}_p, (p)) \to (W(S^\flat), I)$ by 2.21. Then, $I = pW(S^\flat)$ by the rigidity lemma 2.24. Hence, $S = W(S^\flat)/I = S^\flat$ is perfect. ☐

Lemma 2.39 (perfectoid valuation ring)**.** *Let V be a p-complete valuation ring extension of $\mathbb{Z}_p$. Then, V is perfectoid if and only if the Frobenius is surjective on V/pV and V is not absolutely unramified. In particular, if the fraction field of V is algebraically closed, then V is perfectoid.*

Proof. We only need to prove the sufficiency. Firstly, we claim that there exists $\pi \in V$ with $\pi^p \in pV^\times$. As V is not absolutely unramified, we can write $p = \pi_1\pi_2$ for some elements $\pi_1, \pi_2 \in \mathfrak{m}_V$. Then, the surjectivity of the Frobenius on V/pV implies that $\pi_i = x_i^p + py_i$ for some $x_i, y_i \in V$ (where $i = 1, 2$). Notice that $x_i^p = \pi_i - py_i \in \pi_i V^\times$ by construction. We get $x_1^p x_2^p \in pV^\times$.

Since V is integrally closed in $V[1/p]$, the Frobenius induces an injection $V/\pi V \to V/pV$ (see [He24a, 5.21]). Thus, the conclusion follows from the perfectoidness criterion 2.35. □

Lemma 2.40 (torsion-free quotient)**.** *Let S be a perfectoid ring. Then, its maximal p-torsion-free quotient $\overline{S} = S/S[p^\infty]$ is also perfectoid.*

Proof. Let ϖ be a strict pseudo-uniformizer of S. Then, $S[p^\infty] = S[\varpi^\infty] = S[\varpi^{1/p^\infty}]$ by 2.33. In particular, $S[p^\infty] \cap \varpi^{1/p^n} S = 0$ for any $n \in \mathbb{N}$. The exact sequence $0 \to S[p^\infty] \to S \to \overline{S} \to 0$ induces exact sequences $0 \to S[p^\infty] \to S/\varpi^{1/p^n} S \to \overline{S}/\varpi^{1/p^n}\overline{S} \to 0$. In particular, the Frobenius induces an isomorphism $\overline{S}/\varpi^{1/p}\overline{S} \xrightarrow{\sim} \overline{S}/\varpi\overline{S}$ by 2.32. Hence, the conclusion follows from the perfectoidness criterion 2.35. □

Lemma 2.41 (integral closure)**.** *Let S be a perfectoid ring. Then, its integral closure S^+ in $S[1/p]$ is also perfectoid.*

Proof. After 2.40, we may assume that S is p-torsion-free. The injectivity of the Frobenius $S/\varpi^{1/p}S \to S/\varpi S$ implies that S is p-integrally closed, i.e., for any $x \in S[1/p]$ with $x^p \in S$ we have $x \in S$ (see [He24a, 5.21]). Then, $S \to S^+$ is an almost isomorphism, i.e., $\varpi^{1/p^\infty} S^+ \subseteq S$ (see [He24a, 5.25]).

We claim that the Frobenius induces an isomorphism $S^+/\varpi^{1/p}S^+ \xrightarrow{\sim} S^+/\varpi S^+$. It is injective as S^+ is p-integrally closed. For any $z \in S^+$, the previous discussion allows us to write $\varpi^{1/p}z = x^p + \varpi y$ for some $x, y \in S$. As $z = (x/\varpi^{1/p^2})^p + \varpi^{1-1/p}y$, we see that $x' = x/\varpi^{1/p^2} \in S^+$. We continue to write $z = x'^p + \varpi^{1-1/p}y = x'^p + \varpi^{1-1/p}(y'^p + \varpi^{1-1/p}z')$ for some $y', z' \in S^+$. Thus, $z = (x' + \varpi^{1/p-1/p^2}y')^p + \varpi z''$ for some $z'' \in S^+$. This shows the surjectivity of the Frobenius on $S^+/\varpi S^+$.

In conclusion, S^+ is perfectoid by the perfectoidness criterion 2.35. □

Lemma 2.42 (direct product)**.** *Let $\{S_i\}_{i\in I}$ be a family of perfectoid rings. Then, the product $\prod_{i\in I} S_i$ is perfectoid.*

Proof. Let ξ_i be a distinguished generator of $\mathrm{Ker}(W(S_i^\flat) \to S_i)$ for any $i \in I$. Then, $\prod_{i\in I} S_i^\flat$ is a $\xi = (\xi_i)_{i\in I}$-complete perfect $\mathbb{F}_p$-algebra. By universal p-deformation 2.7, it is easy to see that $W(\prod_{i\in I} S_i^\flat) = \prod_{i\in I} W(S_i^\flat)$ and that ξ is a distinguished element. Thus, $\prod_{i\in I} S_i = \prod_{i\in I}(W(S_i)/\xi_i) = (\prod_{i\in I} W(S_i))/\xi = W(\prod_{i\in I} S_i^\flat)/\xi$ is a perfectoid ring. □

Lemma 2.43 (tensor product)**.** *Let $S_2 \leftarrow S_1 \to S_3$ be morphisms of perfectoid rings. Then, the p-completed tensor product $S_2\widehat{\otimes}_{S_1}S_3$ is perfectoid.*

Proof. Let $\xi = [\xi_0] + p\xi'$ be a distinguished generator of $\mathrm{Ker}(W(S_1^\flat) \to S_1)$ (and thus also a distinguished generator of for S_2 and S_3 by the rigidity lemma 2.24). The given morphisms $S_2 \leftarrow S_1 \to S_3$ induce morphisms of perfect $\mathbb{F}_p$-algebras $S_2^\flat \leftarrow S_1^\flat \to S_3^\flat$. It is clear that the ξ_0-completed tensor product $S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat$ is still perfect.

Firstly, we claim that $W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p^n = W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)/p^n$ for any $n \in \mathbb{N}$. This holds for $n = 1$. In general, it follows from taking induction and the following exact sequences

(2.43.1)

$$\begin{array}{ccccccccc} & & W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p^n & \longrightarrow & W(S_2^\flat) \otimes_{W(S_1^\flat)} W(S_3^\flat)/p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)/p^n & \longrightarrow & S_2^\flat \otimes_{S_1^\flat} S_3^\flat & \longrightarrow & 0. \end{array}$$

Therefore, we have $W(S_2^\flat)\widehat{\otimes}^p_{W(S_1^\flat)}W(S_3^\flat) = W(S_2^\flat \otimes_{S_1^\flat} S_3^\flat)$, where the completion is p-adic, and thus there is an exact sequence $0 \to W(S_2^\flat)\widehat{\otimes}^p_{W(S_1^\flat)}W(S_3^\flat) \xrightarrow{\cdot p} W(S_2^\flat)\widehat{\otimes}^p_{W(S_1^\flat)}W(S_3^\flat) \to S_2^\flat \otimes_{S_1^\flat} S_3^\flat \to 0$. Since $S_2^\flat \otimes_{S_1^\flat} S_3^\flat$ is ξ-torsion-bounded by 2.16, taking ξ-completion still produces an exact sequence $0 \to W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat) \xrightarrow{\cdot p} W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat) \to S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat \to 0$.

To show that $W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p^n = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/p^n$ for any $n\in\mathbb{N}$. We still take induction on n. The case for $n=1$ is proved above. In general, it follows from the following exact sequences

(2.43.2)
$$\begin{array}{ccccccccc} & & W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p^n & \longrightarrow & W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/p^{n-1} & \xrightarrow{\cdot p} & W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/p^n & \longrightarrow & S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat & \longrightarrow & 0. \end{array}$$

Therefore, we have $W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat) = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)$. In particular, we have $S_2\otimes_{S_1}S_3/p^n = W(S_2^\flat)\widehat{\otimes}_{W(S_1^\flat)}W(S_3^\flat)/(p^n,\xi) = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/(p^n,\xi)$. Taking inverse limit on $n\in\mathbb{N}$, we see that $S_2\widehat{\otimes}_{S_1}S_3 = W(S_2^\flat\widehat{\otimes}_{S_1^\flat}S_3^\flat)/\xi$ is perfectoid. □

3. Cohomological Descent of Perfectoids

Recall that the descent of commutative algebras in Zariski topology can be stated as: for any affine scheme $X=\mathrm{Spec}(A)$, we have

$$A = \mathrm{R}\Gamma_{\mathrm{Zar}}(X,\mathcal{O}_X), \tag{3.0.1}$$

i.e., $A=H^0(X,\mathcal{O}_X)$ and $H^q(X,\mathcal{O}_X)=0$ for any nonzero number q. This is equivalent to the fact that for any family of generators $(f_1,\dots,f_n)$ of the unit ideal of A, the Čech complex

$$0\to A\to \prod_{1\le i\le n}A_{f_i}\to \prod_{1\le i,j\le n}A_{f_if_j}\to \prod_{1\le i,j,k\le n}A_{f_if_jf_k}\to\cdots \tag{3.0.2}$$

is exact (see [Sta26, 01X8]). Moreover, Grothendieck [FGA] established the faithfully flat descent: for any faithfully flat ring homomorphism $A\to B$, the Čech complex

$$0\to A\to B\to B\otimes_A B\to B\otimes_A B\otimes_A B\to\cdots \tag{3.0.3}$$

is exact (see [Sta26, 023F]). Equivalently, we have

$$A = \mathrm{R}\Gamma_{\mathrm{fppf}}(X,\mathcal{O}) \tag{3.0.4}$$

where the cohomology is the cohomology of the category of X-schemes endowed with topology generated by locally finitely presented faithfully flat coverings (see [Sta26, 03P2]). This seems the best result in general. But if A satisfies some extra condition (such as perfect or perfectoid), could we obtain a better cohomological descent result?

3.a. **h, v and arc topologies.**

Definition 3.1. A *valuation ring* is a local domain V such that for any $x,y\in V$, either x divides y or y divides x. An *extension of valuation rings* is an injective local (or equivalently, faithfully flat, see [He24b, 3.1]) homomorphism $V\to W$ of valuation rings.

We refer to [Sta26, 00I8] for some basic properties of valuation rings, and to [Bou06a] for a systematic development. We gather some basic properties here:

Proposition 3.2. *Let V be a valuation ring with fraction field K.*

1. *The quotient $K^\times/V^\times$ is a totally ordered abelian group with respect to divisibility relation.*
2. *The quotient map $v:K^\times\to K^\times/V^\times$ is a valuation map, i.e., $v(1)=0$, $v(xy)=v(x)+v(y)$, $v(x+y)\ge\min(v(x),v(y))$ for any $x,y\in K^\times$.*
3. *All the ideals of V form a chain by inclusions.*
4. *The dimension of V is ≤ 1 if and only if the value group $K^\times/V^\times$ identifies with an ordered subgroup of $\mathbb{R}$.*
5. *V is a discrete valuation ring if and only if the value group $K^\times/V^\times$ is isomorphic to $\mathbb{Z}$.*

Proof. (1) is proved in [Bou06a, VI, §1.2, Théorème 1.(d)], see also [Sta26, 00ID].
(2) follows immediately from (1), see also [Sta26, 00IF].
(3) is proved in [Bou06a, VI, §1.2, Théorème 1.(e)].
(4) is proved in [Bou06a, VI, §4.5, Proposition 7].
(5) is proved in [Bou06a, VI, §3.6, Proposition 9]. □

Definition 3.3. A ring homomorphism $A\to B$ is called

(1) a *v-covering* if for any valuation ring V with a homomorphism $A \to V$, there exists an extension of valuation rings $V \to W$ and a commutative diagram of rings (see [Sta26, 0ETN])

$$\begin{array}{ccc} B & \longrightarrow & W \\ \uparrow & & \uparrow \\ A & \longrightarrow & V \end{array} \tag{3.3.1}$$

(2) an (resp. d-complete (where $d \in A$)) *arc-covering* if the same condition in (1) holds for every (resp. d-complete) valuation ring V over A with dimension ≤ 1 (see [BM21, 1.2] or [CS24, 2.2.1]).

(3) an *h-covering* if it is a v-covering of finite presentation (see [Sta26, 0ETS]).

These different types of coverings endow the category of schemes **Sch** with h, v and arc topologies (see [He24a, 3.3]). It follows directly from the definition that h-topology is coarser than v-topology, and the latter is coarser than arc-topology. To get familiar with these topologies, we include the following result about h-topology although we will not make use of it.

Proposition 3.4 ([Sta26, 0ETK, 0ETU])**.** *The h-topology on* **Sch** *is generated by locally finitely presented faithfully flat coverings and finitely presented proper surjective coverings.*

Example 3.5 (Canonical v-covering)**.** For any ring A, consider the set $\mathscr{V} = \{V\}$ of all the valuation rings $V \subseteq \mathrm{Frac}(V) = \kappa(x)$ of all the residue fields $\kappa(x)$ of A (where $x \in \mathrm{Spec}(A)$). Notice that any homomorphism from A to a valuation ring W must factors through a unique $V \in \mathscr{V}$ such that the induced morphism $V \to W$ is an extension of valuation rings (indeed let x be the image of $\mathrm{Spec}(\mathrm{Frac}(W)) \to X$ then $V = \kappa(x) \cap W$, see [He24b, 3.1]). This shows that $A \to \prod_{\mathscr{V}} V$ is a v-covering. We remark that every connected component of $\prod_{\mathscr{V}} V$ (endowed with the reduced closed subscheme structure) is the spectrum of a valuation ring by [BS17, 6.2].

Moreover, let $\overline{V}$ be an algebraic valuation extension of V with algebraically closed fraction field. Then, $A \to \prod_{\mathscr{V}} \overline{V}$ is still a v-covering and every connected component of $\prod_{\mathscr{V}} \overline{V}$ is the spectrum of a valuation ring with algebraically closed fraction field.

Let $\mathscr{V}^{\leq 1}$ be the subset of $\mathscr{V}$ consisting of valuation rings of dimension ≤ 1. Then, the same argument as above implies that $A \to \prod_{\mathscr{V}^{\leq 1}} \overline{V}$ is an arc covering. Fixing $d \in A$, let $\widehat{\overline{V}}$ be the d-completion of $\overline{V}$. We see that $\widehat{\overline{V}} = 0$ if $d \in \overline{V}^{\times}$ and $\overline{V} \to \widehat{\overline{V}}$ is an extension of valuation rings otherwise ([Bou06a, VI, §5.3, Proposition 5]). In particular, $A \to \prod_{\mathscr{V}^{\leq 1}} \widehat{\overline{V}}$ is a d-complete arc covering (by taking d-completion of an arc covering). Note that the fraction field of $\widehat{\overline{V}}$ is still algebraically closed ([BGR84, §3.4.1, Proposition 3]). However, we don't know if every connected component of $\prod_{\mathscr{V}^{\leq 1}} \widehat{\overline{V}}$ is the spectrum of a valuation ring of dimension ≤ 1.

3.b. **Cohomological descent of perfect $\mathbb{F}_p$-algebras in arc topology.** For any $\mathbb{F}_p$-algebra A, we denote by

$$A_{\mathrm{perf}} = \operatorname*{colim}_{\mathrm{Frob}} A = \operatorname{colim}(A \xrightarrow{\mathrm{Frob}} A \xrightarrow{\mathrm{Frob}} A \xrightarrow{\mathrm{Frob}} \cdots) \tag{3.5.1}$$

the initial perfect A-algebra (i.e., any homomorphism from A to a perfect $\mathbb{F}_p$-algebra B factors uniquely through the perfect algebra A_{perf}). We should distinguish it from the perfect algebra $A^{\flat} = \varprojlim_{\mathrm{Frob}} A$.

Theorem 3.6 (Gabber, see [BST17, 3.3])**.** *For any h-covering of $\mathbb{F}_p$-algebras $A \to B$, the Čech complex*

$$0 \to A_{\mathrm{perf}} \to B_{\mathrm{perf}} \to (B \otimes_A B)_{\mathrm{perf}} \to (B \otimes_A B \otimes_A B)_{\mathrm{perf}} \to \cdots \tag{3.6.1}$$

is exact. In particular, if A and B are both perfect, then the Čech complex

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots \tag{3.6.2}$$

is exact.

Let's firstly look at two essential examples on h-descent of perfect $\mathbb{F}_p$-algebras.

Example 3.7 ([Sta26, 0EVK])**.** Let A be an $\mathbb{F}_p$-algebra with $f \in A$ and a finitely generated ideal $J \subseteq A$ such that $f^r J = 0$ for some $r \in \mathbb{N}$. Then, in the h-topology, the scheme $X = \mathrm{Spec}(A)$ is covered by two closed subschemes $Z = \mathrm{Spec}(A/fA)$ and $X' = \mathrm{Spec}(A/J)$ whose scheme theoretic intersection is $E = \mathrm{Spec}(A/fA + J)$.

$$\begin{array}{ccc} E = \mathrm{Spec}(A/fA+J) & \longrightarrow & \mathrm{Spec}(A/J) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/fA) & \longrightarrow & \mathrm{Spec}(A) = X \end{array} \tag{3.7.1}$$

Taking $B = A/fA \times A/J$, we claim that the alternating Čech complex (cf. (3.6.1))

$$0 \longrightarrow A_{\mathrm{perf}} \xrightarrow{\alpha} (A/fA)_{\mathrm{perf}} \oplus (A/J)_{\mathrm{perf}} \xrightarrow{\beta} (A/fA + J)_{\mathrm{perf}} \longrightarrow 0 \tag{3.7.2}$$

is exact. Since filtered colimit preserves surjection, γ is surjective. It remains to show that $\alpha : A_{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Ker}(\beta)$. To be more precise, for any filtered colimit $M = \mathrm{colim}_{n\in\mathbb{N}} M_n$ of abelian groups, we denote by $\phi_n : M_n \to M$ the canonical map. We take $\phi_n(x, y) \in (A/fA)_{\mathrm{perf}} \oplus (A/J)_{\mathrm{perf}} = \mathrm{colim}_{\mathrm{Frob}} A/fA \times A/J$, where $x \in A/fA$ and $y \in A/J$. We take some liftings of x, y in A and denote them still by x, y. The condition $\gamma(\phi_n(x, y)) = 0$ means that $\mathrm{Frob}^m(x) - \mathrm{Frob}^m(y) = fz + g$ for some $m \in \mathbb{N}$, $z \in A$ and $g \in J$. We take $a = \mathrm{Frob}^m(x) - fz = \mathrm{Frob}^m(y) + g$. Then, we see that $\phi_n(x, y) = \phi_{n+m}(\mathrm{Frob}^m(x), \mathrm{Frob}^m(y)) = \alpha(\phi_{n+m}(a))$, i.e., $\alpha : A_{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Ker}(\beta)$ is surjective. Moreover, if there is $b \in A$ and $l \in \mathbb{N}$ such that $\alpha(\phi_l(b)) = \phi_n(x, y)$, then after applying Frobenius to a and b, we may assume that $a \equiv b \equiv \mathrm{Frob}^k(x) \mod fA$ and $a \equiv b \equiv \mathrm{Frob}^k(y) \mod J$ for some $k \in \mathbb{N}$. Since $f^r J = 0$, we have $(a - b)^r \in (fA \cap J)^r = 0$ so that $\mathrm{Frob}^r(a) = \mathrm{Frob}^r(b)$, i.e., $\alpha : A_{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Ker}(\beta)$ is injective.

Example 3.8 ([Sta26, 0EVJ])**.** Let A be an $\mathbb{F}_p$-algebra with $f_1, f_2 \in A$. Then, in the h-topology, the scheme $X = \mathrm{Spec}(A)$ is covered by the closed subscheme $Z = \mathrm{Spec}(A/(f_1, f_2))$ and the projective A-scheme $X' = \mathrm{Proj}(A[T_1, T_2]/(T_1f_2 - T_2f_1))$ whose fibred product is $E = \mathrm{Proj}(A/(f_1, f_2)[T_1, T_2])$.

$$\begin{array}{ccc} E = \mathrm{Proj}(A/(f_1,f_2)[T_1,T_2]) & \longrightarrow & \mathrm{Proj}(A[T_1,T_2]/(T_1f_2 - T_2f_1)) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/(f_1,f_2)) & \longrightarrow & \mathrm{Spec}(A) = X \end{array} \tag{3.8.1}$$

We claim that

(1) $H^0(E, \mathcal{O}_E) = \mathrm{R}\Gamma(E, \mathcal{O}_E)$ and $H^0(X', \mathcal{O}_{X'}) = \mathrm{R}\Gamma(X', \mathcal{O}_{X'})$, that
(2) $H^0(Z, \mathcal{O}_Z) \xrightarrow{\sim} H^0(E, \mathcal{O}_E)$ and $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ is surjective with square-zero kernel, and that
(3) the sequence

$$0 \longrightarrow A_{\mathrm{perf}} \longrightarrow H^0(Z, \mathcal{O}_Z)_{\mathrm{perf}} \oplus H^0(X', \mathcal{O}_{X'})_{\mathrm{perf}} \longrightarrow H^0(E, \mathcal{O}_E)_{\mathrm{perf}} \longrightarrow 0 \tag{3.8.2}$$

is exact.

It is clear that (3) follows immediately from (2).

Since $E = \mathbb{P}^1_Z$, we get $A/(f_1, f_2) = H^0(Z, \mathcal{O}_Z) = H^0(E, \mathcal{O}_E) = \mathrm{R}\Gamma(E, \mathcal{O}_E)$ by the standard calculation of cohomology of projective spaces ([Sta26, 01XT]).

Consider the universal case where $A = \mathbb{Z}[f_1, f_2]$ is a polynomial algebra over $\mathbb{Z}$ with variables f_1 and f_2. Then, $T_1f_2 - T_2f_1 \in H^0(\mathbb{P}^1_X, \mathcal{O}_{\mathbb{P}^1_X}(1)) = (A[T_1, T_2])_1$ is a degree-1 homogeneous nonzero divisor of $A[T_1, T_2]$. Thus, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_X}(-1) \xrightarrow{\cdot(T_1f_2 - T_2f_1)} \mathcal{O}_{\mathbb{P}^1_X} \longrightarrow \mathcal{O}_{X'} \longrightarrow 0. \tag{3.8.3}$$

Combining the associated long exact sequence of cohomology groups with the standard calculation of cohomology of projective spaces ([Sta26, 01XT]), we get $A = H^0(X, \mathcal{O}_X) = H^0(X', \mathcal{O}_{X'}) = \mathrm{R}\Gamma(X', \mathcal{O}_{X'})$.

For general A, consider the base change along $\mathbb{Z}[f_1, f_2] \to A$. Then, the base change property for the top-degree cohomology implies (1) and (2) except for the "square-zero" property of the kernel (see [Sta26, 0EVJ] for a detailed proof). To complete the proof, it remains to show that the kernel of

$$A \to A[X]/(f_2 - f_1X) \oplus A[Y]/(f_2Y - f_1) \tag{3.8.4}$$

is square-zero, since the two principal open subsets associated with $A[X] \xrightarrow{\sim} A[T_1, T_2]_{(T_1)}$, $X \mapsto T_2/T_1$ and $A[Y] \xrightarrow{\sim} A[T_1, T_2]_{(T_2)}, Y \mapsto T_1/T_2$ cover $\mathbb{P}^1_X$. Let a and b be two elements of the kernel of

(3.8.4). Then, we can write

$$a = (f_2 - f_1X)(a_0 + a_1X + \cdots + a_nX^n) \tag{3.8.5}$$

$$b = (f_2Y - f_1)(b_0 + b_1Y + \cdots + b_nY^n). \tag{3.8.6}$$

We see that

$$a = f_2a_0,\ f_1a_0 = f_2a_1,\ \cdots,\ f_1a_{n-1} = f_2a_n,\ f_1a_n = 0, \tag{3.8.7}$$

$$b = f_1b_0,\ f_2b_0 = f_1b_1,\ \cdots,\ f_2b_{n-1} = f_1b_n,\ f_2b_n = 0, \tag{3.8.8}$$

and thus

$$ab = f_1f_2a_0b_0 = f_1f_2a_1b_1 = \cdots = f_1f_2a_{n-1}b_{n-1} = f_1f_2a_nb_n = 0, \tag{3.8.9}$$

which completes the proof.

Proof ideas of 3.6. In fact, by some abstract formalism (”decomposing h-coverings into almost blowups”), we reduce to 3.7 and 3.8. See [Sta26, 0EWU] for a detailed proof. □

Corollary 3.9 ([BM21, 1.9])**.** *For any arc-covering of $\mathbb{F}_p$-algebras $A \to B$, the Čech complex*

$$0 \to A_{\mathrm{perf}} \to B_{\mathrm{perf}} \to (B \otimes_A B)_{\mathrm{perf}} \to (B \otimes_A B \otimes_A B)_{\mathrm{perf}} \to \cdots \tag{3.9.1}$$

is exact.

Proof ideas of 3.9. The h-descent of perfect algebras implies the v-descent by a limit argument (see [BS17, 4.1]). Then, the arc-descent follows from by decomposing a valuation rings into arcs (see [BM21, 1.7]). We also refer to [He24a, 4.10] for a detailed proof. □

Remark 3.10. Let $\mathcal{O}$ be the sheafification of the presheaf $\mathcal{O}^{\mathrm{pre}}$ on **Sch** sending a scheme X to $\Gamma(X, \mathcal{O}_X)$ in the arc topology. Then, for any affine $\mathbb{F}_p$-scheme $X = \mathrm{Spec}(R)$, we have

$$R_{\mathrm{perf}} = \mathrm{R}\Gamma_{\mathrm{arc}}(X, \mathcal{O}). \tag{3.10.1}$$

Indeed, 3.9 implies that the presheaf $\mathcal{O}^{\mathrm{pre}}_{\mathrm{perf}} = \mathrm{colim}_{\mathrm{Frob}}\, \mathcal{O}^{\mathrm{pre}} : X \mapsto \Gamma(X, \mathcal{O}_X)_{\mathrm{perf}}$ is a sheaf in the arc-topology over $\mathbf{Sch}_{/\mathbb{F}_p}$. Moreover, since perfect affine $\mathbb{F}_p$-schemes form a topological base of $(\mathbf{Sch}_{/X})_{\mathrm{v}}$ (and thus of $(\mathbf{Sch}_{/X})_{\mathrm{arc}}$) by 3.5, we have $\mathcal{O} \xrightarrow{\sim} \mathrm{colim}_{\mathrm{Frob}}\, \mathcal{O} = \mathcal{O}_{\mathrm{perf}} = \mathcal{O}^{\mathrm{pre}}_{\mathrm{perf}}$ over $\mathbb{F}_p$. Then, by the relation between Čech cohomology and cohomology, 3.9 implies that $H^q_{\mathrm{arc}}(X, \mathcal{O}) = H^q_{\mathrm{arc}}(X, \mathcal{O}^{\mathrm{pre}}_{\mathrm{perf}}) = 0$ for any $q \neq 0$ ([Sta26, 03F9]).

3.c. **Tilting correspondence of perfectoid valuation rings.**

Lemma 3.11. *Let S be a perfectoid ring with tilt $S^\flat = \varprojlim_{\mathrm{Frob}} S/pS$. Then, S is a valuation ring if and only if $S^\flat$ is a valuation ring. In this case, there is a canonical isomorphism of their value groups*

$$\mathrm{Frac}(S^\flat)^\times/(S^\flat)^\times \xrightarrow{\sim} \mathrm{Frac}(S)^\times/S^\times \tag{3.11.1}$$

induced by the composition $(-)^\sharp : S^\flat \xrightarrow{[\ ]} W(S^\flat) \xrightarrow{\theta} S$.

Proof. Suppose firstly that S is a valuation ring. Since S is p-complete (2.34), the canonical projection $\varprojlim_{\mathrm{Frob}} S \to \varprojlim_{\mathrm{Frob}} S/pS$ is a multiplicative bijection by 2.20. In particular, we see that $S^\flat$ is a domain and for any $x, y \in S^\flat$, either x divides y or y divides x. Indeed, if we denote by $(\cdots, x_2, x_1, x_0)$ and $(\cdots, y_2, y_1, y_0)$ the corresponding elements of $\varprojlim_{\mathrm{Frob}} S$, as S is a valuation ring, we may assume that there are infinitely many $i \in \mathbb{N}$ such that x_i divides y_i. But the transition morphisms force that x_i divides y_i for any $i \in \mathbb{N}$ and thus x divides y. The same argument shows that $S^\flat$ is a domain.

On the other hand, let ϖ be a strict pseudo-uniformizer of S. Since $S^\flat$ is $\varpi^\flat$-complete and $S/\varpi S = S^\flat/\varpi^\flat S^\flat$ is local (see 2.30), we see that $S^\flat$ is also local. Therefore, $S^\flat$ is a valuation ring by definition 3.1.

Conversely, suppose that $S^\flat$ is a valuation ring. let ϖ be a strict pseudo-uniformizer of S. Since S is ϖ-complete and $S/\varpi S = S^\flat/\varpi^\flat S^\flat$ is local, we see that S is also local.

We claim that S is either ϖ-torsion-free or $S = S^\flat$ (in the latter case, S is a valuation ring). Indeed, if $\varpi^\flat \neq 0$ in the valuation ring $S^\flat$, then $S[\varpi^\infty] = S^\flat[(\varpi^\flat)^\infty] = 0$ (2.33). Otherwise, if $\varpi^\flat = 0$ in $S^\flat$, then $\varpi = 0$ as the image of $[\varpi^\flat] = 0$ by definition 2.29. As $\varpi \in pS^\times$, we see that $p = 0$ in S and thus $S = S^\flat$ by 2.38.

Then, we assume that S is ϖ-torsion-free. For any nonzero element $x \in S$, we can write $x = \varpi^{n/p} \cdot y$ for some unique $n \in \mathbb{N}$ such that $y \in S \setminus \varpi^{1/p}S$. Let $\overline{y} \in S/\varpi S$ be the image of $y \in S$. Then, $y = \theta([\overline{y}]) + \varpi \cdot z$ for some unique $z \in S$, where $\theta : W(S^\flat) \to S$ is the canonical surjection (see the proof of 2.21). Since $S^\flat$ is a valuation and $\overline{y}$ is nonzero in $S^\flat/\varpi^\flat S^\flat$ (actually nonzero in $S^\flat/(\varpi^\flat)^{1/p}S^\flat$), we see that $\overline{y}$ divides $\varpi^\flat$ in $S^\flat$. As $\varpi = \theta([\varpi^\flat])$ (see 2.30), we can write

$$x = \varpi^{n/p} \cdot \theta([\overline{y}]) \cdot w, \text{ where } w = 1 + \theta([\varpi^\flat/\overline{y}])z \in S^\times. \tag{3.11.2}$$

This expression implies that S is a domain and for any $x, x' \in S$, either x divides x' or x' divides x. Indeed, if we write $x' = \varpi^{m/p} \cdot \theta([\overline{y'}]) \cdot w'$ as above, then $xx' = 0$ implies that $\theta([\overline{yy'}]) = 0$ as S is ϖ-torsion-free. Then, $\overline{yy'} \in \varpi^\flat S^\flat$ which contradicts with the fact that $\overline{y}, \overline{y'} \notin (\varpi^\flat)^{1/p}S^\flat$. Moreover, since $x = \theta([(\varpi^\flat)^{n/p}\overline{y}]) \cdot w$ and $x' = \theta([(\varpi^\flat)^{m/p}\overline{y'}]) \cdot w'$, we see that x divides x' or x' divides x. This also shows that S and $S^\flat$ share the same value group. In conclusion, S is also a valuation ring. □

Remark 3.12. One can show further that $\mathrm{Frac}(S)$ is algebraically closed if and only if $\mathrm{Frac}(S^\flat)$ is algebraically closed (see [CS24, 2.1.9]).

Proposition 3.13. *Let S be a perfectoid ring with tilt $S^\flat = \varprojlim_{\mathrm{Frob}} S/pS$ and strict pseudo-uniformizer ϖ. Then, there is a canonical equivalence of categories*

$$\{\textit{perfectoid valuation ring over } S\} \xrightarrow{\sim} \{\varpi^\flat\textit{-complete perfect valuation ring over } S^\flat\} \tag{3.13.1}$$
$$V \longmapsto V^\flat.$$

Proof. It follows directly from 2.25 and 3.11. □

Corollary 3.14. *Let $R \to S$ be a morphism of perfectoid rings, $R^\flat \to S^\flat$ the morphism of their tilts, ϖ a strict pseudo-uniformizer of R. Then, $R \to S$ is a ϖ-complete arc covering if and only if $R^\flat \to S^\flat$ is a $\varpi^\flat$-complete arc covering.*

Proof. Suppose firstly that $R^\flat \to S^\flat$ is a $\varpi^\flat$-complete arc covering. Let $R \to V$ be a morphism to a ϖ-complete valuation ring V of dimension ≤ 1. After replacing V by an algebraic valuation extension, we may assume that the fraction field of V is algebraically closed .

If $\varpi = 0$ in V, then $S \to V$ factors through $S/\varpi \cong S^\flat/\varpi^\flat$. Thus, there exists a valuation ring extension $V \to W$ with a morphism $S'/\varpi \cong S'^\flat/\varpi^\flat \to W$ lifting $S/\varpi \cong S^\flat/\varpi^\flat \to V$.

$$\begin{array}{ccccccccc} S' & \longrightarrow & S'/\varpi & \longrightarrow & W & \longleftarrow & S'^\flat/\varpi^\flat & \longleftarrow & S'^\flat \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ S & \longrightarrow & S/\varpi & \longrightarrow & V & \longleftarrow & S^\flat/\varpi^\flat & \longleftarrow & S^\flat \end{array} \qquad S'/\varpi \xrightarrow{\sim} S'^\flat/\varpi^\flat,\quad S/\varpi \xrightarrow{\sim} S^\flat/\varpi^\flat \tag{3.14.1}$$

If $\varpi \neq 0$ in V, then V is ϖ-torsion-free and ϖ-complete with algebraically closed fraction field. Thus, V is perfectoid by 2.39 (note that $\varpi^{1/p} \in V$). Then, $V^\flat$ is a $\varpi^\flat$-complete perfectoid valuation ring over $S^\flat$ by 3.13. There exists a valuation ring extension $V^\flat \to W$ with a morphism $S'^\flat \to W$ lifting $S^\flat \to V^\flat$ by assumption. By tilting correspondence 2.25 and 3.13 again, $W^\sharp$ is a perfectoid valuation ring with a commutative diagrams

$$\begin{array}{ccc} S' & \longrightarrow & W^\sharp \\ \uparrow & & \uparrow \\ S & \longrightarrow & V \end{array} \qquad \begin{array}{ccc} W & \longleftarrow & S'^\flat \\ \uparrow & & \uparrow \\ V^\flat & \longleftarrow & S^\flat. \end{array} \tag{3.14.2}$$

Since the value group $\mathrm{Frac}(V)^\times/V^\times \cong \mathrm{Frac}(V^\flat)^\times/(V^\flat)^\times$ is canonically embedded into $\mathrm{Frac}(W^\sharp)^\times/(W^\sharp)^\times \cong \mathrm{Frac}(W)^\times/W^\times$. We see that $V \to W^\sharp$ is an extension of valuation rings (see [Bou06a, VI, §3.5, Corollaire]).

The converse part follows from a similar argument. □

Remark 3.15. A ring homomorphism $R \to S$ is a ϖ-complete arc covering if and only if $R \to S \times R[1/\varpi]$ is an arc covering. For the necessity, let $R \to V$ be a morphism to a valuation ring V of dimension ≤ 1. If $V = V[1/\varpi]$, then $R \to V$ lifts to $R[1/\varpi] \to V$. Otherwise, $V \to \widehat{V}$ is faithfully flat (i.e., an extension of valuation rings) so that $R \to V$ lifts to $S \to W$ for some valuation extension W of $\widehat{V}$. For the sufficiency, let $R \to V$ be a morphism to a nonzero ϖ-complete valuation ring V of dimension ≤ 1 (so that the closed point of V does not lie over $R[1/\varpi]$). It lifts to $S \times R[1/\varpi] \to W$ where W is a valuation ring extension of V. Then, the closed point of W does not lie over $R[1/\varpi]$ so that $S \times R[1/\varpi] \to W$ factors through $S \to W$.

3.d. Cohomological descent of perfectoids in p-complete arc topology.

Theorem 3.16 ([BS22, 8.10]). *Let $R \to S$ be a p-complete arc covering of perfectoid rings. Then, the p-completed Čech complex*

$$0 \to R \to S \to S\widehat{\otimes}_R S \to S\widehat{\otimes}_R S\widehat{\otimes}_R S \to \cdots \tag{3.16.1}$$

is exact.

Proof. Let ϖ be a strict pseudo-uniformizer of R. Then, $R^\flat \to S^\flat \times R^\flat[1/\varpi^\flat]$ is an arc covering of $\varpi^\flat$-complete perfect $\mathbb{F}_p$-algebras by 3.14 and 3.15. Thus, the Čech complex

$$0 \to R^\flat \to S^\flat \times R^\flat[1/\varpi^\flat] \to (S^\flat \times R^\flat[1/\varpi^\flat]) \otimes_{R^\flat} (S^\flat \times R^\flat[1/\varpi^\flat]) \to \cdots \tag{3.16.2}$$

is exact by 3.9. As each term is a perfect $\mathbb{F}_p$-algebra (thus $\varpi^\flat$-torsion-bounded (2.16)), its derived $\varpi^\flat$-completion coincides with the classical $\varpi^\flat$-completion (see [Sta26, 0BKG]). Hence, taking derived $\varpi^\flat$-completion of the exact sequence (3.16.2), we obtain an exact sequence ([Sta26, 091V])

$$0 \to R^\flat \to S^\flat \to S^\flat\widehat{\otimes}_{R^\flat} S^\flat \to S^\flat\widehat{\otimes}_{R^\flat} S^\flat\widehat{\otimes}_{R^\flat} S^\flat \to \cdots . \tag{3.16.3}$$

By universal p-deformation and dévissage, we see that

$$0 \to W(R^\flat) \to W(S^\flat) \to W(S^\flat\widehat{\otimes}_{R^\flat} S^\flat) \to W(S^\flat\widehat{\otimes}_{R^\flat} S^\flat\widehat{\otimes}_{R^\flat} S^\flat) \to \cdots \tag{3.16.4}$$

is still exact. Since a distinguished generator ξ is a nonzero divisor of each term in the above exact sequence by 2.19, modulo ξ we still get an exact sequence (see 2.43)

$$0 \to R \to S \to S\widehat{\otimes}_R S \to S\widehat{\otimes}_R S\widehat{\otimes}_R S \to \cdots . \tag{3.16.5}$$

□

Definition 3.17 (Perfectoidization). Let $\mathcal{O}$ be the sheafification of the presheaf $\mathcal{O}^{\mathrm{pre}}$ on **Sch** sending a scheme X to $\Gamma(X, \mathcal{O}_X)$ in the p-complete arc topology. Then, for any ring S, we put

$$S_{\mathrm{perfd}} = \mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(S), \mathcal{O}) \in \mathbf{D}(S) \tag{3.17.1}$$

and call it the *perfectoidization* of S.

Lemma 3.18. *If S is perfectoid, then $S = S_{\mathrm{perfd}}$.*

Proof. Theorem 3.16 implies that the presheaf $\mathcal{O}^{\mathrm{pre}} = \mathrm{colim}_{\mathrm{Frob}}\, \mathcal{O}^{\mathrm{pre}} : X \mapsto \Gamma(X, \mathcal{O}_X)$ is a sheaf in the arc-topology over the opposite category $\mathbf{Perfd}_{/S}^{\mathrm{op}}$ of perfectoid S-algebras (note that the fibred product in $\mathbf{Perfd}_{/S}^{\mathrm{op}}$ is given by the p-completed tensor product of perfectoid rings by 2.43). Moreover, since $\mathbf{Perfd}_{/S}^{\mathrm{op}}$ forms a topological base of $(\mathbf{Sch}_{/S})_{p\text{-arc}}$ by 3.5, 2.39 and 2.42, we have $\mathcal{O} = \mathcal{O}^{\mathrm{pre}}$ over $\mathbf{Perfd}_{/S}^{\mathrm{op}}$. Then, by the relation between Čech cohomology and cohomology, 3.16 implies that $H^q_{p\text{-arc}}(\mathrm{Spec}(S), \mathcal{O}) = 0$ for any $q \neq 0$ ([Sta26, 03F9]). This shows that $S = S_{\mathrm{perfd}}$. □

Lemma 3.19. *Let R be a perfectoid ring, S an R-algebra. If S_{perfd} is concentrated in degree 0 (i.e., $H^q(S_{\mathrm{perfd}}) = 0$ for any $q \neq 0$), then $S_{\mathrm{perfd}} = H^0(S_{\mathrm{perfd}})$ is perfectoid.*

Proof. We write $S_0 = H^0(S_{\mathrm{perfd}})$. Let ϖ be a strict pseudo-uniformizer of R. Since $\mathbf{Perfd}_{/S}^{\mathrm{op}}$ forms a topological base of $(\mathbf{Sch}_{/S})_{p\text{-arc}}$, 3.18 implies that

$$0 \longrightarrow \mathcal{O} \xrightarrow{\cdot \varpi^{1/p^n}} \mathcal{O} \longrightarrow \mathcal{O}/\varpi^{1/p^n}\mathcal{O} \longrightarrow 0 \tag{3.19.1}$$

is exact and that the Frobenius induces an isomorphism $\mathcal{O}/\varpi^{1/p}\mathcal{O} \xrightarrow{\sim} \mathcal{O}/\varpi\mathcal{O}$. Taking cohomology at S, the condition $S_0 = S_{\mathrm{perfd}} = \mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(S), \mathcal{O})$ implies that $S_0/\varpi^{1/p^n}S_0 = \mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(S), \mathcal{O}/\varpi^{1/p^n}\mathcal{O})$ and that the Frobenius induces an isomorphism $S_0/\varpi^{1/p}S_0 \xrightarrow{\sim} S_0/\varpi S_0$. We put $\mathcal{O}^\flat = \mathrm{R}\varprojlim_{\mathrm{Frob}} \mathcal{O}/p\mathcal{O}$. Then,

(3.19.2)

$$\mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(S), \mathcal{O}^\flat) = \mathrm{R}\varprojlim_{\mathrm{Frob}} \mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(S), \mathcal{O}/p\mathcal{O}) = \mathrm{R}\varprojlim_{\mathrm{Frob}} S_0/\varpi S_0 = \varprojlim_{\mathrm{Frob}} S_0/\varpi S_0 = S_0^\flat$$

where the first equality follows from [Sta26, 0D6K] and the third equality follows from the surjectivity of the Frobenius on $S_0/\varpi S_0$ ([Sta26, 07KW]).

Consider the presheaf $W(\mathcal{O}^\flat)^{\mathrm{pre}} : \mathbf{Perfd}_{/R}^{\mathrm{op}} \to \mathbf{Set}$ sending a perfectoid R-algebra R' to $W(R'^\flat)$. By dévissage, we see that $W(\mathcal{O}^\flat)^{\mathrm{pre}}$ is a sheaf with respect to the p-complete arc topology by 3.18 and moreover $H^q(\mathrm{Spec}(R'), W(\mathcal{O}^\flat)^{\mathrm{pre}}) = 0$ for any $q \neq 0$. Let $W(\mathcal{O}^\flat)$ be the sheafification of $W(\mathcal{O}^\flat)^{\mathrm{pre}}$ over $\mathbf{Sch}_{/R}$. Then, one can check that $W(\mathcal{O}^\flat) = \mathrm{R}\lim_{n\to\infty} W(\mathcal{O}^\flat)/p^n W(\mathcal{O}^\flat)$ and that

$0 \to W(\mathcal{O}^\flat)/p^{n-1}W(\mathcal{O}^\flat) \xrightarrow{\cdot p} W(\mathcal{O}^\flat)/p^nW(\mathcal{O}^\flat) \to \mathcal{O}^\flat \to 0$ is exact by working over $\mathbf{Perfd}^{\mathrm{op}}_{/R}$. Therefore, by a similar argument as in the first paragraph, we have

$$\mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(S), W(\mathcal{O}^\flat)) = W(S_0^\flat). \tag{3.19.3}$$

Finally, let ξ be a distinguished generator of $\mathrm{Ker}(W(R^\flat) \to R)$ and take cohomology of the exact sequence

$$0 \longrightarrow W(\mathcal{O}^\flat) \xrightarrow{\cdot\xi} W(\mathcal{O}^\flat) \longrightarrow \mathcal{O} \longrightarrow 0, \tag{3.19.4}$$

which can be checked over $\mathbf{Perfd}^{\mathrm{op}}_{/R}$. We obtain an exact sequence

$$0 \longrightarrow W(S_0^\flat) \xrightarrow{\cdot\xi} W(S_0^\flat) \longrightarrow S_0 \longrightarrow 0, \tag{3.19.5}$$

which shows that S_0 is a perfectoid ring. □

Remark 3.20. For a general ring S, we don't know if $S = S_{\mathrm{perfd}}$ could imply the perfectoidness of S or not.

Remark 3.21. If we consider the category of p-adic formal schemes endowed with arc topology, then $S_{\mathrm{perfd}} = \mathrm{R}\Gamma_{\mathrm{arc}}(\mathrm{Spf}(S), \mathcal{O})$, which coincides with Bhatt-Scholze's original definition [BS22, 8.11].

Lemma 3.22. *Let R be a ring. If the category of perfectoid R-algebras admits an initial object S, then $R_{\mathrm{perfd}} = S$. In particular, if R is an $\mathbb{F}_p$-algebra, then $R_{\mathrm{perfd}} = R_{\mathrm{perf}}$.*

Proof. Firstly, we claim that the category of perfectoid $\otimes_R^n S$-algebras admits an initial object S. Indeed, the multiplication map $m : S \otimes_R \cdots \otimes_R S \to S$ defines S as a perfectoid $\otimes_R^n S$-algebra. On the other hand, for any perfectoid $\otimes_R^n S$-algebra S', the universal property of S implies that there exists a unique morphism $f : S \to S'$ fitting into the following commutative diagram

$$\begin{array}{ccc} S \otimes_R \cdots \otimes_R S & \xrightarrow{\alpha} & S' \\ \uparrow{\scriptstyle \iota_n} & & \uparrow{\scriptstyle f} \\ R & \xrightarrow{\iota_1} & S. \end{array} \tag{3.22.1}$$

Since the morphisms $S \to S \otimes_R \cdots \otimes_R S$ sending S to the i-th component with the rest coordinates to be 1 also induces a morphism $S \to S'$ (by composing with α) making the diagram commutative, the uniqueness of f implies that $\alpha(x_1 \otimes \cdots x_n) = f(x_1) \cdots f(x_n) = f(m(x_1 \otimes \cdots x_n))$. Hence, α factors uniquely through $m : S \otimes_R \cdots \otimes_R S \to S$, which verifies the claim.

Then, we claim that $R \to S$ is a p-complete arc covering. Indeed, for any morphism $R \to V$ to a p-complete valuation ring of dimension ≤ 1, we may assume that the fraction field of V is algebraically closed after extension. Thus, V is a perfectoid ring by 2.39 so that $R \to V$ factors uniquely through S by assumption.

Now we prove by induction that $H^0(R_{\mathrm{perfd}}) = S$ and $H^q(R_{\mathrm{perfd}}) = 0$ for any $q > 0$. Notice that ([Sta26, 03AZ])

$$R_{\mathrm{perfd}} = \mathrm{Tot}(S_{\mathrm{perfd}} \overset{d_0}{\underset{d_1}{\rightrightarrows}} (S \otimes_R S)_{\mathrm{perfd}} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots). \tag{3.22.2}$$

Since $S_{\mathrm{perfd}} = S$ is the initial object of the category of perfectoid R-algebras, we see that $d_0 = d_1$, i.e., $H^0(R_{\mathrm{perfd}}) = S$. Combining with our first claim, we get $H^0((S \otimes_R \cdots \otimes_R S)_{\mathrm{perfd}}) = S$. This implies that $H^1(R_{\mathrm{perfd}}) = H^1(S \to H^0((S \otimes_R S)_{\mathrm{perfd}}) \to H^0((S \otimes_R S \otimes_R S)_{\mathrm{perfd}})) = H^1(S \xrightarrow{0} S \xrightarrow{\mathrm{id}} S) = 0$. Combining with our first claim again, we get $H^1((S \otimes_R \cdots \otimes_R S)_{\mathrm{perfd}}) = 0$. This implies that $H^2(R_{\mathrm{perfd}}) = H^1(S \to H^0((S \otimes_R S)_{\mathrm{perfd}}) \to H^0((S \otimes_R S \otimes_R S)_{\mathrm{perfd}}) \to H^0((S \otimes_R S \otimes_R S \otimes_R S)_{\mathrm{perfd}})) = H^2(S \xrightarrow{0} S \xrightarrow{\mathrm{id}} S \xrightarrow{0} S) = 0$. Thus, we can get $R_{\mathrm{perfd}} = S$ by induction.

The "in particular" part follows from 2.38 (or directly from 3.10). □

4. Almost Purity Theorem

4.a. **The ideas.** Recall that Tate [Tat67] showed that for any finite field extension L of $\mathbb{Q}_p(\zeta_{p^\infty})$, its valuation ring $\mathcal{O}_L$ is almost unramified over $\mathbb{Z}_p[\zeta_{p^\infty}]$ (see 1.4).

(4.0.1)
$$\begin{array}{ccc}
L & \longleftarrow & \mathcal{O}_L \\
\text{finite}\uparrow & & \uparrow\text{almost unramified} \\
\mathbb{Q}_p(\zeta_{p^\infty}) & \longleftarrow & \mathbb{Z}_p[\zeta_{p^\infty}] \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\mathbb{Q}_p(\zeta_p) & \longleftarrow & \mathbb{Z}_p[\zeta_p] \\
\uparrow & & \uparrow \\
\mathbb{Q}_p & \longleftarrow & \mathbb{Z}_p
\end{array}$$

Later, Faltings [Fal88, Fal02] extended Tate's result to smooth varieties. More precisely, consider the smooth $\overline{\mathbb{Z}}_p$-algebra $\overline{\mathbb{Z}}_p[T_1, \ldots, T_d]$. After adding p-power roots of the local coordinates, we consider a finite étale $\overline{\mathbb{Q}}_p[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]$-algebra $R[1/p]$. Faltings proved that the integral closure R of $\overline{\mathbb{Z}}_p[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]$ in $R[1/p]$ is almost finite étale.

(4.0.2)
$$\begin{array}{ccc}
R[1/p] & \longleftarrow & R \\
\text{finite étale}\uparrow & & \uparrow\text{almost finite étale} \\
\overline{\mathbb{Q}}_p[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}] & \longleftarrow & \overline{\mathbb{Z}}_p[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}] \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\overline{\mathbb{Q}}_p[T_1^{1/p}, \ldots, T_d^{1/p}] & \longleftarrow & \overline{\mathbb{Z}}_p[T_1^{1/p}, \ldots, T_d^{1/p}] \\
\uparrow & & \uparrow \\
\overline{\mathbb{Q}}_p[T_1, \ldots, T_d] & \longleftarrow & \overline{\mathbb{Z}}_p[T_1, \ldots, T_d]
\end{array}$$

These results are known as *almost purity theorem* nowadays, following the perfectoidness of $\mathbb{Z}_p[\zeta_{p^\infty}]$ and $\overline{\mathbb{Z}}_p[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]$, whose proofs in the literature are all very profound and technical. For instance, Tate's proof makes use of local class field theory and higher ramification groups, which are specialized techniques tied to the discrete valuation structures; Faltings' original proof is even more intricate and I regret being unable to give a concise summary of it. One of the fundamental difficulty is that we don't know the precise structure of "integral closures". Indeed, we are able to compute the integral closures in some special cases, e.g. $\mathbb{Z}_p[\zeta_{p^\infty}]$ and $\overline{\mathbb{Z}}_p[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]$. But almost purity theorem requires a deeper understanding of integral closures in the general case. We will see how this can be done through the proof.

A trivial but essential case where we are able to compute the integral closure is that the integral closure of a p-torsion-free normal ring A in the $A[1/p]$-algebra $\prod_{i=1}^n A[1/p]$ is simply $\prod_{i=1}^n A$, since geometrically $\mathrm{Spec}(\prod_{i=1}^n A[1/p])$ is just the disjoint union of n copies of $\mathrm{Spec}(A[1/p])$.

(4.0.3)
$$\begin{array}{ccc}
\prod_{i=1}^n A[1/p] & \longleftarrow & \prod_{i=1}^n A \\
\text{finite étale}\uparrow & & \uparrow\text{finite étale} \\
A[1/p] & \longleftarrow & A
\end{array}$$

In general, we would like to reduce to this trivial case but not every finite Galois $A[1/p]$-algebra splits. The idea is that if there exists a faithfully flat covering of perfectoid rings $A \to B$ such that B is *absolutely integrally closed* (i.e., any monic polynomial has a root), then we may use faithfully flat descent to reduce to the trivial case. We are thus led to establish the following variant of André's flatness theorem.

Theorem 4.1. *Let A be a perfectoid ring. Then, there exists a perfectoid A-algebra B such that*

(1) *B is absolutely integrally closed (i.e., any monic polynomial has a root) and*
(2) *$A \to B$ is p-completely faithfully flat (i.e., $A/p^nA \to B/p^nB$ is faithfully flat for any $n \in \mathbb{N}$).*

André's flatness theorem was originally used to prove the direct summand conjecture [And18b, And18a] (see [Bha18a, 1.5]). It turns out to be a crucial feature of perfectoids and there appear many variants and many proofs. Especially, a conceptual proof is given in [BS22, 7.14] using prismatic cohomology. But in this lecture series, we adopt a constructive proof given in [CS24, 2.3.4] and we will see how the structures of integral closures are understood.

4.b. **Construct a perfectoid covering by adding p-power roots.** The key to André's flatness theorem is that we need to add roots of every monic polynomial while keeping perfectoidness and flatness.

Let $P \in A[T]$ be a monic polynomial. To add a root of P, we need to consider $A \to A[T]/(P)$. To get a perfectoid, we may consider $A \to A[T]/(P) \to A\langle T^{1/p^\infty}\rangle/(P)$ as in 2.37. However, although $A\langle T^{1/p^\infty}\rangle$ is perfectoid, its quotient $A\langle T^{1/p^\infty}\rangle/(P)$ isn't. To make the Frobenius surjective, we may consider the subring $A\langle T^{1/p^\infty}\rangle[\frac{P}{p^\infty}]$ of $A\langle T^{1/p^\infty}\rangle[1/p]$ generated by $P, \frac{P}{p}, \frac{P}{p^2}, \dots$ and its quotient

$$\frac{A\langle T^{1/p^\infty}\rangle[P, \frac{P}{p}, \frac{P}{p^2}, \dots]}{(P, \frac{P}{p}, \frac{P}{p^2}, \dots)}. \tag{4.1.1}$$

Note that the p-adic completion of (4.1.1) coincides with that of $A\langle T^{1/p^\infty}\rangle[\frac{P}{p^\infty}]$ (as the ideal $(P, \frac{P}{p}, \frac{P}{p^2}, \dots)$ is p-divisible) and that the canonical map

$$A\langle T^{1/p^\infty}\rangle/(p) \longrightarrow A\langle T^{1/p^\infty}\rangle[\frac{P}{p^\infty}]/(p) \tag{4.1.2}$$

is surjective (which implies that the Frobenius is surjective on (4.1.1) modulo p).

Lemma 4.2 ([CS24, 2.1.8])**.** *Let A be a ring such that there exists $\varpi \in pA^\times$ with a compatible system of p-power roots $(\varpi^{1/p^n})_{n\in\mathbb{N}}$ (e.g., if A is perfectoid, see 2.29) and let A^+ be the p-integral closure of A in $A[1/p]$ (cf. 2.41). Assume that* $\mathrm{Frob} : A/pA \to A/pA$ *is surjective. Then, the p-adic completion $\widehat{A^+}$ is perfectoid.*

Proof. After replacing A by its image in $A[1/p]$, we may assume that A is p-torsion-free. We want to eliminate the kernel of $\mathrm{Frob} : A/\varpi^{1/p}A \to A/\varpi A$. As the Frobenius is surjective, for any $a \in A$ with $a^p \in \varpi A$, we take a sequence $a = a_1, a_2, a_3, \dots$ of elements of A such that $a_{n+1}^p \equiv a_n \mod \varpi A$ for any $n \geq 1$. Then, $a_n^{p^n} \equiv a^p \equiv 0 \mod \varpi A$ so that $a_n/\varpi^{1/p^n} \in A^+$. Consider the A-subalgebra of A^+ generated by such sequences $(a_n/\varpi^{1/p^n})_{n\geq 1}$,

$$A_1 = A[a_n/\varpi^{1/p^n} \mid a \in A \text{ with } a^p \in \varpi A,\ n \geq 1] \subseteq A^+. \tag{4.2.1}$$

Then, we see that $a \in \varpi^{1/p}A_1$ and $\mathrm{Frob} : A_1/pA_1 \to A_1/pA_1$ is still surjective. Repeating this construction, we get $A \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A^+$ and let $A_\infty = \bigcup_{n\geq 1} A_n$ so that the Frobenius induces a bijection $\mathrm{Frob} : A_\infty/\varpi^{1/p}A_\infty \xrightarrow{\sim} A_\infty/\varpi A_\infty$. This shows that A_∞ is p-integrally closed (see [He24a, 5.21]) so that $A^+ = A_\infty$ and moreover $\widehat{A^+}$ is perfectoid by 2.35. □

Back to the construction of perfectoid cover by adding a root of P, we should consider

$$A \longrightarrow \frac{A\langle T^{1/p^\infty}\rangle[P, \frac{P}{p}, \frac{P}{p^2}, \dots]^+}{(P, \frac{P}{p}, \frac{P}{p^2}, \dots)}, \tag{4.2.2}$$

where $(-)^+$ means taking the p-integral closure in $(-)[1/p]$. The p-adic completion of the latter coincides with that of $A\langle T^{1/p^\infty}\rangle[P, \frac{P}{p}, \frac{P}{p^2}, \dots]^+$, which is perfectoid by (4.1.2) and 4.2. Then, it remains to show that (4.2.2) is p-completely faithfully flat. For this, we need a refined analysis on the structure of $A\langle T^{1/p^\infty}\rangle[\frac{P}{p^\infty}]^+$ rather than the inductive construction in 4.2.

4.c. Structure of $A\langle T^{1/p^\infty}\rangle[\frac{P}{p^m}]^+$.

Lemma 4.3. *Let A be a ring with a nonzero divisor ϖ and let $Q \in A\langle T^{1/p^\infty}\rangle = A[T^{1/p^\infty}]^\wedge$ (where the completion is ϖ-adic) be an element that is monic and non-constant in $(A/\varpi A)[T^{1/p^\infty}]$, $m \in \mathbb{N}_{>0}$. Then,*

$$A\langle T^{1/p^\infty}\rangle[\frac{Q}{\varpi^m}] \cong A\langle T^{1/p^\infty}\rangle[X]/(\varpi^m X - Q) \tag{4.3.1}$$

is ϖ-completely faithfully flat over A.

Proof. Note that ϖ^m, Q form a regular sequence in $A\langle T^{1/p^\infty}\rangle$, i.e., ϖ^m is a nonzero divisor in $A\langle T^{1/p^\infty}\rangle$ and Q is a non-invertible nonzero divisor in $(A/\varpi^m A)[T^{1/p^\infty}]$ (as Q is monic and non-constant). Thus, the isomorphism (4.3.1) follows from [Sta26, 0BIQ].

To see $A/\varpi^n A \to (A/p^n A)[T^{1/p^\infty}][X]/(\varpi^m X - Q)$ is faithfully flat, we take a lifting $q_n \in A[T^{1/p^\infty}]$ of Q. In particular, q_n is monic and non-constant in $(A/\varpi A)[T^{1/p^\infty}]$ so that $\varpi^m X - q_n$ forms an A-regular sequence in $A[T^{1/p^\infty}][X]$, i.e., for any residue field κ of A, $\varpi^m X - q_n$ is a non-invertible nonzero divisor in $\kappa[T^{1/p^\infty}][X]$. This implies that $A[T^{1/p^\infty}][X]/(\varpi^m X - q_n)$ is faithfully flat over A by [Sta26, 046Z] and a colimit argument. Modulo ϖ^n, we see that $(A/\varpi^n A)[T^{1/p^\infty}][X]/(\varpi^m X - Q)$ is faithfully flat over $A/\varpi^n A$. □

Lemma 4.4. *Let A be a p-torsion-free perfectoid ring with a strict pseudo-uniformizer ϖ and let $Q \in A\langle T^{1/p^\infty}\rangle$ be an element that is monic and non-constant in $(A/\varpi A)[T^{1/p^\infty}]$, $m \in \mathbb{N}_{>0}$. If Q admits a compatible system of p-power roots $(Q^{1/p^n})_{n\in\mathbb{N}}$ in $A\langle T^{1/p^\infty}\rangle$, then*

$$A\langle T^{1/p^\infty}\rangle[\frac{Q}{p^m}]^+ = A\langle T^{1/p^\infty}\rangle[\frac{Q^{1/p^\infty}}{\varpi^{m/p^\infty}}] \tag{4.4.1}$$

is p-completely faithfully flat over A whose p-adic completion is perfectoid.

Proof. Applying 4.3 (which holds for p-power roots of Q), we see that

$$A\langle T^{1/p^\infty}\rangle[\frac{Q^{1/p^n}}{\varpi^{m/p^n}}] \cong A\langle T^{1/p^\infty}\rangle[X^{1/p^n}]/(\varpi^{m/p^n} X^{1/p^n} - Q^{1/p^n}) \tag{4.4.2}$$

is p-completely faithfully flat over A. After taking filtered colimit over $n \in \mathbb{N}$, consider the exact sequences

(4.4.3)

$$\begin{array}{ccccccccc}
0 & \to & (\varpi^{m/p^\infty} X^{1/p^\infty} - Q^{1/p^\infty})/\varpi^{1/p} & \to & A\langle T^{1/p^\infty}\rangle[X^{1/p^\infty}]/\varpi^{1/p} & \to & A\langle T^{1/p^\infty}\rangle[\frac{Q^{1/p^\infty}}{\varpi^{m/p^\infty}}]/\varpi^{1/p} & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle \mathrm{Frob}} & & \\
0 & \to & (\varpi^{m/p^\infty} X^{1/p^\infty} - Q^{1/p^\infty})/\varpi & \to & A\langle T^{1/p^\infty}\rangle[X^{1/p^\infty}]/\varpi & \to & A\langle T^{1/p^\infty}\rangle[\frac{Q^{1/p^\infty}}{\varpi^{m/p^\infty}}]/\varpi & \to & 0.
\end{array}$$

It is clear that the left vertical arrow is surjective and the middle vertical arrow is bijective (2.32). Thus, the right vertical arrow is bijective by snake lemma. In particular, $A\langle T^{1/p^\infty}\rangle[\frac{Q^{1/p^\infty}}{\varpi^{m/p^\infty}}]$ is p-integrally closed and thus equal to $A\langle T^{1/p^\infty}\rangle[\frac{Q}{p^m}]^+$ (see [He24a, 5.21]), whose p-adic completion is perfectoid by 2.35. □

4.d. Approximation lemma.

To reduce to the special case 4.4, we need the following technical lemma, which captures a crucial feature of perfectoids: any element "approximately admits" a compatible system of p-power roots in view of valuation rings. Such an approximation lemma appears in Scholze's tilting correspondence of étale/analytic sites of perfectoid spaces, which might explain the underlying geometric intuition, see [Sch12, 6.7].

Proposition 4.5 (Approximation lemma, [CS24, 2.3.1])**.** *Let A be a perfectoid ring with a strict pseudo-uniformizer ϖ. For any $a \in A$ and $m \in \mathbb{N}$, there exists $b \in A^\flat$ with $b^\sharp \in A$ (which admits a compatible system of p-power roots, see* (2.28.2)*) such that for any valuation ring V over A extension of $\mathbb{Z}_p$, we have*

$$|a - b^\sharp|_V \leq |p|_V \cdot \max(|b^\sharp|_V, |p^m|_V), \tag{4.5.1}$$

where $|\ |_V$ is an associated absolute value on V.

Corollary 4.6. *With the notation in* 4.5*, we have $A[\frac{a}{p^m}]^+ = A[\frac{b^\sharp}{p^m}]^+$ (p-integral closures).*

Proof. From $|\frac{a}{p^m} - \frac{b^\sharp}{p^m}|_V < \max(|\frac{b^\sharp}{p^m}|_V, 1)$, we see that $|\frac{a}{p^m}|_V \leq 1$ if and only if $|\frac{b^\sharp}{p^m}|_V \leq 1$. Therefore, for any valuation ring V over $A[\frac{a}{p^m}]$ (or equivalently over $A[\frac{b^\sharp}{p^m}]$) extension of $\mathbb{Z}_p$, we have $|\frac{a}{p^m} - \frac{b^\sharp}{p^m}|_V < 1$. This shows that $\frac{a}{p^m} - \frac{b^\sharp}{p^m}$ is topologically nilpotent with respect to the p-adic topology (see [CS24, 2.3.2] for a detailed proof). In particular, $(\frac{a}{p^m} - \frac{b^\sharp}{p^m})^{p^r} \in pA[\frac{a}{p^m}] \cap pA[\frac{b^\sharp}{p^m}]$, which implies that $\frac{a}{p^m} \in A[\frac{b^\sharp}{p^m}]^+$ and $\frac{b^\sharp}{p^m} \in A[\frac{a}{p^m}]^+$. □

Proof of 4.5. After replacing V by an algebraic extension together with a p-adic completion, we may assume that V is perfectoid. Then, $V^\flat$ is a valuation ring with an absolute value given by $b \mapsto |b^\sharp|_V$ (see 3.11). It induces a norm on $W(A^\flat)$ by associating each $x = [x_0] + p[x_1] + \cdots$ with

(4.6.1) $$|x|_{\sup} = \sup_{i \geq 0} |x_i|_{V^\flat} = \sup_{i \geq 0} |x_i^\sharp|_V.$$

Indeed, one can use the arithmetic of Witt rings 2.13 to check that $|x|_{\sup} = 0$ if and only if $x = 0$, that $|x+y|_{\sup} \leq \max(|x|_{\sup}, |y|_{\sup})$ and that $|xy|_{\sup} \leq |x|_{\sup}|y|_{\sup}$. In the following, we denote $\overline{x} = x_0$ and $x' = [x_1] + p[x_2] + \cdots$. Note that

(4.6.2) $$|\theta(x - [\overline{x}])|_V = |px_1^\sharp + p^2 x_2^\sharp + \cdots|_V \leq \sup_{i \geq 1} |p^i|_V \cdot |x_i^\sharp|_V \leq |p|_V \cdot |x - [\overline{x}]|_{\sup}.$$

We need to find a lifting $x \in W(A^\flat)$ of $a \in A$ with $\overline{x} \in A^\flat$ satisfying

(4.6.3) $$|a - \overline{x}^\sharp|_V = |\theta(x - [\overline{x}])|_V \leq |p|_V \cdot \max(|\overline{x}^\sharp|_V, |p^m|_V).$$

At first, we fix a lifting $x_0 = [\overline{x_0}] + px'_0 \in W(A^\flat)$ of $a \in A$ and a distinguished element $\xi = [\overline{\xi}] + p\xi' \in \mathrm{Ker}(\theta : W(A^\flat) \to A)$ (where $\xi' \in W(A^\flat)^\times$). Then, we construct inductively another liftings for $n \in \mathbb{N}$,

(4.6.4) $$\begin{aligned} x_{n+1} :&= x_n - \xi\xi'^{-1}x'_n \\ &= [\overline{x_n}] + px'_n - ([\overline{\xi}]\xi'^{-1} + p)x'_n \\ &= [\overline{x_n}] - [\overline{\xi}]\xi'^{-1}x'_n. \end{aligned}$$

Note that

(4.6.5) $$|x_{n+1} - [\overline{x_n}]|_{\sup} \leq |[\overline{\xi}]|_{\sup} \cdot |\xi'^{-1}|_{\sup} \cdot |x'_n|_{\sup} \leq |\varpi|_V \cdot 1 \cdot |x_n|_{\sup} = |p|_V \cdot |x_n|_{\sup}.$$

Now, we take $0 \leq N \leq \infty$ the least element such that $|[\overline{x_N}]|_{\sup} > |p^{N+1}|_V$. Then, repeatedly using (4.6.5), we get

(4.6.6) $$\begin{aligned} |p|_V \geq |[\overline{x_0}]|_{\sup} \leq |x_0|_{\sup} \leq |1|_V, \\ |p^2|_V \geq |[\overline{x_1}]|_{\sup} \leq |x_1|_{\sup} \leq |p|_V, \\ |p^3|_V \geq |[\overline{x_2}]|_{\sup} \leq |x_2|_{\sup} \leq |p^2|_V, \\ \cdots \\ |p^N|_V \geq |[\overline{x_{N-1}}]|_{\sup} \leq |x_{N-1}|_{\sup} \leq |p^{N-1}|_V, \\ |p^{N+1}|_V < |[\overline{x_N}]|_{\sup} \leq |x_N|_{\sup} \leq |p^N|_V. \end{aligned}$$

Applying (4.6.5) once more, we get $|x_{N+1}|_{\sup} = |[\overline{x_N}]|_{\sup} \in (|p^{N+1}|_V, |p^N|_V]$. Moreover, as $|\overline{x_{N+1}} - \overline{x_N}|_{V^\flat} = |\overline{x_{N+1} - [\overline{x_N}]}|_{V^\flat} \leq |x_{N+1} - [\overline{x_N}]|_{\sup} \leq |p|_V \cdot |x_N|_{\sup} \leq |p^{N+1}|_V$, we also get $|[\overline{x_{N+1}}]|_{\sup} = |[\overline{x_N}]|_{\sup} \in (|p^{N+1}|_V, |p^N|_V]$. Thus,

(4.6.7) $$|p^{N+1}|_V < |[\overline{x_{N+1}}]|_{\sup} = |x_{N+1}|_{\sup} \leq |p^N|_V.$$

Repeating this argument, we get

(4.6.8) $$\begin{aligned} |p^{N+1}|_V < |[\overline{x_{N+2}}]|_{\sup} = |x_{N+2}|_{\sup} \leq |p^N|_V, \\ |p^{N+1}|_V < |[\overline{x_{N+3}}]|_{\sup} = |x_{N+3}|_{\sup} \leq |p^N|_V, \\ \cdots. \end{aligned}$$

In conclusion, if $m \leq N$, then we have $|x_m - [\overline{x_m}]|_{\sup} \leq |p^m|_V$; and if $m > N$, then we have $|x_m - [\overline{x_m}]|_{\sup} \leq |[\overline{x_m}]|_{\sup} = |\overline{x_m}^\sharp|_V$. Thus,

(4.6.9) $$|\theta(x_m - [\overline{x_m}])|_V \leq |p|_V \cdot |x_m - [\overline{x_m}]|_{\sup} \leq |p|_V \cdot \max(|\overline{x_m}^\sharp|_V, |p^m|_V).$$

□

Corollary 4.7. *Let A be a p-torsion-free perfectoid ring, $P \in A[T]$ a monic polynomial of positive degree, $m \in \mathbb{N}$. Then, $A\langle T^{1/p^\infty}\rangle[\frac{P}{p^m}]^+$ is p-completely faithfully flat over A whose p-adic completion is perfectoid. In particular,*

$$\frac{A\langle T^{1/p^\infty}\rangle[\frac{P}{p^\infty}]^+}{(\frac{P}{p^\infty})} = \frac{A\langle T^{1/p^\infty}\rangle[P, \frac{P}{p}, \frac{P}{p^2}, \dots]^+}{(P, \frac{P}{p}, \frac{P}{p^2}, \dots)} \tag{4.7.1}$$

is p-completely faithfully flat over A whose p-adic completion is perfectoid.

Proof. By approximation lemma 4.5 and its corollary 4.6, there exists $Q \in A\langle T^{1/p^\infty}\rangle$ admitting a compatible system of p-power roots such that $A\langle T^{1/p^\infty}\rangle[\frac{P}{p^m}]^+ = A\langle T^{1/p^\infty}\rangle[\frac{Q}{p^m}]^+$. Moreover, the proof of 4.6 also shows that $(P-Q)^{p^r} \in pA\langle T^{1/p^\infty}\rangle$ (taking $m = 0$) so that $P - Q \in \varpi^{1/p^r} A\langle T^{1/p^\infty}\rangle$ by perfectoidness. In particular, Q is monic and non-constant. Thus, the first assertion follows from 4.4. The "in particular" part follows from taking filtered colimit over $m \in \mathbb{N}$ and the p-divisibility of the ideal $(\frac{P}{p^\infty})$. ☐

4.e. **André's flatness theorem.**

Proof of 4.1. Firstly, since every perfectoid ring is a quotient of a p-torsion-free perfectoid ring ([CS24, 2.1.12]), we may assume that A is p-torsion-free. Then, we put

$$A_1 = \widehat{\bigotimes}_{P \in A[T] \text{ monic}} \frac{A\langle T^{1/p^\infty}\rangle[\frac{P}{p^\infty}]^+}{(\frac{P}{p^\infty})}, \tag{4.7.2}$$

which is a perfectoid ring p-completely faithfully flat over A (and thus p-torsion-free) by 4.7 and the arguments of 2.43. Notice that every monic A-polynomial has a root in A_1.

Then, we proceed a transfinite recursion: for any ordinal $\mathfrak{a} < \aleph_1 = \omega^\omega$, if it is a limit ordinal, i.e., $\mathfrak{a} = \bigcup_{\mathfrak{b} < \mathfrak{a}} \mathfrak{b}$, then we put $A_\mathfrak{a} = (\operatorname{colim}_{\mathfrak{b} < \mathfrak{a}} A_\mathfrak{b})^\wedge$; if it has a predecessor, i.e., $\mathfrak{a} = \mathfrak{b} + 1$, then we put $A_\mathfrak{a} = (A_\mathfrak{b})_1$ as in (4.7.2). By construction, $A_\mathfrak{a}$ is a perfectoid ring p-completely faithfully flat over A and every monic $A_\mathfrak{b}$-polynomial splits for any ordinal $\mathfrak{b} < \mathfrak{a}$.

Finally, we take $B = \operatorname{colim}_{\mathfrak{a} < \aleph_1} A_\mathfrak{a}$. Then, every monic B-polynomial splits over B and $A \to B$ is still p-completely faithfully flat. Since the well-ordered set of ordinals $\{\mathfrak{a} < \aleph_1\}$ is $\aleph_1$-filtered (see [Lur09, 5.3.1.7]), the filtered colimit over it commutes with $\aleph_1$-small limits ([Lur09, 5.3.3.3]). In particular, B is still p-adically complete and thus perfectoid by 2.35. ☐

4.f. **Almost purity theorem.**

Theorem 4.8 (Almost purity, [Sch12, 7.9])**.** *Let R be a perfectoid ring, S an integral R-algebra such that*

1. *$S[1/p]$ is finite étale over $R[1/p]$ and*
2. *S is integrally closed in $S[1/p]$.*

Then, S is a perfectoid ring almost finite étale over R.

Proof ideas of 4.8. **Step 1: Apply André's flatness theorem**. By 4.1, there exists an absolutely integrally closed and p-completely faithfully flat perfectoid R-algebra R'. As $R'[1/p]$ is also absolutely integrally closed, the finite étale $R'[1/p]$-algebra $R'[1/p] \otimes_{R[1/p]} S[1/p]$ is isomorphic to a finite product of copies of $R'[1/p]$ Zariski locally on $\operatorname{Spec}(R'[1/p])$ (see [Sta26, 0DCS, 04GG]). For simplicity, we focus on the special case where (for the general case, we refer to [BS22, 10.9])

$$R'[1/p] \otimes_{R[1/p]} S[1/p] \cong \prod_{i=1}^r R'[1/p]. \tag{4.8.1}$$

Note that $R' \otimes_R S$ is integral over R'. Its integral closure S' in $R'[1/p] \otimes_{R[1/p]} S[1/p] \cong \prod_{i=1}^r R'[1/p]$ is isomorphic to the integral closure of $\prod_{i=1}^r R'$ in $\prod_{i=1}^r R'[1/p]$. Since $\prod_{i=1}^r R'$ is perfectoid, we see that S' is perfectoid and $\prod_{i=1}^r R' \to S'$ is an almost isomorphism by 2.41 and its proof.

Step 2: Show that perfectoidization is discrete and almost finite étale over R'. As $R' \otimes_R S \to S' \times (R' \otimes_R S/p)$ is an arc-covering, there is a distinguished triangle in the derived category (see [Sta26, 0EVY, 0EVD])

$$(R' \otimes_R S)_{\mathrm{perfd}} \longrightarrow S'_{\mathrm{perfd}} \oplus (R' \otimes_R S/p)_{\mathrm{perfd}} \longrightarrow (S'/p)_{\mathrm{perfd}} \longrightarrow (R' \otimes_R S)_{\mathrm{perfd}}[1]. \tag{4.8.2}$$

Note that $(R' \otimes_R S/p)_{\mathrm{perfd}} = (R' \otimes_R S/p)_{\mathrm{perf}}$ and $(S'/p)_{\mathrm{perfd}} = (S'/p)_{\mathrm{perf}}$ as the underlying rings are in characteristic p (see 3.22). Moreover, as S' is perfectoid, $S'_{\mathrm{perfd}} = S'$ (3.18) and $(S'/p)_{\mathrm{perf}} = S'/\varpi^{1/p^\infty}$ (2.32) where ϖ is a strict pseudo-uniformizer of R. In particular, we see that

$$(R' \otimes_R S)_{\mathrm{perfd}} = \operatorname{Ker}(S' \oplus (R' \otimes_R S/p)_{\mathrm{perf}} \to S'/\varpi^{1/p^\infty}), \tag{4.8.3}$$

which is almost isomorphic to S', as $(R'\otimes_R S/p)_{\mathrm{perf}}$ and $S'/\varpi^{1/p^\infty}$ are both almost zero (i.e., killed by ϖ^{1/p^∞}). This shows that $(R'\otimes_R S)_{\mathrm{perfd}}$ is concentrated in degree 0 and thus is a perfectoid ring by 3.19. Moreover, it is almost isomorphic to $\prod_{i=1}^r R'$ and thus almost finite étale over R' of rank r.

Step 3: Base change of perfectoidization. Recall that a simplicial covering $S_\bullet$ of S by perfectoid S-algebras induces

$$S_{\mathrm{perfd}} = (S_0 \to S_1 \to S_2 \to \cdots). \tag{4.8.4}$$

Applying derived base change along $R\to R'$ and derived p-completion, we obtain

$$(R'\otimes^{\mathrm{L}}_R S_{\mathrm{perfd}})^\wedge = ((R'\otimes^{\mathrm{L}}_R S_0)^\wedge \to (R'\otimes^{\mathrm{L}}_R S_1)^\wedge \to (R'\otimes^{\mathrm{L}}_R S_2)^\wedge \to \cdots). \tag{4.8.5}$$

Notice that

$$\begin{aligned}(R'\otimes^{\mathrm{L}}_R S_i)^\wedge &= \mathrm{R}\lim_{n\to\infty}(R'\otimes^{\mathrm{L}}_R S_i)\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}/p^n\\ &= \mathrm{R}\lim_{n\to\infty} R'/p^n\otimes^{\mathrm{L}}_{R/p^n} S_i/p^n \quad \text{(as } R',\ R,\ S_i \text{ have bounded } p^\infty\text{-torsion)}\\ &= \mathrm{R}\lim_{n\to\infty} R'/p^n\otimes_{R/p^n} S_i/p^n \quad \text{(as } R\to R' \text{ is } p\text{-completely flat)}\\ &= R'\widehat{\otimes}_R S_i \quad \text{(the classical } p\text{-completion)}\\ &= (R'\otimes_R S_i)_{\mathrm{perfd}} \quad \text{(by 2.43)}.\end{aligned} \tag{4.8.6}$$

Hence, we see that (one can also argue by [BS22, 8.13])

$$\begin{aligned}(R'\otimes^{\mathrm{L}}_R S_{\mathrm{perfd}})^\wedge &= ((R'\otimes_R S_0)_{\mathrm{perfd}} \to (R'\otimes_R S_1)_{\mathrm{perfd}} \to (R'\otimes_R S_2)_{\mathrm{perfd}} \to \cdots)\\ &= (R'\otimes_R S)_{\mathrm{perfd}}.\end{aligned} \tag{4.8.7}$$

Step 4: Descend the properties of perfectoidization. We claim that S_{perfd} is a perfectoid ring almost finite étale over R of rank r. Since we have shown that $(R'\otimes_R S)_{\mathrm{perfd}}$ is a perfectoid ring, denoted by S'', we see that

$$(S_{\mathrm{perfd}}\otimes^{\mathrm{L}}_R R/p^n)\otimes_{R/p^n} R'/p^n = (R'\otimes^{\mathrm{L}}_R S_{\mathrm{perfd}})\otimes^{\mathrm{L}}_{R'} R'/p^n = S''\otimes^{\mathrm{L}}_{R'} R'/p^n \tag{4.8.8}$$

is connective (i.e., concentrated in non-positive degrees). Moreover, as $R/p^n\to R'/p^n$ is faithfully flat, we see that $S_{\mathrm{perfd}}\otimes^{\mathrm{L}}_R R/p^n$ is connective and $H^0(S_{\mathrm{perfd}}\otimes^{\mathrm{L}}_R R/p^n)\otimes_{R/p^n} R'/p^n = S''/p^n$ so that there exists a p-complete R-module M such that $H^0(S_{\mathrm{perfd}}\otimes^{\mathrm{L}}_R R/p^n) = M/p^nM$ ([Sta26, 09B8]). Therefore, the coconnective complex $S_{\mathrm{perfd}} = \mathrm{R}\lim_{n\to\infty} S_{\mathrm{perfd}}\otimes^{\mathrm{L}}_R R/p^n$ is also connective (see [Sta26, 07KW]) so that it is concentrated in degree 0 and thus a perfectoid ring by 3.19. Moreover, we see that $S_{\mathrm{perfd}}/p^n = H^0(S_{\mathrm{perfd}}\otimes^{\mathrm{L}}_R R/p^n)$ is almost finite étale over R/p^nR of rank r by faithfully flat descent, as S''/p^n is almost finite étale over R'/p^nR' of rank r. Thus, the p-complete ring S_{perfd} is also almost finite étale over R of rank r by deformation ([GR03, 5.3.27]).

Step 5: Show that perfectoidization coincides with integral closure by valuative detection. Finally, we need to show that $S = S_{\mathrm{perfd}}$. It suffices to see that $S[1/p] = S_{\mathrm{perfd}}[1/p]$ and S_{perfd} is integral over S (and then the conclusion follows from that S is integrally closed). For the first, notice that $S_{\mathrm{perfd}}[1/p]$ and $S[1/p]$ are both finite étale over $R[1/p]$ of rank r. Thus, $S_{\mathrm{perfd}}[1/p]$ is finite étale over $S[1/p]$ of rank 1 and thus $S[1/p] = S_{\mathrm{perfd}}[1/p]$. For the latter, we need the almost purity theorem for valuation rings ([GR03, 6.6.16]). This implies that for any residue field L of $S[1/p]$ (which is a finite extension of a residue field K of $R[1/p]$), any valuation ring W of L extension of $\mathbb{Z}_p$ (which is an extension of the valuation ring $V = K\cap W$, where the latter is pre-perfectoid as R is so, see [He24b, 10.19]) is pre-perfectoid (i.e., its p-adic completion $\widehat{W}$ is perfectoid and thus over S_{perfd}). In other words, for any p-complete valuation ring U extension of $\mathbb{Z}_p$, any $\mathbb{Z}_p$-homomorphism $S\to U$ factors through S_{perfd}. This shows that S_{perfd} is integral over S ([Hub93, 3.3.(i)], see also the proof of [Sta26, 090P]): indeed, suppose that $x\in S_{\mathrm{perfd}}\subseteq S[1/p]$ is not integral over S, then $x\notin S[1/x]$ in $S[1/px]$. This shows that $1/x$ is not a unit in $S[1/x]$ and thus there exists a maximal ideal $\mathfrak{m}$ of $S[1/x]$ containing $1/x$. Then, we take a p-torsion-free p-complete valuation ring U dominating the p-torsion-free local ring $S[1/x]_{\mathfrak{m}}$ ([EGA II, 7.1.4]). We see that $1/x$ lies in the maximal ideal of U so that $x\in S_{\mathrm{perfd}}$ does not lies in U, which is a contradiction. □

5. Galois Cohomology over Smooth Varieties

Recall that for any ring R, we defined the perfectoidization of R as the cohomology in the p-complete arc topology $R_{\mathrm{perfd}} = \mathrm{R}\Gamma_{p\text{-arc}}(\mathrm{Spec}(R),\mathcal{O})$. Notably, if R is perfectoid, then $R = R_{\mathrm{perfd}}$ (3.18). Thus, the cohomology groups of R_{perfd} can be generally regarded as the distance from R to perfectoids. In this section, we will discuss how we compute it when R is smooth following Faltings [Fal88].

5.a. **Perfectoidization and Galois cohomology.** Firstly, we show that perfectoidization can be computed by Galois cohomology if R admits a good perfectoid tower.

Example 5.1. Consider $R = \overline{\mathbb{Z}}_p[T^{\pm 1}]$ and

$$R_\infty = \overline{\mathbb{Z}}[T^{\pm 1/p^\infty}] \cong \bigoplus_{l/p^n \in \mathbb{Z}[1/p]} \overline{\mathbb{Z}}_p \cdot T^{l/p^n} \cong \bigoplus_{0 \le l/p^n < 1} R \cdot T^{l/p^n} \tag{5.1.1}$$

which is a faithfully flat cover of R. Moreover, it is easy to see that $R_\infty[1/p]$ is ind-finite étale over $R[1/p]$ and R_∞ (ind-smooth over R) is the integral closure of R in $R_\infty[1/p]$ (so that we understand the explicit structure of this special integral closure completely).

$$\begin{array}{ccc} R_\infty[1/p] & \longleftarrow & R_\infty \\ \text{ind-finite étale}\uparrow & & \uparrow\text{integral closure} \\ R[1/p] & \longleftarrow & R \end{array} \tag{5.1.2}$$

Note that the Frobenius induces an isomorphism $R_\infty/p^{1/p}R_\infty \xrightarrow{\sim} R_\infty/pR_\infty$ and thus the p-adic completion $\widehat{R_\infty}$ is perfectoid (2.35).

The perfectoidization of R can be computed by the following Čech complex associated to the arc covering $R \to R_\infty$ (see [Sta26, 03AZ]),

$$R_{\mathrm{perfd}} = ((R_\infty)_{\mathrm{perfd}} \to (R_\infty \otimes_R R_\infty)_{\mathrm{perfd}} \to \cdots). \tag{5.1.3}$$

Thus, we have computed out the first term $(R_\infty)_{\mathrm{perfd}} = \widehat{R_\infty}$. To compute the second term, we firstly introduce some notation:

$$\begin{array}{ccccc} K_\infty = \overline{\mathbb{Q}}_p(T^{\pm 1/p^\infty}) & \longleftarrow & \overline{\mathbb{Q}}_p[T^{\pm 1/p^\infty}] = A_\infty = R_\infty[1/p] & \longleftarrow & R_\infty \\ \uparrow & & \uparrow & & \uparrow \\ K_n = \overline{\mathbb{Q}}_p(T^{\pm 1/p^n}) & \longleftarrow & \overline{\mathbb{Q}}_p[T^{\pm 1/p^n}] = A_n = R_n[1/p] & \longleftarrow & R_n \\ \uparrow & & \text{finite Galois}\uparrow & & \uparrow\text{integral closure} \\ K = \overline{\mathbb{Q}}_p(T^{\pm 1}) & \longleftarrow & \overline{\mathbb{Q}}_p[T^{\pm 1}] = A = R[1/p] & \longleftarrow & R \end{array} \tag{5.1.4}$$

Note that K_n is a finite Galois extension of K of Galois group $\mathbb{Z}/p^n\mathbb{Z}$, whose generator σ sends T^{1/p^n} to $\zeta_{p^n} T^{1/p^n}$ for some primitive p^n-th root of unity ζ_{p^n}. By Galois theory, we have

$$\begin{aligned} K_n \otimes_K K_n &\xrightarrow{\sim} \prod_{\mathbb{Z}/p^n\mathbb{Z}} K_n \\ f \otimes g &\longmapsto (fg, f\sigma(g), \dots, f\sigma^{p^n-1}(g)). \end{aligned} \tag{5.1.5}$$

As étale base change preserves normality ([Sta26, 03GV]), we have

$$\begin{array}{ccccc} \prod_{\mathbb{Z}/p^n\mathbb{Z}} K_n & \longleftarrow & \prod_{\mathbb{Z}/p^n\mathbb{Z}} A_n & \longleftarrow & \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_n \\ \| & & \| & & \uparrow \\ K_n \otimes_K K_n & \longleftarrow & A_n \otimes_A A_n & \longleftarrow & R_n \otimes_R R_n \\ \uparrow & & \uparrow & & \uparrow \\ K_n & \longleftarrow & A_n & \longleftarrow & R_n \\ \uparrow & & \text{finite Galois}\uparrow & & \uparrow\text{integral closure} \\ K & \longleftarrow & A & \longleftarrow & R \end{array} \tag{5.1.6}$$

and taking filtered colimit over $n \in \mathbb{N}$ we obtain

$$\begin{array}{ccccc} \operatorname{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} K_\infty & \longleftarrow & \operatorname{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} A_\infty & \longleftarrow & \operatorname{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_\infty \\ \| & & \| & & \uparrow\text{integral closure} \\ K_\infty \otimes_K K_\infty & \longleftarrow & A_\infty \otimes_A A_\infty & \longleftarrow & R_\infty \otimes_R R_\infty. \end{array} \tag{5.1.7}$$

Similar as the Step 2 in the proof of 4.8, there is a canonical almost isomorphism

$$(R_\infty \otimes_R R_\infty)_{\mathrm{perfd}} \longrightarrow (\operatorname*{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_\infty)_{\mathrm{perfd}}. \tag{5.1.8}$$

Notice that the Frobenius induces an isomorphism $\mathrm{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_\infty/p^{1/p} \xrightarrow{\sim} \mathrm{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_\infty/p$. Thus, the latter perfectoidization coincides with

$$\begin{aligned}(\operatorname*{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_\infty)^\wedge &= \lim_{r\to\infty} \operatorname*{colim}_{n\in\mathbb{N}} \prod_{\mathbb{Z}/p^n\mathbb{Z}} R_\infty/p^r R_\infty \\ &= \lim_{r\to\infty} \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Map}(\mathbb{Z}/p^n\mathbb{Z}, R_\infty/p^r R_\infty) \\ &= \lim_{r\to\infty} \mathrm{Cont}(\mathbb{Z}_p, R_\infty/p^r R_\infty) \\ &= \mathrm{Cont}(\mathbb{Z}_p, \widehat{R_\infty}),\end{aligned} \tag{5.1.9}$$

where Cont denotes the set of continuous maps and we endow $R_\infty/p^r R_\infty$ (resp. $\widehat{R_\infty}$) with the discrete (resp. p-adic) topology.

In general, one can check by hand that (5.1.3) is almost isomorphic to

$$C^\bullet_{\mathrm{cont}}(\mathbb{Z}_p, \widehat{R_\infty}) = (\widehat{R_\infty} \to \mathrm{Cont}(\mathbb{Z}_p, \widehat{R_\infty}) \to \mathrm{Cont}(\mathbb{Z}_p^2, \widehat{R_\infty}) \to \dots) \tag{5.1.10}$$

the complex of continuous non-homogeneous cochains of the profinite group $\mathbb{Z}_p$ with values in $\widehat{R_\infty}$ ([AGT16, II.3.8]), where $\mathrm{d} : \widehat{R_\infty} \to \mathrm{Cont}(\mathbb{Z}_p, \widehat{R_\infty})$ sends x to $\mathrm{d}x : \sigma \mapsto \sigma x - x$ and $\mathrm{d} : \mathrm{Cont}(\mathbb{Z}_p, \widehat{R_\infty}) \to \mathrm{Cont}(\mathbb{Z}_p^2, \widehat{R_\infty})$ sends f to $\mathrm{d}f : (\sigma, \tau) \mapsto \sigma f(\tau) - f(\sigma\tau) + f(\sigma)$ and so on. Therefore, we obtain an isomorphism between perfectoidization and the Galois cohomology complex in the derived category of almost modules

$$R_{\mathrm{perfd}} \xrightarrow{\sim} \mathrm{R}\Gamma(\mathbb{Z}_p, \widehat{R_\infty}) \in \mathbf{D}^{\mathrm{al}}(R). \tag{5.1.11}$$

The same arguments prove the following proposition.

Proposition 5.2 (Faltings, [Fal88]). *Let R be an étale $\overline{\mathbb{Z}}_p[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$-algebra and we define R_∞ to be the tensor product fitting into the following pushout square*

$$\begin{array}{ccc} \overline{\mathbb{Z}}_p[T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}] & \longrightarrow & R_\infty \\ \uparrow & & \uparrow \\ \overline{\mathbb{Z}}_p[T_1^{\pm 1}, \dots, T_d^{\pm 1}] & \longrightarrow & R. \end{array} \tag{5.2.1}$$

Then, there is an isomorphism in the derived category of almost R-modules

$$R_{\mathrm{perfd}} \xrightarrow{\sim} \mathrm{R}\Gamma(\mathbb{Z}_p^d, \widehat{R_\infty}) \in \mathbf{D}^{\mathrm{al}}(R), \tag{5.2.2}$$

where we identify $\mathbb{Z}_p^d$ with the Galois group of the ind-étale homomorphism $R[1/p] \to R_\infty[1/p]$.

5.b. **Galois cohomology computation.** In fact, the basic theory of group cohomology enables us to explicitly compute $\mathrm{R}\Gamma(\mathbb{Z}_p^d, \widehat{R_\infty})$. For simplicity, we only present the case where $d = 1$.

Proposition 5.3 ([AGT16, II.3.25]). *Let M be a p-complete abelian group endowed with a continuous action of the profinite group $\mathbb{Z}_p$. Then, $\mathrm{R}\Gamma(\mathbb{Z}_p, M) \cong (M \xrightarrow{\sigma - 1} M)$, where σ is a topological generator of $\mathbb{Z}_p$.*

Example 5.4. Consider $R = \overline{\mathbb{Z}}_p[T^{\pm 1}]$ and

$$R_\infty = \overline{\mathbb{Z}}[T^{\pm 1/p^\infty}] \cong \bigoplus_{0 \le l/p^n < 1} R \cdot T^{l/p^n} = R \oplus \bigoplus_{0 < l/p^n < 1} R \cdot T^{l/p^n} = R \oplus D. \tag{5.4.1}$$

Then, from 5.3 we obtain

$$\mathrm{R}\Gamma(\mathbb{Z}_p, \widehat{R_\infty}) \cong (\widehat{R} \oplus \widehat{D} \xrightarrow{\sigma - 1} \widehat{R} \oplus \widehat{D}). \tag{5.4.2}$$

Notice that $\sigma - 1$ acts as multiplication by $\zeta_{p^n}^l - 1$ on $R \cdot T^{l/p^n}$. Thus, $\sigma - 1 : \widehat{R} \to \widehat{R}$ is the zero map and $\sigma - 1 : \widehat{D} \to \widehat{D}$ is injective with cokernel killed by $\zeta_p - 1$. This shows that

$$H^0(\mathbb{Z}_p, \widehat{R_\infty}) \cong \widehat{R}, \quad H^1(\mathbb{Z}_p, \widehat{R_\infty}) \cong \widehat{R} \oplus \widehat{D}/(\sigma - 1)\widehat{D}. \tag{5.4.3}$$

To obtain a cleaner result, we invert p on everything:

$$H^0(\mathbb{Z}_p, \widehat{R_\infty}[1/p]) \cong \widehat{R}[1/p], \quad H^1(\mathbb{Z}_p, \widehat{R_\infty}[1/p]) \cong \widehat{R}[1/p]. \tag{5.4.4}$$

The same arguments prove the following corollary.

Corollary 5.5. *With the notation in* 5.2*, for any* $q \in \mathbb{N}$*, we have*

$$H^q(\mathbb{Z}_p^d, \widehat{R_\infty}[1/p]) \cong \bigwedge\nolimits^q_{\widehat{R}[1/p]} \widehat{R}[1/p]^{\oplus d}. \tag{5.5.1}$$

Notice that both sides actually admit interpretation independent on the chosen chart (5.2.1). For simplicity, we focus on the case where $\mathrm{Spec}(R)$ is connected. Then, for the left we may consider the "universal ind-étale covering" $\overline{R}[1/p]$ of $R[1/p]$, that is, the integral closure in the maximal field extension of the fraction field of R such that the integral closure is ind-étale over $R[1/p]$. Thus, the Galois group G_R of $\overline{R}[1/p]$ of $R[1/p]$ is actually the fundamental group of $\mathrm{Spec}(R[1/p])$. Let $\overline{R}$ be the integral closure of R in $\overline{R}[1/p]$. Then, the left hand side is isomorphic to $H^q(G_R, \widehat{\overline{R}}[1/p])$ by the same proof of 5.1. On the other hand, we have $\Omega^1_{R[1/p]/\overline{\mathbb{Q}}_p} \cong R[1/p]^{\oplus d}$ and thus the right hand side is isomorphic to the p-completion of the module of q-th differentials $\widehat{\Omega}^q_{R[1/p]/\overline{\mathbb{Q}}_p}$. Then, a natural question arises as in 1.2,

Question 5.6. *Is there a canonical isomorphism* $H^q(G_R, \widehat{\overline{R}}[1/p]) \xrightarrow{\sim} \widehat{\Omega}^q_{R[1/p]/\overline{\mathbb{Q}}_p}$ *(independent of the chart* (5.2.1)*) fitting into the following commutative diagram?*

$$\begin{array}{ccc} H^q(\mathbb{Z}_p^d, \widehat{R_\infty}[1/p]) & \xrightarrow{\sim} & \bigwedge^q_{\widehat{R}[1/p]} \widehat{R}[1/p]^{\oplus d} \\ \uparrow\wr & & \uparrow\wr \\ H^q(G_R, \widehat{\overline{R}}[1/p]) & \xrightarrow{\sim} & \widehat{\Omega}^q_{R[1/p]/\overline{\mathbb{Q}}_p} \end{array} \tag{5.6.1}$$

5.c. **Faltings extension and canonical comparison.** The question 5.6 has an affirmative answer: for degree $q = 1$, the canonical isomorphism is given by the following theorem; and for general degree, it is given by the q-th wedge product of the canonical isomorphism in degree 1.

Theorem 5.7 (Faltings, see [Sch13a, 6.19], [AGT16, II.10.3.5, II.10.15] or [He25a, 8.9])**.** *Let R be a connected smooth $\overline{\mathbb{Z}}_p$-algebra which admits an étale homomorphism $f : \overline{\mathbb{Z}}_p[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \to R$. Then, there exists a canonical G_R-equivariant exact sequence (independent of f), called the* Faltings extension *of R,*

$$0 \longrightarrow \widehat{\overline{R}}[1/p] \longrightarrow \mathcal{F} \longrightarrow \widehat{\overline{R}}[1/p] \otimes_R \Omega^1_{R/\overline{\mathbb{Z}}_p}(-1) \longrightarrow 0 \tag{5.7.1}$$

such that the long exact sequence associated by taking G_R-invariants gives

$$0 \longrightarrow \widehat{R}[1/p] \xrightarrow{\sim} \mathcal{F}^{G_R} \xrightarrow{0} \widehat{R}[1/p] \otimes_R \Omega^1_{R/\overline{\mathbb{Z}}_p}(-1) \xrightarrow{\sim} H^1(G_R, \widehat{\overline{R}}[1/p]). \tag{5.7.2}$$

Proof ideas of 5.7. We follow the construction in [He25b, 9.36]. For simplicity, we assume that R is a smooth $\mathbb{Z}_p$-algebra admitting an étale homomorphism $f : \mathbb{Z}_p[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \to R$ (in general, it descends to a smooth algebra over a finite extension of $\mathbb{Z}_p$ and the following arguments still work after slight modifications) and we consider its base change to $\overline{\mathbb{Z}}_p$ and $\overline{\mathbb{Q}}_p$:

$$\begin{array}{ccccc} \overline{R}_{\overline{\mathbb{Q}}_p} & \longleftarrow & \overline{R} & & \\ \uparrow & & \uparrow & & \\ R_{\overline{\mathbb{Q}}_p} & \longleftarrow & R_{\overline{\mathbb{Z}}_p} & \longleftarrow & R \\ \uparrow & & \uparrow & & \uparrow \\ \overline{\mathbb{Q}}_p & \longleftarrow & \overline{\mathbb{Z}}_p & \longleftarrow & \mathbb{Z}_p \end{array} \tag{5.7.3}$$

We claim that f induces a morphism of exact sequences

(5.7.4)

$$\begin{array}{ccccccc} 0 \longrightarrow & 0 \oplus \overline{R}^{\oplus d} & \longrightarrow & (\overline{R}[1/p]/\overline{R}) \oplus \overline{R}[1/p]^{\oplus d} & \longrightarrow & (\overline{R}[1/p]/\overline{R}) \oplus (\overline{R}[1/p]/\overline{R})^{\oplus d} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \overline{R} \otimes_R \Omega^1_{R/\mathbb{Z}_p} & \longrightarrow & \Omega^1_{\overline{R}/\mathbb{Z}_p} & \longrightarrow & \Omega^1_{\overline{R}/R} & \longrightarrow 0 \end{array}$$

where the vertical homomorphisms are almost isomorphisms induced by sending the i-th standard base $p^{-n} \cdot \mathbf{e}_i$ to $\mathrm{d}\log \zeta_{p^n}$ if $i = 0$ and $\mathrm{d}\log T_i^{1/p^n}$ if $1 \leq i \leq d$.

In fact, the first vertical arrow induced by f is an isomorphism since f is étale.

For the middle vertical arrow, we decompose $\mathbb{Z}_p \to \overline{R}$ into the tower $\mathbb{Z}_p \to \overline{\mathbb{Z}}_p \to \overline{R}$. Then, we can compute $\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}$ by the tower $\mathbb{Z}_p \to \mathbb{Z}_p[\zeta_{p^\infty}] \to \overline{\mathbb{Z}}_p$ (note that $\overline{\mathbb{Z}}_p$ is almost unramified over $\mathbb{Z}_p[\zeta_{p^\infty}]$ by almost purity). This shall induce the part $(\overline{R}[1/p]/\overline{R})$. Similarly, we can also compute $\Omega^1_{\overline{R}/\overline{\mathbb{Z}}_p}$ by the tower $\overline{\mathbb{Z}}_p \to (R_\infty)_{\overline{\mathbb{Z}}_p} \to \overline{R}$ (note that $\overline{R}$ is almost unramified over $(R_\infty)_{\overline{\mathbb{Z}}_p}$ by almost purity). This shall induce the part $\overline{R}[1/p]^{\oplus d}$.

Finally, we get the third vertical arrow by taking quotients. We refer to [He25b, 9.32] for a detailed proof.

The claim implies the second row in (5.7.4) is almost exact. Applying $\mathrm{RHom}(\mathbb{Z}/p^n\mathbb{Z}, -)$ (note that $\mathbb{Z}/p^n\mathbb{Z}$ admits a projective resolution $0 \to \mathbb{Z} \xrightarrow{\cdot p^n} \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$), we get an almost exact sequence

$$0 = \overline{R} \otimes \Omega^1_{R/\mathbb{Z}_p}[p^n] \to \Omega^1_{\overline{R}/\mathbb{Z}_p}[p^n] \to \Omega^1_{\overline{R}/R}[p^n] \to \overline{R} \otimes \Omega^1_{R/\mathbb{Z}_p}/p^n \to \Omega^1_{\overline{R}/\mathbb{Z}_p}/p^n = 0. \tag{5.7.5}$$

Taking inverse limit over $n \in \mathbb{N}$, we obtain an almost exact sequence

$$0 \longrightarrow \lim_{n\to\infty} \Omega^1_{\overline{R}/\mathbb{Z}_p}[p^n] \longrightarrow \lim_{n\to\infty} \Omega^1_{\overline{R}/R}[p^n] \longrightarrow \widehat{\overline{R}} \otimes_R \Omega^1_{R/\mathbb{Z}_p} \longrightarrow 0. \tag{5.7.6}$$

Note that there is a canonical almost isomorphism $\widehat{\overline{R}}(1) := \widehat{\overline{R}} \cdot (\mathrm{d}\log \zeta_{p^n})_{n\in\mathbb{Z}} \longrightarrow \lim_{n\to\infty} \Omega^1_{\overline{R}/\mathbb{Z}_p}[p^n]$ by (5.7.4). Thus, inverting p and twist by $\widehat{\overline{R}}(-1)$, we get a canonical exact sequence

$$0 \longrightarrow \widehat{\overline{R}}[1/p] \longrightarrow \mathcal{F} \longrightarrow \widehat{\overline{R}}[1/p] \otimes_R \Omega^1_{R/\mathbb{Z}_p}(-1) \longrightarrow 0, \tag{5.7.7}$$

where $\mathcal{F} = (\lim_{n\to\infty} \Omega^1_{\overline{R}/R}[p^n])[1/p](-1)$. Since we have an explicit coordinate-form (5.7.4) for this construction, it is easy to check that the coboundary map associated to the G_R-invariant part of this exact sequence is an isomorphism (see the proof of [He25a, 8.7]). ☐

5.d. **Hodge-Tate decomposition.** Now, we can summarize Faltings' computation of the perfectoidization of smooth algebras, which ultimately leads us to an affirmative answer of our initial question (1.1.7), i.e., the Hodge-Tate decomposition for proper smooth p-adic varieties.

Theorem 5.8 (Faltings' computation of Galois cohomology)**.** *For any smooth $\overline{\mathbb{Z}}_p$-algebra R and any $q \in \mathbb{Z}$, there is a canonical isomorphism*

$$\widehat{\Omega}^q_{R[1/p]/\overline{\mathbb{Q}}_p}(-q) \xrightarrow{\sim} H^q(R_{\mathrm{perfd}})[1/p] = H^q_{p\text{-arc}}(\mathrm{Spec}(R), \mathcal{O})[1/p]. \tag{5.8.1}$$

Proof. Note that R locally admits an étale homomorphism from $\overline{\mathbb{Z}}_p[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$ and that each term in (5.8.1) satisfies the Zariski descent (as $\widehat{\Omega}^q_{R[1/p]/\overline{\mathbb{Q}}_p} = \widehat{R}[1/p] \otimes_R \Omega^q_{R/\overline{\mathbb{Z}}_p}$). Thus, the conclusion follows from the local case discussed in the previous two subsections. ☐

Theorem 5.9 (Faltings' main comparison theorem, see [Sch13a, 5.1])**.** *For any proper smooth $\overline{\mathbb{Z}}_p$-scheme X and any $q \in \mathbb{Z}$, there is a canonical isomorphism*

$$H^q_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{C}_p) \xrightarrow{\sim} H^q_{p\text{-arc}}(X, \mathcal{O})[1/p]. \tag{5.9.1}$$

Proof ideas of 5.9. **Step 1: transfer the étale cohomology to p-complete arc cohomology**. In fact, for any $\overline{\mathbb{Z}}_p$-scheme X, we have $H^q_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\sim} H^q_{p\text{-arc}}(X, \mathbb{Z}/p^n\mathbb{Z})$. This can be checked by reducing X to valuation ring with algebraically closed fraction field (see [He24a, 3.27]).

Step 2: almost finiteness of étale cohomology when X is proper. Using the local Galois cohomology computation, one can prove that $H^q_{p\text{-arc}}(X, \mathcal{O}/p)$ is almost finitely generated over $\mathcal{O}_{\mathbb{C}_p} = \widehat{\overline{\mathbb{Z}}}_p$ (see [Sch13a, 5.8]). Using some almost algebra lemma [Sch13a, 2.12], one can show that $H^q_{p\text{-arc}}(X, \mathcal{O}^\flat)$ is almost isomorphic to $(\mathcal{O}^\flat_{\mathbb{C}_p})^{\oplus r}$ (see [Sch13a, page 34]). In particular, $H^q_{p\text{-arc}}(X, \mathcal{O}^\flat[1/p^\flat]) = (\mathbb{C}_p^\flat)^{\oplus r}$ and $H^q_{p\text{-arc}}(X, \mathcal{O}/p) = H^q_{p\text{-arc}}(X, \mathcal{O}^\flat/p^\flat)$ is almost isomorphic to $(\mathcal{O}_{\mathbb{C}_p}/p)^{\oplus r} = (\mathcal{O}^\flat_{\mathbb{C}_p}/p^\flat)^{\oplus r}$.

Step 3: apply Artin-Schreier sequence $0 \to \mathbb{F}_p \to \mathcal{O}^\flat[1/p^\flat] \xrightarrow{X^p - X} \mathcal{O}^\flat[1/p^\flat] \to 0$ in characteristic p. We get $H^q_{p\text{-arc}}(X, \mathbb{F}_p) = H^q_{p\text{-arc}}(X, \mathcal{O}^\flat[1/p^\flat])^{\mathrm{Frob}=1} = ((\mathbb{C}_p^\flat)^{\oplus r})^{\mathrm{Frob}=1} = \mathbb{F}_p^{\oplus r}$. Therefore, $H^q_{p\text{-arc}}(X, \mathcal{O}_{\mathbb{C}_p}/p) = (\mathcal{O}_{\mathbb{C}_p}/p)^{\oplus r}$ is canonically almost isomorphic to $H^q_{p\text{-arc}}(X, \mathcal{O}/p)$. The conclusion follows from dévissage and inverting p. ☐

Theorem 5.10 (Hodge-Tate decomposition)**.** *For any proper smooth $\overline{\mathbb{Z}}_p$-scheme X and any $n \in \mathbb{Z}$, there is a canonical* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-equivariant decomposition*

$$H^n_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{C}_p) \cong \bigoplus_{i+j=n} H^i(X_{\mathbb{C}_p}, \Omega^j_{X_{\mathbb{C}_p}/\mathbb{C}_p})(-j). \tag{5.10.1}$$

Proof ideas of 5.10. Consider the morphism of ringed sites $\nu : (\mathbf{Sch}_{/X})_{p\text{-arc}} \to (\mathbf{Sch}_{/X})_{\mathrm{Zar}}$. By 5.8, we have $\mathrm{R}^j\nu_*\mathcal{O}[1/p] \cong \widehat{\Omega}^j_{X/\overline{\mathbb{Z}}_p}[1/p](-j)$. The Cartan-Leray spectral sequence induces a convergent $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant spectral sequence ([Sta26, 015N])

$$H^i_{\mathrm{Zar}}(X, \widehat{\Omega}^j_{X/\overline{\mathbb{Z}}_p}[1/p](-j)) \Rightarrow H^{i+j}_{p\text{-arc}}(X, \mathcal{O}[1/p]) = H^{i+j}_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{C}_p), \tag{5.10.2}$$

where the last equality follows from 5.9. As X is proper and $\Omega^j_{X/\overline{\mathbb{Z}}_p}$ is coherent, the comparison between formal geometry and algebraic geometry gives $H^i_{\mathrm{Zar}}(X, \widehat{\Omega}^j_{X/\overline{\mathbb{Z}}_p}) = H^i(X, \Omega^j_{X/\overline{\mathbb{Z}}_p}) \otimes_{\overline{\mathbb{Z}}_p} \mathcal{O}_{\mathbb{C}_p}$ (see [Abb10, 2.12.2]). Thus, we obtain a canonical $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant convergent spectral sequence

$$H^i(X_{\mathbb{C}_p}, \Omega^j_{X_{\mathbb{C}_p}/\mathbb{C}_p})(-j) \Rightarrow H^{i+j}_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{C}_p). \tag{5.10.3}$$

By Tate's computation [Tat67, §3.3] of Galois cohomology of $\mathbb{C}_p$ and its twists (see also 1.4 and 5.8), one can check that the differential maps in the second page of the spectral sequence are zero and that the filtration associated to the spectral sequence splits uniquely. □

Remark 5.11. Since Tate [Tat67] proposed the conjecture on Hodge-Tate decomposition for p-adic varieties, there have been tremendous brilliant work on this project. This conjecture was settled by Faltings [Fal88, Fal02], Nizioł [Niz98, Niz08] and Tsuji [Tsu99, Tsu02] through different approaches. Later, Faltings' approach was generalized to perfectoid method by Scholze [Sch12] who thus extended Hodge-Tate decomposition to proper smooth rigid analytic varieties [Sch13a, Sch13b]. Over the decade since Scholze's revolutionary theory, there appeared many different variants of Hodge-Tate decompositions in different contexts, including integral versions [BMS18] (leading to prismatic cohomology), relative versions [AG24], non-smooth versions [Guo23]... Although the readers may find the existing literature on Hodge-Tate decomposition quite different from our presentation, the essences of how we play with perfectoids share the same.

6. Galois Cohomology over Valuation Rings

In the last section, we discuss how to understand the ramification over p-adic smooth varieties. In this section, we move to the ramification over general (non-discrete) valuation rings. The difficulty for this extension already lies in the construction of a good perfectoid cover. In the smooth case, one can construct $R_\infty = R[T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}]$ for a system of local coordinates of R. However, for a general valuation ring, it's a question that which coordinates we should pick and what the structure of the integral closure is after adding the p-power roots of those coordinates. To understand better the case of valuation rings, we first extend basic ramification theory for discrete valuation rings to the most general case.

6.a. **Different ideals and differentials.** We fix a finite separable extension L/K of non-discrete Henselian valuation fields of rank 1 (i.e., $\dim \mathcal{O}_L = \dim \mathcal{O}_K = 1$). Although we can treat discrete and non-discrete case together as in [He25a], we focus on the latter to simplify the notation. A key feature in the non-discrete case is that the maximal ideals of the valuation rings satisfy

$$\mathfrak{m}_K^2 = \mathfrak{m}_K, \quad \mathfrak{m}_L = \mathfrak{m}_K\mathcal{O}_L. \tag{6.0.1}$$

As L is finite free over K, we can still consider the trace morphism

$$\mathrm{Tr}_{L/K} : L \longrightarrow K \tag{6.0.2}$$

which sends x to the trace of the K-linear homomorphism given by multiplication by x. By Galois theory (as L/K is separable), $\mathrm{Tr}_{L/K}(x) = \sum_{\sigma: L\hookrightarrow \overline{K}} \sigma(x)$, where σ runs through all the field embeddings of L into an algebraic closure of K. Moreover, the trace morphism induces a perfect pairing $L \times L \to K$ sending (x, y) to $\mathrm{Tr}_{L/K}(xy)$ ([GR03, 4.1.14]), namely it induces an isomorphism

$$L \xrightarrow{\sim} \mathrm{Hom}_K(L, K),\ x \mapsto (y \mapsto \mathrm{Tr}_{L/K}(xy)). \tag{6.0.3}$$

We define the *codifferent ideal* of L/K to be the $\mathcal{O}_L$-submodule of L,

$$\mathcal{O}_L^* = \{x \in L \mid \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_K,\ \forall y \in \mathcal{O}_L\}. \tag{6.0.4}$$

Under the above isomorphism (6.0.3), we see that $\mathcal{O}_L^*$ is identified with $\{f \in \mathrm{Hom}_K(L, K) \mid f(\mathcal{O}_L) \subseteq \mathcal{O}_K\}$. It is clear that $\mathcal{O}_L \subseteq \mathcal{O}_L^* \subseteq L$. We define the *different ideal* of L/K to be the "inverse" of the codifferent ideal

$$\mathscr{D}_{L/K} = \{x \in L \mid x\mathcal{O}_L^* \subseteq \mathcal{O}_L\} \subseteq \mathcal{O}_L. \tag{6.0.5}$$

This is actually the "inverse" in the almost sense as we have

$$\mathfrak{m}_L \subseteq \mathcal{O}_L^* \cdot \mathscr{D}_{L/K} \subseteq \mathcal{O}_L. \tag{6.0.6}$$

Proposition 6.1 (Relation with trace, [He25a, 4.3], cf. the discrete case [Ser79, III.§3, Proposition 7])**.** *For any nonzero fractional ideals $\mathfrak{a}$ of $\mathcal{O}_K$ and $\mathfrak{b}$ of $\mathcal{O}_L$, we have*

$$\mathrm{Tr}_{L/K}(\mathfrak{m}_L\mathfrak{b}) \subseteq \mathfrak{a} \Longleftrightarrow \mathfrak{m}_L\mathfrak{b}\mathscr{D}_{L/K} \subseteq \mathfrak{a}\mathcal{O}_L. \tag{6.1.1}$$

Proof. We put $\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subseteq \mathcal{O}_K\}$. Then, one can easily check from the definitions the following equivalences

$$\mathrm{Tr}_{L/K}(\mathfrak{m}_L\mathfrak{b}) \subseteq \mathfrak{a} \Leftrightarrow \mathrm{Tr}_{L/K}(\mathfrak{m}_L\mathfrak{a}^{-1}\mathfrak{b}) \subseteq \mathcal{O}_K \Leftrightarrow \mathfrak{m}_L\mathfrak{b} \subseteq \mathfrak{a}\mathcal{O}_L^* \Leftrightarrow \mathfrak{m}_L\mathfrak{b}\mathscr{D}_{L/K} \subseteq \mathfrak{a}\mathcal{O}_L \tag{6.1.2}$$

using the fact that $\mathfrak{m}_K^2 = \mathfrak{m}_K$, $\mathfrak{m}_L = \mathfrak{m}_K\mathcal{O}_L$, $\mathfrak{m}_K \subseteq \mathfrak{a}\cdot\mathfrak{a}^{-1} \subseteq \mathcal{O}_K$ and $\mathfrak{m}_L \subseteq \mathcal{O}_L^* \cdot \mathscr{D}_{L/K} \subseteq \mathcal{O}_L$. □

Corollary 6.2 ([He25a, 4.5])**.** *For any $x \in L$, there is an inequality of absolute values*

$$|\mathrm{Tr}_{L/K}(x)| \leq |\mathscr{D}_{L/K}| \cdot |x|. \tag{6.2.1}$$

Proof. Since the valuations on K and L are non-discrete, the absolute values of the test fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ are dense in $\mathbb{R}$. Thus, the conclusion follows directly from 6.1. □

Corollary 6.3. *We have*

$$|[L : K]| \leq |\mathscr{D}_{L/K}| \leq 1. \tag{6.3.1}$$

Proof. Take $x = 1$ in 6.2. □

Theorem 6.4 (Relation with differentials, [GR03, 6.3.8, 6.3.23])**.** *The $\mathcal{O}_K$-module $\mathcal{O}_L$ is almost finite projective and the module of differentials $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is uniformly almost finitely generated. Moreover, $\mathscr{D}_{L/K}$ is almost isomorphic to $\prod_{q=1}^{\infty} \mathrm{Ann}_{\mathcal{O}_L}(\Omega^q_{\mathcal{O}_L/\mathcal{O}_K})$. In particular, we have*

$$\mathfrak{m}_L \cdot \mathscr{D}_{L/K} \subseteq \mathrm{Ann}_{\mathcal{O}_L}(\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}). \tag{6.4.1}$$

The idea of its proof is to understand the structure of $\mathcal{O}_L$ when $[L : K]$ is prime, see [GR03, 6.3.13]. Moreover, using the same idea, Gabber-Ramero computed the cotangent complex:

Theorem 6.5 ([GR03, 6.3.32, 6.5.20])**.** *We have*

$$\mathbb{L}_{\mathcal{O}_L/\mathcal{O}_K} = \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}. \tag{6.5.1}$$

Moreover, if K is algebraically closed, then $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is torsion-free.

This suggests that a general valuation ring extension $\mathcal{O}_K \to \mathcal{O}_L$ behaves like a smooth morphism. Using this idea, we can explore further the structure of $\mathcal{O}_K$ and its extensions in the following.

6.b. **Structure of $\Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ and control of $\mathscr{D}_{K_\infty/K}$.** We fix a Henselian valuation field K of rank 1 extension of $\overline{\mathbb{Q}}_p$. After 6.5, we know that $\Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ is a torsion-free $\mathcal{O}_K$-submodule of $\Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}[1/p] = \Omega^1_{K/\overline{\mathbb{Q}}_p}$, where the latter is a K-module whose dimension is the transcendental degree $\mathrm{trdeg}(K/\overline{\mathbb{Q}}_p)$ of $K/\overline{\mathbb{Q}}_p$. We identify additional properties of $\Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ that make $\mathcal{O}_K$ behave more like a smooth algebra over $\overline{\mathbb{Z}}_p$.

Theorem 6.6 (Structure of torsion-free modules, [He25a, 3.16, 3.20])**.** *Let M be a torsion-free $\mathcal{O}_K$-module with $n = \dim_K M[1/p] < \infty$. Then, there is an exact sequence*

$$0 \longrightarrow \widehat{K}^{\oplus r} \longrightarrow \mathcal{O}_{\widehat{K}} \otimes_{\mathcal{O}_K} M \longrightarrow \widehat{M} \longrightarrow 0. \tag{6.6.1}$$

Moreover, the p-adic completion $\widehat{M}$ is uniformly almost finite free over $\mathcal{O}_{\widehat{K}}$ (i.e., for any $\epsilon \in \mathfrak{m}_K$ there exists a finite free $\mathcal{O}_K$-submodule M_ϵ of rank $n - r$ of M with cokernel M/M_ϵ killed by ϵ).

Corollary 6.7 ([He25a, 7.1])**.** *Assume that* $\mathrm{trdeg}(K/\overline{\mathbb{Q}}_p) < \infty$*. Then, the canonical morphism*

$$\mathcal{O}_{\widehat{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} \longrightarrow \widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} \tag{6.7.1}$$

is surjective. Moreover, there exists a number $d \leq$ $\mathrm{trdeg}(K/\overline{\mathbb{Q}}_p)$ such that for any $\epsilon \in \mathfrak{m}_K$ there exists $t_1, \dots, t_d \in \mathcal{O}_K^\times$ (depending on ϵ) such that

$$\epsilon \cdot \widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} \subseteq \mathcal{O}_{\widehat{K}} \cdot \mathrm{d}t_1 \oplus \cdots \mathcal{O}_{\widehat{K}} \cdot \mathrm{d}t_d \subseteq \widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}. \tag{6.7.2}$$

For simplicity, we assume that $d = 1$ in the following so that for any $\epsilon \in \mathfrak{m}_K \setminus p\mathcal{O}_K$ there exists $t \in \mathcal{O}_K^\times$ (depending on ϵ) such that

$$\epsilon \cdot \widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} \subseteq \mathcal{O}_{\widehat{K}} \cdot \mathrm{d}t \subseteq \widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}. \tag{6.7.3}$$

We want to understand the ramification of $K_n = K(t^{1/p^n})$ over K. Our expectation is that $\mathcal{O}_{K_n}$ should be close to $\mathcal{O}_K[t^{1/p^n}]$.

Lemma 6.8 ([He25a, 6.2])**.** *For* $\mathrm{d}t^{1/p^n} \in \Omega^1_{\mathcal{O}_{K_n}/\mathcal{O}_K}$*, we have*

$$|p^n| \leq |\mathrm{Ann}_{\mathcal{O}_{K_n}}(\mathrm{d}t^{1/p^n})| \leq |p^n/\epsilon|. \tag{6.8.1}$$

Proof. One the one hand, as $p^n t^{\frac{p^n-1}{p^n}} \mathrm{d}t^{\frac{1}{p^n}} = \mathrm{d}t$, we have $p^n \in \mathrm{Ann}_{\mathcal{O}_{K_n}}(\mathrm{d}t^{1/p^n})$ (as t is a unit). On the other hand, consider the exact sequence

$$0 = H_1(\mathbb{L}_{\mathcal{O}_{K_n}/\mathcal{O}_K}) \longrightarrow \mathcal{O}_{K_n} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} \xrightarrow{f} \Omega^1_{\mathcal{O}_{K_n}/\overline{\mathbb{Z}}_p} \longrightarrow \Omega^1_{\mathcal{O}_{K_n}/\mathcal{O}_K} \longrightarrow 0. \tag{6.8.2}$$

Suppose that there exists a nonzero element $\epsilon' \in \mathcal{O}_K$ with $|\epsilon'| < |\epsilon|$ such that $p^n/\epsilon' \in \mathrm{Ann}_{\mathcal{O}_{K_n}}(\mathrm{d}t^{1/p^n})$. Then, $p^n/\epsilon' \cdot \mathrm{d}t^{1/p^n} = f(\omega)$ for some $\omega \in \mathcal{O}_{K_n} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$. This implies that $f(\mathrm{d}t) = p^n t^{\frac{p^n-1}{p^n}} \mathrm{d}t^{\frac{1}{p^n}} = \epsilon' t^{\frac{p^n-1}{p^n}} f(\omega) = f(\epsilon' t^{\frac{p^n-1}{p^n}} \omega)$. Since f is injective, we get $\mathrm{d}t \in \epsilon' \cdot \mathcal{O}_{K_n} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ and thus $\mathrm{d}t \in \epsilon' \cdot \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ as $\Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ is flat over $\mathcal{O}_K$. This contradicts with (6.7.3). Therefore, $|\mathrm{Ann}_{\mathcal{O}_{K_n}}(\mathrm{d}t^{1/p^n})| \leq |p^n/\epsilon|$. □

Proposition 6.9 ([He25a, 6.3])**.** *We have*

$$|p^n| \leq |\mathscr{D}_{K_n/K}| \leq |p^n/\epsilon|. \tag{6.9.1}$$

Proof. It follows from 6.4 and 6.8 that $\mathfrak{m}_{K_n}\mathscr{D}_{K_n/K} \subseteq \mathrm{Ann}_{\mathcal{O}_{K_n}}(\Omega^1_{\mathcal{O}_{K_n}/\mathcal{O}_K}) \subseteq \mathrm{Ann}_{\mathcal{O}_{K_n}}(\mathrm{d}t^{1/p^n}) \subseteq p^n/\epsilon \cdot \mathcal{O}_{K_n}$, which implies that $|\mathscr{D}_{K_n/K}| \leq |p^n/\epsilon|$. The other inequality follows from 6.3 as $[K_n : K] = p^n$ (see [He25a, 6.1]). □

Corollary 6.10 ([He25a, 6.6])**.** *For any* $x \in K_n$*, we have*

$$|p^{-n}\mathrm{Tr}_{K_n/K}(x)| \leq |x/\epsilon|. \tag{6.10.1}$$

Proof. It follows directly from 6.2 and 6.9. □

6.c. **Structure of $\mathcal{O}_{K_\infty}$.** As Qu-Yu pointed out, we can already control the structure of $\mathcal{O}_{K_n}$.

Proposition 6.11 ([He25a, §6], see [QY25, 7.6])**.** *We have*

$$\epsilon \cdot \mathcal{O}_{K_n} \subseteq \mathcal{O}_K[t^{1/p^n}] \subseteq \mathcal{O}_{K_n}. \tag{6.11.1}$$

Proof. For any $x \in \epsilon \cdot \mathcal{O}_{K_n}$, we write $x = a_0 + a_1 t^{\frac{1}{p^n}} + \cdots a_{p^n-1} t^{\frac{p^n-1}{p^n}}$ for some unique $a_i \in K$ where $0 \leq i \leq p^n - 1$ (as $K_n = K(t^{1/p^n})$ is of degree p^n over K). Notice that for any $a \in K$,

$$\mathrm{Tr}_{K_n/K}(a \cdot t^{\frac{i}{p^n}}) = a \cdot \sum_{j=0}^{p^n-1} (\zeta_{p^n}^j t^{\frac{1}{p^n}})^i = a t^{\frac{i}{p^n}} \cdot \sum_{j=0}^{p^n-1} (\zeta_{p^n}^i)^j = \begin{cases} p^n a, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases} \tag{6.11.2}$$

Thus, we have

$$\begin{aligned} |a_i| &= |p^{-n}\mathrm{Tr}_{K_n/K}(x \cdot t^{-\frac{i}{p^n}})| \quad \text{(by (6.11.2))} \\ &\leq |x \cdot t^{-\frac{i}{p^n}}/\epsilon| \quad \text{(by 6.10)} \\ &= |x/\epsilon| \leq 1, \end{aligned} \tag{6.11.3}$$

i.e., $a_i \in \mathcal{O}_K$. □

In general, for d not necessarily equal to 1, one can still prove the following theorem using the same arguments.

Theorem 6.12 ([He25a, §7], see [QY25, 7.6])**.** *For any Henselian valuation field K of rank 1 extension of $\overline{\mathbb{Q}}_p$ with $\mathrm{trdeg}(K/\overline{\mathbb{Q}}_p) < \infty$, there exists a number $d \leq \mathrm{trdeg}(K/\overline{\mathbb{Q}}_p)$ such that for any $\epsilon \in \mathfrak{m}_K$, there exist $t_1, \ldots, t_d \in \mathcal{O}_K^\times$ with*

$$\epsilon \cdot \mathcal{O}_{K_\infty} \subseteq \mathcal{O}_K[t_1^{1/p^\infty}, \ldots, t_d^{1/p^\infty}] \subseteq \mathcal{O}_{K_\infty}, \tag{6.12.1}$$

where $K_\infty = K(t_1^{1/p^\infty}, \ldots, t_d^{1/p^\infty})$ is a Galois extension of K with Galois group canonically isomorphic to $\mathbb{Z}_p^d$ and valuation ring $\mathcal{O}_{K_\infty}$ pre-perfectoid.

Then, one can use the same arguments as in 5.5 to compute the Galois cohomology.

Corollary 6.13. *For any $q \in \mathbb{N}$, we have*

$$H^q(\mathbb{Z}_p^d, \widehat{K_\infty}) \cong \bigwedge\nolimits_{\widehat{K}}^q \widehat{K}^{\oplus d}. \tag{6.13.1}$$

6.d. **Faltings extension and canonical comparison.**

Theorem 6.14 ([He25a, 8.6, 8.9])**.** *Let K be a Henselian valuation field of rank 1 extension of $\overline{\mathbb{Q}}_p$ with* $\mathrm{trdeg}(K/\overline{\mathbb{Q}}_p) < \infty$*, $\overline{K}$ an algebraic closure of K with Galois group G_K. Then, there exists a canonical G_K-equivariant exact sequence, called the* Faltings extension *of K,*

$$0 \longrightarrow \widehat{\overline{K}} \longrightarrow \mathcal{F}_K \longrightarrow \widehat{\overline{K}} \otimes_K \Omega^1_{K/\overline{\mathbb{Q}}_p}(-1) \longrightarrow 0 \tag{6.14.1}$$

such that the long exact sequence associated by taking G_K-invariants gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{K} & \longrightarrow & \mathcal{F}_K^{G_K} & \longrightarrow & \widehat{K} \otimes_K \Omega^1_{K/\overline{\mathbb{Q}}_p}(-1) & \xrightarrow{\quad\delta\quad} & H^1(G_K, \widehat{\overline{K}}). \\ & & & & & & \searrow^{(6.7.1)} & & \nearrow_{\sim} \\ & & & & & & & \widehat{\Omega}^1_{K/\overline{\mathbb{Q}}_p}(-1) & \end{array} \tag{6.14.2}$$

6.e. **Perfectoidness criterion.** In p-adic arithmetic geometry, we frequently encounter the following question: given an $\overline{\mathbb{Z}}_p$-algebra R and a specific ind-finite étale $R[1/p]$-algebra $\widetilde{R}[1/p]$, is there a simple condition of $\widetilde{R}[1/p]$ that forces the integral closure of R in $\widetilde{R}[1/p]$ pre-perfectoid?

$$\begin{array}{ccc} \widetilde{R}[1/p] & \longleftarrow & \widetilde{R} \\ {\scriptstyle \text{ind-finite étale}}\uparrow & & \uparrow{\scriptstyle \text{integral closure}} \\ R[1/p] & \longleftarrow & R \\ \uparrow & & \uparrow \\ \overline{\mathbb{Q}}_p & \longleftarrow & \overline{\mathbb{Z}}_p \end{array} \tag{6.14.3}$$

We haven't found a satisfying criterion for this question, since we know little about the structure of the integral closure $\widetilde{R}$ in general. Instead, my recent work [He26] adopted a valuative point of view which finally leads to a satisfactory answer. More precisely, we want a suitable condition on $\widetilde{R}[1/p]$ such that $\widetilde{R}$ is *pointwise perfectoid*, namely for any residue field K of $\widetilde{R}[1/p]$, any valuation ring $\mathcal{O}_K$ of K containing $\widetilde{R}$ is pre-perfectoid. To see its possibility, a trivial case is that when $\widetilde{R}[1/p]$ is absolutely integrally closed, every residue field K is algebraically closed so that the Frobenius induces an isomorphism $\mathcal{O}_K/p^{1/p}\mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_K/p\mathcal{O}_K$, i.e., $\mathcal{O}_{\widehat{K}}$ is perfectoid. Although this example is too simple, at least we see that some algebraic properties of $\widetilde{R}[1/p]$ or K are able to guarantee the perfectoidness of any valuation structure $\mathcal{O}_K$. This was my original faith, which kept me going until I arrived at the following perfectoid criterion.

Corollary 6.15 ([He26, 4.23], see also [He25a, 9.7])**.** *With the notation in* 6.14*, if* $\dim_{\widehat{K}} \mathcal{F}_K^{G_K} = 1 + \mathrm{trdeg}(K/\overline{\mathbb{Q}}_p)$*, then $\mathcal{O}_{\widehat{K}}$ is perfectoid.*

Proof. As $\dim_K \Omega^1_{K/\overline{\mathbb{Q}}_p} = \mathrm{trdeg}(K/\overline{\mathbb{Q}}_p)$, the assumption implies that the coboundary map $\delta = 0$ in 6.14. Thus, $\widehat{\Omega}^1_{K/\overline{\mathbb{Q}}_p} = 0$. This implies that $\widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} = 0$ as it is almost finite free by 6.7. Therefore, $\mathcal{O}_{\widehat{K}}$ is perfectoid by [GR03, 6.6.6]: indeed, consider the morphism of exact sequences (see 6.5)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} & \longrightarrow & \Omega^1_{\mathcal{O}_{\overline{K}}/\overline{\mathbb{Z}}_p} & \longrightarrow & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot p^n} & & \downarrow{\scriptstyle \cdot p^n} & & \downarrow{\scriptstyle \cdot p^n} & & \\ 0 & \longrightarrow & \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} & \longrightarrow & \Omega^1_{\mathcal{O}_{\overline{K}}/\overline{\mathbb{Z}}_p} & \longrightarrow & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} & \longrightarrow & 0. \end{array} \tag{6.15.1}$$

As $\Omega^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p}$ is torsion-free (6.5) and $\widehat{\Omega}^1_{\mathcal{O}_K/\overline{\mathbb{Z}}_p} = 0$, the first vertical arrow is an isomorphism. Similarly, the second vertical arrow is also an isomorphism. Thus, so is the third vertical arrow. In particular, $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} = \bigcup_{n \geq 1} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^n] = 0$. Hence, for any finite field extension L of K, we still have $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} = 0$ (6.5). This implies that the different ideal $\mathscr{D}_{L/K}$ is almost trivial by 6.4. Hence, the almost finite projective $\mathcal{O}_K$-algebra $\mathcal{O}_L$ is also almost finite étale by [GR03, 4.1.27]. As the Frobenius induces a surjection on $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, by almost faithfully flat descent we see that the Frobenius also induces an almost surjection on $\mathcal{O}_K/p\mathcal{O}_K$ [GR03, 3.5.13.(ii)]. One can easily check that it actually implies further that the Frobenius on $\mathcal{O}_K/p\mathcal{O}_K$ is surjective. Hence, $\mathcal{O}_{\widehat{K}}$ is perfectoid. □

6.f. **Application to Shimura varieties: Calegari-Emerton conjecture.** We fix a Shimura datum (G, X) ([Del79, 2.1.1], see also [Mil05, 5.5]) and let $E \subseteq \mathbb{C}$ be its reflex field (which is a finite extension of $\mathbb{Q}$, [Del79, 2.2.1], see also [Mil05, 12.2]). We denote by $\mathbb{A}_f$ (resp. $\mathbb{A}_f^p$) the ring of (resp. prime-to-p) finite adèles of $\mathbb{Q}$. For any neat compact open subgroup $K \subseteq G(\mathbb{A}_f)$ ([Pin90, 0.6]), we denote by Sh_K the canonical model of the Shimura variety associated to (G, X) of level K (see [Mil05, page 128]). It is a quasi-projective smooth E-scheme, whose $\mathbb{C}$-points are canonically identified with

$$\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))/K. \tag{6.15.2}$$

Moreover, these canonical models form a directed inverse system of E-schemes $(\mathrm{Sh}_K)_{K\subseteq G(\mathbb{A}_f)}$ (note that open subgroups of K are also neat) with finite étale transition morphisms (see [Del79, 2.1.2]).

We fix a compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. Consider the directed inverse system of E-schemes $(\mathrm{Sh}_{K^pK_p})_{K_p\subseteq G(\mathbb{Q}_p)}$, where K_p runs through all the neat compact open subgroups of $G(\mathbb{Q}_p)$ ([HJ23, 2.12]). Its limit

$$\mathrm{Sh}_{K^p} = \lim_{K_p\subseteq G(\mathbb{Q}_p)} \mathrm{Sh}_{K^pK_p} \tag{6.15.3}$$

is called the *Shimura variety at infinite level* K^p. Motivated by the p-adic Langlands program, Calegari-Emerton [CE12] predict the vanishing of the étale cohomology in higher degrees:

Conjecture 6.16 (Calegari-Emerton [CE12, 1.5], cf. [HJ23, 1.3])**.** *For any integer* $q > \dim \mathrm{Sh}_{K^p}$*, we have*

$$H^q_{\text{ét}}(\mathrm{Sh}_{K^p,\mathbb{C}}, \mathbb{Z}_p) = 0. \tag{6.16.1}$$

Scholze [Sch15] made the first fundamental progress on this conjecture. Indeed, for Shimura varieties of Hodge type, he proved that they are perfectoid as p-adic analytic spaces at infinite level at p. This established a profound connection between the étale cohomology of Shimura varieties with the analytic cohomology of certain coherent sheaves, leading to resolutions of numerous conjectures including the higher vanishing of the compactly supported completed cohomology for (G, X) of Hodge type (a variant of Conjecture 6.16). A natural question arises:

Question 6.17. *Does* Sh_{K^p} *define a perfectoid space in general?*

Using our perfectoidness criterion derived from p-adic Hodge theory for valuation rings 6.15, we can provide an affirmative answer from the valuative point of view:

Theorem 6.18 ([He26, 11.4])**.** *In general,* Sh_{K^p} *is pointwise perfectoid. More precisely, for any residue field* K *of* $\mathrm{Sh}_{K^p,\overline{\mathbb{Q}}_p}$*, any valuation ring* $\mathcal{O}_K$ *of* K *extension of* $\overline{\mathbb{Z}}_p$ *is pre-perfectoid.*

It turns out that pointwise perfectoidness is sufficient to relate étale cohomology and analytic cohomology. Finally, we can prove that

Corollary 6.19 ([He26, 11.3])**.** *The conjecture* 6.16 *is true.*

Epilogue

Looking back at these six lectures, we began our journey by understanding ramification over $\mathbb{Q}_p$ and introducing the theory of perfectoid rings. Using deformation theory and reducing problems to characteristic p, we established various properties of these rings, viewing them as the "affine objects" of the p-adic world. This provided us with the necessary tools to understand ramification on smooth algebraic varieties.

Simultaneously, we adopted the perspective of general valuation rings to examine cohomological descent and almost purity for perfectoid rings, and even the ramification of valuation rings themselves.

Although this course was brief, it touched on the major themes of p-adic Hodge theory from the last sixty years. However, precisely because of this broad scope, you may find that many proofs in the lecture notes are condensed or lack absolute rigor.

Nevertheless, our hope is that this brief overview has given you a sense of the cutting-edge ideas and techniques in the field. We hope this helps you avoid detours and see the big picture clearly when you dive into your own research later on.

But most importantly, there is a spirit I want to pass on to you, the younger generation: No matter how "cutting-edge" the field is, or how obscure and difficult the proofs in the literature may seem, do not be intimidated. As long as you have the courage and patience to break the details down, blow them up, and dig deep, piece by piece, step by step, you can understand anything, no matter how hard it seems at first.

It is not a matter of innate brilliance, but of whether you can settle down and commit to understanding every detail with courage, patience, and honesty.

I wish you all the best in your future studies. Keep working hard and keep moving forward!

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