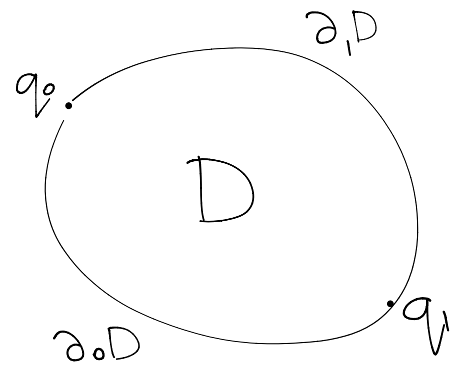
J-holomorphic discs

Take D = closed disc $\subset \mathbb{C}$ with distinct $q_0, q_1 \in \partial D$. Using the orientation of $\mathbb{C}$, further write $\partial D - \{q_i\} = \partial_0 D \amalg \partial_1 D$ as shown



We want to define a moduli space of J-holomorphic maps $v : D \longrightarrow M$ such that

1. $v(q_0) = p_i$, $v(q_1) = p_j$,
2. $v(\partial_k D) \subset L_k$ for $k = 0, 1$

As before we want it to be compact, and have good properties. By analogy with GW theory, we choose $\beta \in H_2(M, L_0 \cup L_1, \mathbb{Z})$, the "relative" homology group, and require

$$③ \quad v_*([D]) = \beta$$

Finally, we allow v to be a stable map from a possibly singular $\tilde{D}$, given by a union of discs D and $\mathbb{P}^1_{\mathbb{C}}$s, joined at interior or boundary points.

<u>Rem</u> For details, see the work of Fukaya-Ono and Fukaya-Oh-Ohta-Ono

Notation Write $\mathcal{M}(p_i, p_j, \beta)$ for the resulting moduli space.

If M is Calabi-Yau, and the L_i are "graded", then $\mathcal{M}(p_i, p_j, \beta)$ behaves like a d-dimensional manifold for the purpose of "counting" invariants where $d = \mu(p_i) - \mu(p_j) - 1$ for $\mu(p_i)$ the "Maslov index" of p_i, defined using the data of the Lagrangians. (This should be seen as an analogue of the Morse index.)

If $d=0$, we may hope to define a "counting" invariant. We furthermore require, as before, that the moduli $\mathcal{M}$ has an orientation (this can be constructed from orientations and spin structures on the L_i.)

<u>Notation</u> Write $N_{ij}(\beta) \in \mathbb{Z}$ for the counting invariant associated to the moduli space $\mathcal{M}(p_i, p_j, \beta)$ above, in the case that $\mu(p_i) = \mu(p_j) + 1$.

Lagrangian Floer homology (simplified)

Assume, for simplicity, that

① J is generic, and

② there are no non-trivial holomorphic discs in M with boundary L_0 or L_1, that is J-holomorphic $v: D \to M$ with $v(\partial D) \subset L_i$ some i.

Rem The condition ② may be known as a "no bubbling" condition

Then we find

$$\partial \mathcal{M}(p_i, p_j, \beta) = \coprod_{\beta = \beta_1 + \beta_2} \coprod_{k} \mathcal{M}(p_i, p_k, \beta_1) \times \mathcal{M}(p_k, p_j, \beta_2) \qquad (*)$$

Now let $C_k = \langle p_i \mid \mu(p_i) = k \rangle_{\mathbb{C}}$, and define $\partial_k : C_k \to C_{k-1}$ by

$$\partial p_i = \sum_{\substack{j,\ \beta, \\ \mu(p_j) = k-1}} N_{ij}(\beta)\, e^{-2\pi[\omega]\cdot\beta}\, p_j$$

Take (*) for $\mu(p_i) = \mu(p_j) + 2$ we may deduce that (the coefficient of $e^{-2\pi[\omega]\cdot\beta}$) in $\partial_k \partial_{k+1}$ is zero, and thereby make the following

<u>Definition</u> The Lagrangian Floer homology group $HF_k(L_0, L_1) = \ker \partial_k / \operatorname{Im} \partial_{k+1}$

<u>Rem</u> If conditions 1 and 2 are not satisfied, the right-hand side of $\circledast$ is more complicated, and we say that HF_* is "obstructed", as in general $\partial_k \partial_{k+1} \neq 0$

<u>Rem</u> The factor $e^{-2\pi[\omega]\cdot\beta}$ should be thought of as keeping track of the contributions for different β. It may cause the sum to diverge: in that case we view $[\omega]$ as a formal variable.