LECTURES ON THE GEOMETRIC SATAKE CORRESPONDENCE

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Abstract.

Contents

1. Lecture 1

1.1. **Statement of Satake isomorphism for split groups.** Fix a prime number p. Let F be a nonarchimedian local field with ring of integers $\mathcal{O}$, and with finite residue field $\mathbb{F}_q$ of characteristic p. Let $\varpi \in F^\times$ denote a fixed uniformizer. Fix a separable closure $\bar{F}$ of F. Let $G \supset B \supset T$ be a connected reductive group over F endowed with a choice of Borel subgroup $B = TU$ and maximal torus, both defined and **split** over F. Let W_0 be the finite Weyl group for (G, T). We can assume $G \supset B \supset T$ are all defined over $\mathcal{O}$. Set $K = G(\mathcal{O})$, a maximal compact subgroup of $G(F)$. Choose any prime $\ell \neq p$ and consider the K-spherical Hecke algebra $\mathcal{H}_K(G) = C_c(K\backslash G(F)/K, \bar{\mathbb{Q}}_\ell)$, which is an associative $\bar{\mathbb{Q}}_\ell$-algebra with convolution $*$ defined using the Haar measure dg on $G(F)$ giving K volume 1.

Theorem 1.1.1 (Satake isomorphism for split groups)**.** *There is an isomorphism of $\bar{\mathbb{Q}}_\ell$-algebras*

$$\mathcal{H}_K(G) \ \tilde{\rightarrow}\ \bar{\mathbb{Q}}_\ell[X_*(T)]^{W_0}.$$

In particular, $\mathcal{H}_K(G)$ is a commutative finite-type $\bar{\mathbb{Q}}_\ell$-algebra.

The homomorphism from left to right is given by an explicit rule. Let $|\cdot|_F : F^\times \rightarrow q^{\mathbb{Z}}$ be the normalized absolute value, and choose a square root $q^{1/2} \in \bar{\mathbb{Q}}_\ell$ once and for all. Then $|\cdot|_F^{1/2} : F^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$ is defined, and for $t \in T(F)$ we define $\delta_B^{1/2}(t) = |\det(\mathrm{Ad}(t)\,|\,\mathrm{Lie}(T(F))|_F^{1/2}$, determining an unramified character $\delta_B^{1/2} : T(F) \rightarrow \bar{\mathbb{Q}}_\ell^\times$. Let du be the left Haar measure on $U(F)$ giving $U(\mathcal{O})$ measure 1. Then there is an algebra homomorphism $\mathcal{H}_K(G) \rightarrow \mathcal{H}_{T(\mathcal{O})}(T) \cong \bar{\mathbb{Q}}_\ell[X_*(T)]$, $f \mapsto S(f)$, where

$$S(f)(t) = \delta_B^{1/2}(t) \int_{U(F)} f(tu)\, du$$

which lands in the W_0-invariants and gives the isomorphism above. The identification $X_*(T) = T(F)/T(\mathcal{O})$ here is given by sending a cocharacter $\mu \in X_*(T)$ to the image of $\mu(\varpi) \in T(F)$ in $T(F)/T(\mathcal{O})$.

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1.2. **Statement for general groups.** We need some Bruhat-Tits theory and the Kottwitz homomorphism in order to state the isomorphism in general. Suppose G is an arbitrary connected reductive group over F. In [Ko97], Kottwitz defined (functorially in G) a surjective group homomorphisms

$$\kappa_G : G(\breve{F}) \to \pi_1(G)_I$$

where $\breve{F}$ is the completion of maximal unramified extension of F in $\bar{F}$, and $\pi_1(G)$ is the Borovoi fundamental group of G [Bo98], and finally $I \subset \Gamma := \mathrm{Gal}(\bar{F}/F)$ is the inertia subgroup. For example, if $G = \mathrm{GL}_n$, then $\pi_1(G) = \mathbb{Z}$ and $\kappa_G = \mathrm{val}_{\breve{F}} \circ \det$ where $\mathrm{val}_{\breve{F}} : \breve{F}^\times \to \mathbb{Z}$ is the normalized valuation on $\breve{F}$.

Let $\sigma \in \Gamma$ be any Frobenius element, i.e, any lift of the Frobenius automorphism $x \mapsto x^q$ of $\bar{\mathbb{F}}_q$. Kottwitz also showed that this map remains surjective on taking σ-fixed points, so that

$$\kappa_G : G(F) \to \pi_1(G)_I^\sigma$$

is still surjective. Define the Kottwitz kernels $G(\breve{F})_1 := \ker(\kappa_G)$ and $G(F)_1 := G(\breve{F})_1 \cap G(F)$.

Fix a maximal F-split torus $A \subset G$, and let $M = C_G(A)$ denotes its centralizer in G; then M is a minimal F-Levi subgroup, and is the Levi factor of an F-parabolic subgroup $P = MN$ with unipotent radical N. We have the relative Weyl group $W_0 = W(G, A) := N_G A(F)/M(F)$.

Let $K \subset G(F)$ be a **special maximal parahoric** subgroup, corresponding to a special vertex in the apartment corresponding to A in the Bruhat-Tits building $\mathfrak{B}(G, F)$. Since M is a minimal F-Levi it turns out that $M(F)_1 = K \cap M$, both coinciding with the unique parahoric subgroup of $M(F)$. Let $\Lambda_M := M(F)/M(F)_1$, which is a finitely generated abelian group which can be explicitly described and which carries an obvious action of W_0. The following result shows that the K-spherical Hecke algebra $\mathcal{H}_K(G)$ is still commutative and finite-type in this case.

Theorem 1.2.1. [HaRo10] *There is an isomorphism of $\bar{\mathbb{Q}}_\ell$-algebras*

$$\mathcal{H}_K(G) \ \tilde{\to}\ \bar{\mathbb{Q}}_\ell[\Lambda_M]^{W_0},$$

given by $f \mapsto S(f)$, where for $m \in M(F)$,

$$S(f)(m) = \delta_P^{1/2}(m) \int_{N(F)} f(mn)\, dn.$$

For example, if D is central division algebra over F with $[D : F] = n^2$ and $G = D^\times$, then $G(\breve{F}) = \mathrm{GL}_n(\breve{F})$ and $M = G$, $A = Z(G) \cong \mathbb{G}_m$, $W_0 = 1$, $K = G(F)_1 = \mathcal{O}_D^\times$, and $\Lambda_G = \mathbb{Z}$. Thus $\mathcal{H}_{\mathcal{O}_D^\times}(D^\times) \cong \bar{\mathbb{Q}}_\ell[\mathbb{Z}]$.

Remark 1.2.2. In his Corvallis article [Ca79], Cartier proved a similar result for the Hecke algebra $\mathcal{H}_{\widetilde{K}}(G)$, where $\widetilde{K} \supseteq K$ is a **special maximal compact** subgroup. Then on the right hand side $\Lambda_M = M(F)/M(F)_1$ gets replaced by $\widetilde{\Lambda}_M = M(F)/M(F)^1$, where $M(F)^1$ is the unique maximal compact subgroup of $M(F)$. The proof of Theorem 1.2.1 yields a proof of the result of Cartier, but not vice-versa.

1.3. **Statement in the quasi-split case.** Now assume G is quasi-split over F, and again let K be any special maximal parahoric subgroup of $G(F)$. In this case, we have $M = T$ (a maximal torus), $P = B = TU$, and $\Lambda_M = X_*(T)_I^\sigma$. Now the relative Weyl group is $W_0 = N_G T(F)/T(F)$. Let $\widehat{T}$ be the dual torus over $\bar{\mathbb{Q}}_\ell$, i.e. the unique one such that $X^*(\widehat{T}) = X_*(T)$ as Γ-modules. The theorem above specializes to the following.

Theorem 1.3.1. *There is a canonical isomorphism*

$$\mathcal{H}_K(G) \ \tilde{\to}\ \bar{\mathbb{Q}}_\ell[X^*(\widehat{T}^I)^\sigma]^{W_0},$$

given by a similar formula for $f \mapsto S(f)$.

The appearance of the diagonalizable group $\widehat{T}^I$ appearing here suggests that, ignoring σ, the right hand side is the representation ring of a (possibly disconnected) reductive $\bar{\mathbb{Q}}_\ell$-group $\widehat{G}^I \supset \widehat{T}^I$. We shall see that this is indeed the right point of view.

Remark 1.3.2. Cartier's result translates in this case to an isomorphism $\mathcal{H}_{\widetilde{K}}(G) \cong \bar{\mathbb{Q}}_\ell[X^*(\widehat{T}^{I,\circ})^\sigma]^{W_0}$, involving the connected component $\widehat{T}^{I,\circ}$ of $\widehat{T}^I$, a maximal torus in $\widehat{G}^{I,\circ}$.

1.4. **Transition to dual groups.** Now we change notation slightly. Let k be any algebraically closed field, and let $F = k(\!(t)\!)$ and $\mathcal{O} = k[\![t]\!]$. Let G be any connected reductive group over F. By Steinberg's theorem G is quasi-split over F. Fix a maximal F-split torus $A \subset G$. Write $T := C_G(A)$, a maximal torus defined over F, and fix a Borel pair $G \supset B \supset T$ over F and Levi decomposition $B = TU$. Let $(X^*(T) \supset \Phi, X_*(T) \supset \Phi, \Delta)$ be the based absolute root system enodowed with its Galois action. Consider the dual based root system $(X^*(\widehat{T}) \supset \Phi^\vee, X_*(\widehat{T}) \supset \Phi, \Delta^\vee)$; this comes from a dual Borel pair $\widehat{B} \supset \widehat{T}$.

The group I acts naturally on both systems, preserving a splitting in both. As k is algebraically closed, Frobenius is no longer in the picture. Let $\widehat{G}$ be the dual group of (G, B, T), endowed with its Galois action of I fixing a splitting [Ko84]. As G is quasi-split over F, we may choose an embedding $\widehat{T} \hookrightarrow \widehat{G}$ such that the I-action on $\widehat{T}$ derived from $X_*(T) = X^*(\widehat{T})$ is inherited from the I-action on $\widehat{G}$ [Hai17]. For any possibly non-reduced root system R, let $R_{\rm red}$ denote the reduced root system we get from R by discarding all roots of the form $2a$ when $\{a, 2a\} \subset R$. If R carries an I-action, let $R^\diamond$ denote the set of I-averages of the I-orbits in R (in some ambient real vector space).

Let $\widehat{G}^{I,\circ}$ be the connected component of $\widehat{G}^I$.

Theorem 1.4.1. *The $\bar{\mathbb{Q}}_\ell$-group $\widehat{G}^{I,\circ}$ is reductive with maximal torus $\widehat{T}^{I,\circ}$, and its root system identifies with $(\Phi^\vee)^\diamond_{\rm red}$.*

Proof. See [Hai15, Prop. 4.1] and [Hai18, Prop. 5.2]. □

Thus we may consider the Tannakian category $(\mathrm{Rep}(\widehat{G}^I), \otimes)$.

1.5. **Statement of Main Theorem.** Let $\mathcal{G}$ be the Bruhat-Tits parahoric group scheme over $\mathcal{O}$ with generic fiber G corresponding to a fixed special maximal parahoric subgroup of $G(F)$. We shall define an ind-projective ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ over k with a left-action of the k-group scheme $L^+\mathcal{G}$ and with a corresponding category of equivariant perverse $\bar{\mathbb{Q}}_\ell$-sheaves.

Theorem 1.5.1 (Main Theorem)**.** *Choose any prime ℓ not equal to the characteristic of k. The abelian category $P_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \bar{\mathbb{Q}}_\ell)$ is endowed with a convolution product $\star$, making it into a neutral Tannakian category with fiber functor given by total ℓ-adic cohomology groups*

$$\mathcal{F} \mapsto R^*\Gamma(\mathrm{Gr}_{\mathcal{G}}, \mathcal{F}) := \oplus_{i\in\mathbb{Z}} R^i\Gamma(\mathrm{Gr}_G, \mathcal{F}),$$

and there is a canonical equivalence of Tannakian categories

$$(\mathrm{Rep}(\widehat{G}^I), \otimes) \cong (P_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \bar{\mathbb{Q}}_\ell), \star).$$

The first result of this nature was the celebrated work of Mirković-Vilonen [MV07]. They worked with affine Grassmannians over $\mathbb{C}$ coming from groups defined over $\mathbb{C}$ (and thus their groups over $\mathbb{C}(\!(t)\!)$ are split). See also the exposition of Baumann-Riche [BR18]. Theorem 1.5.1 includes affine Grassmannians over positive characteristic fields, ℓ-adic cohomology, and also non-split groups, which was not adequately covered by [MV07].

Theorem 1.5.1 was proved by Xinwen Zhu [Zhu15] under a tame ramification hypothesis, and in general by Timo Richarz [Ri16b]. Earlier Timo Richarz [Ri14] had proved the split case of this theorem, using a novel approach to identifying the dual group of the Tannakian category, namely by deducing it from a characterization (due to Kazhdan-Larsen-Varshavsky [KLV]) of a Tannakian category in terms of its Grothendieck semiring. Our goal in this course is to show how one can get the general theorem directly, by following the method of Richarz.

The geometric Satake correspondence for ramified groups is essential for arithemetic applications, and we shall mention some, such as the classification of smooth Schubert varieties in affine Grassmannians [HR20a], which plays a role in the He-Pappas-Rapoport classification of pararhoric level Shimura varieties with have good or semistable reduction [HPR20].

1.6. **Group theoretic preliminaries.** Our reference for most of what follows is [HR08].

Definition 1.6.1. The Iwahori-Weyl group of (G, A) over F is defined to be $W = N_GT(F)/T(F)_1$, and maps naturally to the relative Weyl group $W_0 = N_GT(F)/T(F)$ with kernel $T(F)/T(F)_1$.

Using the Kottwitz homomorphism $\kappa_T : T(F) \to X_*(T)_I$, we have an exact sequence

$$0 \to X_*(T)_I \to W \to W_0 \to 0$$

Let $K = \mathcal{G}(\mathcal{O})$; we have $W_0 = (N_G T(F) \cap K)/T(F)_1$, hence the above sequence splits and we have an isomorphism

$$W = X_*(T)_I \rtimes W_0.$$

Let $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The group W acts on V^I as follows: W_0 identifies with the group of linear automorphisms generated by the simple reflections through the hyperplanes H_α corresponding to the **relative roots** α for (G, A); moreover an element $\lambda \in X_*(T)_I$ acts on V by translation by $-\lambda$.

Bruhat-Tits theory provides us with a set of affine-linear functionals $\Phi_{\rm aff}$ on V^I called the *affine roots associated to* (G, A). The zero sets are called the affine hyperplanes. The action of W on V^I induces an action on $\Phi_{\rm aff}$; thus the action of W on V^I permutes the affine hyperplanes and the alcoves in the resulting Coxeter complex in V^I.

Fixing the choice of base alcove $\mathbf{a}$ whose closure contains our fixed special vertex, denote by $\mathcal{G}_{\mathbf{a}}$ the corresponding Bruhat-Tits Iwahori group scheme. Let $\Omega_{\mathbf{a}} \subset W$ denote the stabilizer subgroup of $\mathbf{a}$. Let $(W_{\rm aff}, S_{\rm aff})$ denote the Coxeter system corresponding to the simple affine reflections $S_{\rm aff}$ (those through the walls of $\mathbf{a}$). The group $W_{\rm aff}$ acts simply transitively on the set of all alcoves in the apartment. Thus having chosen $\mathbf{a}$ we have a canonical isomorphism

$$W = W_{\rm aff} \rtimes \Omega_{\mathbf{a}}.$$

The Coxeter group $W_{\rm aff}$ has a length function $\ell : W_{\rm aff} \to \mathbb{Z}_{\geq 0}$ and a Bruhat order $\leq$. We extend both of these to W in the obvious way by (1) declaring $\Omega_{\mathbf{a}}$ to consist of the elements of length 0 in W, and (2) declaring that $w_1\omega_1 \leq w_2\omega_2$ in W if and only if $w_1 \leq w_2$ in $W_{\rm aff}$ and $\omega_1 = \omega_2$ in $\Omega_{\mathbf{a}}$. The group $(W, S_{\rm aff})$ thus inherits what is called a *quasi-Coxeter* structure.

Proposition 1.6.2. *There is a unique reduced root system Σ such that the affine hyperplanes*

$$H_{\beta+r} = \{v \in V^I \mid \langle \beta, v\rangle + r = 0\}$$

for $\beta + r \in \Phi_{\rm aff}$ consist of the affine hyperplanes $H_{\alpha+n}$ attached to functions on V^I of the form $v \mapsto \langle \alpha, v\rangle + n$ for $\alpha \in \Sigma$ and $n \in \mathbb{Z}$. (This set Σ is known as the échelonnage root system *for (G, A) over F.) Thus $W_{\rm aff} = W_{\rm aff}(\Sigma)$, the affine Weyl group associated to the set of affine roots $\alpha + n$ as above.*

Proposition 1.6.3. *The following statements about the root system Σ hold.*

(a) *The root system Σ can be described in terms of the absolute roots Φ as follows. Let $N_I'\Delta$ be the set of* modified norms *of elements in Δ, that is, for $\alpha \in \Delta$ in an I-orbit O,*

$$N_I'\alpha = \begin{cases} \sum_{\beta \in O} \beta, & \text{if } O \text{ consists of pairwise orthogonal roots} \\ 2\sum_{\beta\in O} \beta, & \text{otherwise.} \end{cases}$$

Then the set of simple positive roots in Σ can be identified with $N_I'\Delta$.

(b) *We have an identification of the (based) dual root system $\Sigma^\vee = (\Phi^\vee)^\diamond_{\rm red}$, in other words, the root system Σ is dual to that of $(\widehat{G}^I, \widehat{T}^I)$.*

Proof. See [Hai18, Prop. 5.1, Thm. 6.1]. □

We define the dominant Weyl chamber $\mathcal{C} \subset V^I$ by $\mathcal{C} = \{v \in V^I \mid \langle \alpha, v\rangle \geq 0\,,\ \forall \alpha \in \Delta\}$. We require that $\mathbf{a} \subset \mathcal{C}$. We define $X_*(T)_I^+ = X_*(T)_I \cap \mathcal{C}$.

1.7. **Appendix: lifting dominant translations.** We would like to show that the map $X_*(T) \to X_*(T)_I$, $\mu \mapsto \bar{\mu}$, induces a surjective map on dominant elements

$$X_*(T)^+ \to X_*(T)_I^+.$$

Here dominance in $X_*(T)$ is defined using the absolute roots of G, and in $X_*(T)_I^+$ using the relative roots for (G, A) (equivalently, the échelonnage roots). The fact that dominant elements go to dominant elements is a consequence, for example, of [Hai18]. Indeed, suppose $\mu \in X_*(T)^+$ and let

Δ denote the simple absolute roots. Then for $\alpha \in \Delta$ we have the modified norm $N'_I\alpha$. The elements $N'_I\alpha$ give the simple échelonnage roots acting on $X_*(T)_I$. We have

$$\langle N'_I\alpha, \bar{\mu}\rangle = \langle N'_I\alpha, \mu\rangle \geq 0,$$

which shows that $\bar{\mu}$ is dominant.

The more interesting direction is the converse. **We can prove this at least when the center Z of G is connected**. We assume $\lambda \in X_*(T)_I^+$. We want to show that there is a lift $\mu \in X_*(T)^+$ of λ.

Assume first that G is adjoint, so that the fundamental coweights $\omega_\alpha^\vee,\ \alpha \in \Delta$, form a $\mathbb{Z}$-basis for $X_*(T)$. Let $\mathcal{O}$ denote an I-orbit in Δ. Choose arbitrarily $\alpha \in \mathcal{O}$ for each orbit $\mathcal{O}$. Then the set $\{\bar{\omega}_\alpha^\vee\}_\mathcal{O}$ is a $\mathbb{Z}$-basis for $X_*(T)_I$. We may write $\lambda = \sum_\mathcal{O} n_\mathcal{O}\, \bar{\omega}_\alpha^\vee$ for some $n_\mathcal{O} \in \mathbb{Z}_{\geq 0}$. Clearly a lift of this is $\mu = \sum_\mathcal{O} n_\mathcal{O}\omega_\alpha^\vee$, which is obviously dominant.

Next, we assume Z is connected. Then we have a commutative diagram with exact rows and surjective vertical arrow

$$\begin{tikzcd}
0 \arrow[r] & X_*(Z) \arrow[r] \arrow[d] & X_*(T) \arrow[r] \arrow[d] & X_*(T_{\mathrm{ad}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_*(Z)_I \arrow[r] & X_*(T)_I \arrow[r] & X_*(T_{\mathrm{ad}})_I \arrow[r] & 0.
\end{tikzcd}$$

The lower row is exact on the left because $H_1(I, X_*(T_{\mathrm{ad}})) = 0$ as $X_*(T_{\mathrm{ad}})$ is I-induced. Given $\lambda \in X_*(T)_I^+$, its image $\lambda_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})_I$ is dominant. By the first case above, we may lift λ_{ad} to a dominant $\mu_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})^+$. Let $\mu \in X_*(T)^+$ be any lift of μ_{ad} (it is automatically dominant). Then $\bar{\mu} - \lambda \in X_*(Z)_I$; lift this to an arbitrary $\delta \in X_*(Z)$. Then $\mu - \delta \in X_*(T)^+$ and $\overline{\mu - \delta} = \lambda$. □

We next discuss three fundamental decompositions of the group $G(F)$. Of course all such results can be found in some form or other in [BT84].

1.8. **Bruhat-Tits decomposition.** Let $K_\mathbf{a} = \mathcal{G}_\mathbf{a}(\mathcal{O})$, an Iwahori subgroup of $G(F)$.

Proposition 1.8.1. *For each $w \in W$, choose any lift $\dot{w} \in N_GT(F)$. Then there is a decomposition*

$$G(F) = \coprod_{w \in W} K_\mathbf{a}\dot{w}K_\mathbf{a}.$$

Remark 1.8.2. For $G = \mathrm{Gl}_n$ this can be proved "by hand" using row and column operations. For split groups, it is derived from the BN-pair relations, for example in Brown's book *Buildings*. In general, it is proved in the works of Bruhat-Tits; see for example [HR08]. We shall assume this result without proof.

1.9. **Cartan decomposition.** Recall $K = \mathcal{G}(\mathcal{O})$, a special maximal parahoric subgroup of $G(F)$. For any $\lambda \in X_*(T)_I$, let $t^\lambda \in T(F)$ be any fixed lift, so that $\kappa_T(t^\lambda) = \lambda$.

Proposition 1.9.1. *We have a decomposition*

$$G(F) = \coprod_{\lambda \in X_*(T)_I^+} Kt^\lambda K.$$

Remark 1.9.2. This is a formal consequence of Proposition 1.8.1. See also [HR08].

1.10. **Iwasawa decomposition.** Recall that $B = TU$ with $T = C_G(A)$, and $\mathbf{a}$ is the apartment of the building corresponding to A. With these restrictions, we have the following result.

Proposition 1.10.1. *There is a decomposition*

$$G(F) = \coprod_{\lambda \in X_*(T)_I} U(F)t^\lambda K.$$

Moreover, given $\lambda \in X_*(T)_I$ and $\mu \in X_*(T)_I^+$, we have $U(F)t^\lambda K \cap Kt^\mu K \neq \emptyset$ only if $\mu - \lambda$ is a sum of of positive coroots in $\Sigma^\vee$.

Proof. See [HaRo10, Cor. 9.1.2] for the decomposition. The second part can be deduced from [HaRo10, Lem. 10.2.1] together with [Ra05, Proof of Lem. 3.8].

□

2. Lecture 2

2.1. **Preliminaries on ind-schemes.** Let k be any field. Let Aff_k denote the category of k-algebras. We will endow Aff_k with some standard Grothendieck topology $\mathcal{C}$, such as Zariski, étale, or fpqc.

2.1.1. *Presheaves and sheaves.*

Definition 2.1.1. A *presheaf* F on Aff_k is a covariant functor $F : \mathrm{Aff}_k \to (\mathrm{Sets})$ (or to (Groups), or (Rings), etc.). A presheaf F is a *sheaf* for the $\mathcal{C}$-topology if for every cover $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ in $\mathcal{C}$,

$$F(R) \longrightarrow F(R') \rightrightarrows F(R' \otimes_R R')$$

is an equalizer diagram in the target category.

2.1.2. *Colimits and ind-schemes.*

Definition 2.1.2. Let I be a partially ordered and directed set (any two elements have an upper bound in I) and let F_i, $i \in I$ be a diagram of preheaves on I. Let $\underset{i}{\mathrm{colim}} F_i$ be the *filtered colimit* in the category of presheaves, i.e.,

$$(\underset{i}{\mathrm{colim}} F_i)(R) = \underset{i}{\mathrm{colim}}\, F_i(R)$$

the colimit on the right taken in the target category.

Theorem 2.1.3. *Filtered colimits commute with finite limits, such as equalizers.*

Corollary 2.1.4. *If each F_i is a sheaf, then the presheaf colimit $\underset{i}{\mathrm{colim}}\, F_i$ is a sheaf and is the colimit taken in the category of sheaves: if F' is a sheaf and we have commutative diagrams of presheaves for all $i < j$ in I*

$$\begin{tikzcd} F_i \arrow[r] \arrow[d] & F' \\ F_j \arrow[ur] & \end{tikzcd}$$

then this induces a unique compatible presheaf morphism $\underset{i}{\mathrm{colim}} F_i \to F'$, and this is also a morphism of sheaves.

Corollary 2.1.5. *If the F_i as above are sheaves for the $\mathcal{C}$-topology, so is the presheaf colimit $\underset{i}{\mathrm{colim}}\, F_i$.*

Here is our main example.

Definition 2.1.6. A (strict) ind-scheme is a sheaf for, say, the étale topology isomorphic to one of the form $\underset{i}{\mathrm{colim}}\, F_i$ where each F_i is represented by a scheme X_i (such that $X_i \to X_j$ is a closed immersion, whenever $i \leq j$ in I).

Since X_i begin a scheme implies that it is actually a sheaf for the finer fpqc topology, then the above corollary shows that the ind-scheme is also a sheaf for the fpqc topology.

2.1.3. *Sheafification.* Suppose $F : \mathrm{Aff}_k \to (\mathrm{Sets})$ is a presheaf, and $\mathcal{C}$ is a Grothendieck topology as above. Then there exists a *sheafification* F^{++} of F with respect to $\mathcal{C}$ and a canonical morphism of presheaves $F \to F^{++}$ with the following universal property: for each $\mathcal{C}$-sheaf F',

$$\mathrm{Hom}_{\mathrm{Presheaf}}(F, F') = \mathrm{Hom}_{\mathrm{Sheaf}}(F^{++}, F').$$

2.2. **Loop groups and positive loop groups.** Given G and $\mathcal{G}$ as above, we define the following presheaves on Aff_k with values in (Groups):

$$LG : R \mapsto G(R(\!(t)\!))$$
$$L^+\mathcal{G} : R \mapsto \mathcal{G}(R[\![t]\!]).$$

Exercise. Prove that

(1) LG is represented by an ind-affine group ind-scheme over k

(2) $L^+\mathcal{G}$ is represented by an affine group scheme (of infinite type over k when $\mathcal{G}$ is non-trivial).

Definition 2.2.1. We define the *affine Grassmannian* $\mathrm{Gr}_{\mathcal{G}}$ as the étale-sheafification of the (Set)-valued presheaf

$$R \mapsto LG(R)/L^+\mathcal{G}(R).$$

Remark 2.2.2. Warning: it is NOT always that case that $\mathrm{Gr}_{\mathcal{G}}(R) = LG(R)/L^+\mathcal{G}(R)$. When G is split over F, this does hold for every *local* ring R.

The next task to prove that $\mathrm{Gr}_{\mathcal{G}}$ is represented by an ind-projective ind-scheme over k. We shall start with the case $\mathcal{G} = \mathrm{GL}_{n,\mathcal{O}}$.

2.3. **Representability for $\mathcal{G} = \mathrm{GL}_{n,\mathcal{O}}$.** We write G for GL_n and $\mathcal{G}$ for the group scheme $\mathrm{GL}_{n,\mathcal{O}}$. The starting point is the fact that $LG(R) = \mathrm{GL}_n(R(\!(t)\!))$ acts on the set of $R[\![t]\!]$-lattices in $R(\!(t)\!)^n$.

Definition 2.3.1. Let R be any ring, and set $\Lambda = R[\![t]\!]^n$. An $R[\![t]\!]$-*lattice* is an $R[\![t]\!]$-submodule $\mathcal{L} \subset R(\!(t)\!)^n$ such that $\exists N \in \mathbb{N}$ such that

- $t^N\Lambda \subset \mathcal{L} \subset t^{-N}\Lambda$
- $\mathcal{L}$ is $R[\![t]\!]$-projective.

Now consider only $R \in \mathrm{Aff}_k$. We will show that the functor $R \mapsto \mathrm{Latt}_{n,N}(R)$ os lattices as above is representatble by a projective k-scheme each N. Then it will turn out that

$$\mathrm{Gr}_{\mathcal{G}}(R) = \operatorname*{colim}_N \mathrm{Latt}_{n,N}(R) = \big(\operatorname*{colim}_N \mathrm{Latt}_{n,N}\big)(R).$$

The following discussion is key. will provide some details and corrections for [Goe08, Lem. 2.11] and [Zhu17, Lem. 1.1.5].

Proposition 2.3.2. *Let R be any ring. Let $\Lambda = R[\![t]\!]^n$. The following two conditions on $R[\![t]\!]$-submodules $\mathcal{L} \subset R(\!(t)\!)^n$ which satisfy $t^N\Lambda \subseteq \mathcal{L} \subseteq \Lambda$ for some $N \geq 1$ are equivalent:*

(1) *$\mathcal{L}$ is $R[\![t]\!]$-projective.*

(2) *$\Lambda/\mathcal{L}$ (and hence $\mathcal{L}/t^N\Lambda$) are R-projective.*

Before starting the proof, let us state a well-known lemma.

Lemma 2.3.3. *Let R be any ring, and let M be an R-module of finite presentation. Then M is projective if and only if it is flat. In particular, if R is Noetherian, for finite R-modules flatness and projectivity are equivalent.*

Proof. This is the Corollary to Theorem 7.12 in Matsumura's Commutative Ring Theory. □

Suppose we are given an $R[\![t]\!]$-module $\mathcal{L}$ which is projective and which satisfies $t^N\Lambda \subset \mathcal{L} \subset \Lambda$. We wish to prove that $\Lambda/\mathcal{L}$ and $\mathcal{L}/t^N\Lambda$ are projective as R-modules. Let us first establish that $\mathcal{L}$ is finitely generated as an $R[\![t]\!]$-module. Being finitely generated is a local property, so it is enough to prove that $\mathcal{L}_{\mathfrak{p}}$ is finitely generated over $R[\![t]\!]_{\mathfrak{p}}$, for any $\mathfrak{p} \in \mathrm{Spec}(R[\![t]\!])$. From $t^N\Lambda_{\mathfrak{p}} \subset \mathcal{L}_{\mathfrak{p}} \subset \Lambda_{\mathfrak{p}}$ and flatness of $R[\![t]\!] \to R(\!(t)\!)$, we see that

$$\mathcal{L}_{\mathfrak{p}} \otimes_{R[\![t]\!]_{\mathfrak{p}}} R(\!(t)\!)_{\mathfrak{p}} = R(\!(t)\!)^n_{\mathfrak{p}}.$$

On the other hand, we know $\mathcal{L}_{\mathfrak{p}}$, being projective, is free over $R[\![t]\!]_{\mathfrak{p}}$ (by Kaplansky's theorem), so the above shows it actually has rank n.

Next we claim that $t^{-m}\mathcal{L}/\mathcal{L}$ is R-projective for all $m \in \mathbb{Z}_{\geq 0}$: write $\mathcal{L}$ as a direct summand of a free $R[\![t]\!]$-module and observe that then $t^{-1}\mathcal{L}/\mathcal{L}$ is a direct summand of a free $R = R[\![t]\!]/t$-module, hence is projective. Since $t^{-m}\mathcal{L}/\mathcal{L}$ is filtered by modules isomorphic to $t^{-1}\mathcal{L}/\mathcal{L}$, we see it splits as a direct sum of them as an R-module, hence is projective. Now we have

$$R(\!(t)\!)^n/\mathcal{L} = \varinjlim_m t^{-m}\mathcal{L}/\mathcal{L}$$

and since colimits commute with tensor products, this implies $R(\!(t)\!)^n/\mathcal{L}$ is a flat R-module. The same holds for $R(\!(t)\!)^n/t^N\Lambda$. Now using the characterization of flatness by Tor-vanishing, the exact sequence

$$0 \to \mathcal{L}/t^N\Lambda \to R(\!(t)\!)^n/t^N\Lambda \to R(\!(t)\!)^n/\mathcal{L} \to 0$$

shows that $\mathcal{L}/t^N\Lambda$ is a flat R-module. By the same argument, the exact sequence

$$0 \to \Lambda/\mathcal{L} \to R(\!(t)\!)^n/\mathcal{L} \to R(\!(t)\!)^n/\Lambda \to 0$$

shows that $\Lambda/\mathcal{L}$ is flat. Now the exact sequence

$$0 \to \mathcal{L}/t^N\Lambda \to \Lambda/t^N\Lambda \to \Lambda/\mathcal{L} \to 0$$

shows that $\Lambda/\mathcal{L}$ is finitely presented (note that the middle term is R-free, and the left term is R-finite, and use [StaPro, Tag 0519, Lem. 10.5.3(4)]). We conclude by Lemma 2.3.3 that $\Lambda/\mathcal{L}$ is R-projective. Then the splitting of the above sequence shows that $\mathcal{L}/t^N\Lambda$ is too. Note that this argument shows that $\mathcal{L}/t^N\Lambda$ is not just projective, but is automatically a direct factor of $\Lambda/t^N\Lambda$.

Conversely, we need to show that if $\mathcal{L}$ is an $R[\![t]\!]$-module with $t^N\Lambda \subset \mathcal{L} \subset \Lambda$ and $\Lambda/\mathcal{L}$ is R-projective, then $\mathcal{L}$ is $R[\![t]\!]$-projective. Following [Zhu15, p.8-10], we may assume R is Noetherian. Indeed, write $R = \varinjlim_i R_i$ where the R_i are finitely generated k-algebras. Let Gr_N be the presheaf such that $\mathrm{Gr}_N(R)$ is the set of all projective $R[\![t]\!]$-submodules $\mathcal{L}$ of $R(\!(t)\!)^n$ satisfying $t^N\Lambda \subset \mathcal{L} \subset \Lambda$. Let $\mathrm{Gr}^f_N(R)$ be the set of all $R[\![t]\!]$-submodules $\mathcal{L} \subset R(\!(t)\!)^n$ such that $t^N\Lambda \subset \mathcal{L} \subset \Lambda$, such that $\Lambda/\mathcal{L}$ is R-projective. The above shows that there is an injective map of presheaves $\mathrm{Gr}_N \to \mathrm{Gr}^f_N$. We need to show that it is bijective for every R. Since Gr^f_N is represented by a finite-type (projective) k-scheme, we have $\mathrm{Gr}^f_N(R) = \varinjlim_i \mathrm{Gr}^f_N(R_i)$, and hence it is enough to assume R is Noetherian.

Now this means $R[\![t]\!]$ is Noetherian and $\mathcal{L}$ is a finite $R[\![t]\!]$-module. Since $\mathcal{L}$ is a finite $R[\![t]\!]$-module, to show projectivity we need to show $\mathcal{L}$ is flat as an $R[\![t]\!]$-module, see Lemma 2.3.3.

Let $\Lambda_0 = R[t]^n$ and consider the $R[t]$-submodule $t^N\Lambda_0 \subset \mathcal{L}_0 \subset \Lambda_0$ corresponding to $\mathcal{L}$ under the isomorphism $\Lambda/t^N\Lambda \cong \Lambda_0/t^N\Lambda_0$. As $\mathcal{L}$ is the t-adic completion of $\mathcal{L}_0$ (this uses flatness of $R[t] \to R[\![t]\!]$), it is enough to show that $\mathcal{L}_0$ is $R[t]$-projective. Since it is $R[t]$-finite, again by the above reasoning, it is equivalent to show $\mathcal{L}_0$ is $R[t]$-flat.

It is enough to prove that $\mathcal{L}_{0,\mathfrak{q}}$ is a free $R[t]_\mathfrak{q}$-module, for every *maximal* ideal $\mathfrak{q} \subset R[t]$. Note that $\mathfrak{q}$ lies over a prime ideal $\mathfrak{p} \subset R$ which is not necessarily maximal.

Lemma 2.3.4. *There is an element $a \in R - \mathfrak{p}$ such that $R_a/\mathfrak{p}_a$ is a field and $R_a[t]_{\mathfrak{q}_a} = R[t]_\mathfrak{q}$.*

Proof. We have an inclusion $R/\mathfrak{p} \hookrightarrow R[t]/\mathfrak{q}$, where the former is a domain and the latter is a field. The ring $R[t]/\mathfrak{q}$ is a field generated as a ring by $R/\mathfrak{p}$ and the image of t. If $t \in \mathfrak{q}$, then the inclusion in an equality and we may take $a = 1$. If $t \notin \mathfrak{q}$, then t^{-1} exists in $R[t]/\mathfrak{q}$ and t is algebraic over $R/\mathfrak{p}$. Therefore there exists $a \in R - \mathfrak{p}$ such that the image of t is integral over $R_a/\mathfrak{p}_a$. Now the field $R_a[t]/\mathfrak{q}_a = R[t]/\mathfrak{q}$ is an integral extension of the domain $R_a/\mathfrak{p}_a$, hence the latter is a field. □

Since $R_a/\mathfrak{p}_a \subset R_a[t]/\mathfrak{q}_a$, we see that $\mathfrak{q}_a$ lies over $\mathfrak{p}_a$. We may therefore replace R by R_a and thereby assume that $\mathfrak{p}$ is *maximal* in R. Let us relabel by writing $\mathfrak{m} = \mathfrak{p}$ from now on, and denote the residue field by $k = R/\mathfrak{m}$. To prove $\mathcal{L}_{0,\mathfrak{q}}$ is a free $R[t]_\mathfrak{q}$-module, we will apply Theorem 2.3.5 below to $R_\mathfrak{m} \to R[t]_\mathfrak{q}$ and $M = \mathcal{L}_{0,\mathfrak{q}}$. We need to justify the hypotheses (a) and (b).

To check (b), we need to check that $\mathcal{L}_{0,\mathfrak{q}}$ is $R_\mathfrak{m}$-flat; note this is reasonable, since $\mathbb{A}^1_R \to \mathrm{Spec}(R)$ is flat, and so for example $R[t]_\mathfrak{q}$ is flat over $R_\mathfrak{m}$. Observe that $R[t]_\mathfrak{q} = S^{-1}R[t]_\mathfrak{m}$ for a certain multiplicative subset S. Hence, $\mathcal{L}_{0,\mathfrak{q}} = \mathcal{L}_{0,\mathfrak{m}} \otimes_{R[t]_\mathfrak{m}} R[t]_\mathfrak{q}$. Because $\mathcal{L}_0$ is R-flat (since $\Lambda_0/\mathcal{L}_0$ and Λ_0 are R-projective), we have $\mathcal{L}_{0,\mathfrak{m}}$ is $R_\mathfrak{m}$-flat. We want to show that $\mathcal{L}_{0,\mathfrak{m}} \otimes_{R[t]_\mathfrak{m}} R[t]_\mathfrak{q}$ is $R_\mathfrak{m}$-flat. Let $N \hookrightarrow P$ be an injective $R_\mathfrak{m}$-module map. Then

$$\mathcal{L}_{0,\mathfrak{m}} \otimes_{R_\mathfrak{m}} N \hookrightarrow \mathcal{L}_{0,\mathfrak{m}} \otimes_{R_\mathfrak{m}} P,$$

and as this is clearly $R[t]_\mathfrak{m}$-linear and $R[t]_\mathfrak{q}$ is $R[t]_\mathfrak{m}$-flat, we get

$$(R[t]_\mathfrak{q} \otimes_{R[t]_\mathfrak{m}} \mathcal{L}_{0,\mathfrak{m}}) \otimes_{R_\mathfrak{m}} N \hookrightarrow (R[t]_\mathfrak{q} \otimes_{R[t]_\mathfrak{m}} \mathcal{L}_{0,\mathfrak{m}}) \otimes_{R_\mathfrak{m}} P,$$

as desired.

Finally, we need to prove the hypothesis (a), which is the statement that $\mathcal{L}_{0,\mathfrak{q}}/\mathfrak{m}\mathcal{L}_{0,\mathfrak{q}} = \big(\mathcal{L}_0/\mathfrak{m}\mathcal{L}_0\big)_\mathfrak{q}$ is a free $R[t]_\mathfrak{q}/\mathfrak{m}R[t]_\mathfrak{q} = \big(R[t]/\mathfrak{m}R[t]\big)_\mathfrak{q}$-module. It is enough to prove that $\mathcal{L}_0/\mathfrak{m}\mathcal{L}_0$ is $R[t]/\mathfrak{m}R[t] = k[t]$-free. This follows because by R-projectivity of $\Lambda_0/\mathcal{L}_0$, the sequence

$$0 \to \mathcal{L}_0 \otimes_R R/\mathfrak{m} \to \Lambda_0 \otimes_R R/\mathfrak{m} \to (\Lambda_0/\mathcal{L}_0) \otimes_R R/\mathfrak{m} \to 0$$

is exact and therefore $\mathcal{L}_0/\mathfrak{m}\mathcal{L}_0$ has no t-torsion. The conclusion of Theorem 2.3.5 is that $\mathcal{L}_{0,\mathfrak{q}}$ is free over $R[t]_\mathfrak{q}$, and this completes the proof. $\square$

The following is [StaPro, Tag 00MH]:

Theorem 2.3.5. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let M be a finite S-module. Suppose that*

(a) *$M/\mathfrak{m}M$ is a free $S/\mathfrak{m}S$-module, and*
(b) *M is flat over R.*

Then M is free over S and S is flat over R.

Remark 2.3.6. Fixing n, let $\mathrm{Gr}_N^f(R)$ (resp. $\mathrm{Gr}_N(R)$) be the set of $R[\![t]\!]$-modules $\mathcal{L} \subset R(\!(t)\!)^n$ such that $t^N\Lambda \subset \mathcal{L} \subset t^{-N}\Lambda$ and finally such that $t^{-N}\Lambda/\mathcal{L}$ is R-projective (resp. and finally such that $\mathcal{L}$ is $R[\![t]\!]$-projective). It is clear that Gr_N^f is represented by a closed subscheme of a union of Grassmannian varieties in rank $2nN$ (being stable under the operator t is a closed condition).

In the proof above we used $X(\operatorname{colim}_i R_i) = \operatorname{colim}_i X(R_i)$ for $X = \mathrm{Gr}_N^f$. This holds because X is finite-type over k. In fact, the following appears in EGA IV, 8.14.12 (see also [StaPro, Tag 01ZC]).

Lemma 2.3.7. *A morphism of schemes $X \to S$ is locally of finite presentation if and only if for every directed set I and inverse system $\{T_i\}$ of affine schemes over S, we have*

$$\mathrm{Hom}_S(\varprojlim_i T_i, X) = \varinjlim_i \mathrm{Hom}_S(T_i, X).$$

Exercise: Prove more directly that $\mathrm{Gr}_N^f(R) = \varinjlim_i \mathrm{Gr}_N^f(R_i)$ if $R = \varinjlim_i R_i$. [MAYBE I will give a hint here....]

Finally, we have the following.

Lemma 2.3.8. *If $\mathcal{L} \in \mathrm{Gr}_N(R)$ then there exists a Zariski cover $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ such that $\mathcal{L} \otimes_{R[\![t]\!]} R'[\![t]\!]$ is $R'[\![t]\!]$-free.*

Proof. Write $R_t := R[\![t]\!]$. We proved above that $\mathcal{L}$ is finite over R_t. There exists $g_1, \ldots, g_r \in R_t$ with $(g_1, \ldots, g_r)_{R_t} = 1$ and such that $\mathcal{L}_{g_i}$ is $(R_t)_{g_i}$-free, for all i.

Then $f_i := g_i(0)$ satisfy $(f_1, \ldots, f_r)_R = 1$, and $g_i \in (R_{f_i}[\![t]\!])^\times$ if $f_i \neq 0$. Furthermore

$$\mathcal{L} \otimes_{R_t} R_{f_i}[\![t]\!] = \mathcal{L} \otimes_{R_t} R_{t,g_i} \otimes_{R_{t,g_i}} R_{f_i}[\![t]\!]$$

is free over $R_{f_i}[\![t]\!]$. $\square$

Define $\mathrm{Latt}_n := \operatorname{colim}_N \mathrm{Gr}_N^f$, an ind-projective ind-scheme over k, hence an fpqc sheaf on Aff_k. Note $LG(R)/L^+\mathcal{G}(R)$ identifies with the subset of $R[\![t]\!]$-**free** objects in Latt_n.

We have a diagram

$$\begin{tikzcd} LG/L^+\mathcal{G} \arrow[r, "a"] \arrow[d, "b"'] & \mathrm{Latt}_n \\ (LG/L^+\mathcal{G})^{++} \arrow[ur, "c"'] & \end{tikzcd}$$

where a is a *presheaf monomomrphism, Zariski-locally an epimorphism*, b is the sheafification morphism, and c is a *sheaf monomorphism, locally an epimorphism, hence an epimorphism*. We conclude: c is an isomorphism of fpqc sheaves, proving the desired representability statement.

We used:

Lemma 2.3.9. *The following hold:*

- *$\mathcal{F} \mapsto \mathcal{F}^{++}$ preserves finite limits*
- *In any category with fiber products, $\mathcal{F} \to \mathcal{F}'$ is a monomorphism if and only if*

$$\mathcal{F} \xrightarrow{\mathrm{diag}} \mathcal{F} \times_{\mathcal{F}'} \mathcal{F} \rightrightarrows \mathcal{F}$$

  is an equalizer.
- *Thus, the functor $++$ preserves monomorphisms.*

- *Morphism of sheaves is an isomorphism if and only if it is a monomorphism and an epimorphism.*

To summarize this section, we state:

Corollary 2.3.10. *There is an isomorphism of étale sheaves* $\mathrm{Gr}_{\mathrm{GL}_{n,\mathcal{O}}} \cong \mathrm{Latt}_n$*, and thus* $\mathrm{Gr}_{\mathrm{GL}_{n,\mathcal{O}}}$ *is represented by an ind-projective ind-scheme over* k*.*

3. Lecture 3

3.1. **Torsor description of** $\mathrm{Gr}_{\mathcal{G}}$**.** Let $G \to X$ be any group scheme over a scheme X. Let $\mathcal{C}$ be a Grothendieck topology on Aff_X.

Definition 3.1.1. A (right) $\mathcal{C}$-torsor $\mathcal{E} \to X$ is a $\mathcal{C}$-sheaf on Aff_X with a right G action $\mathcal{E} \times_X G \to \mathcal{E}$ such that for every $\mathrm{Spec}(R) \to X$, $G(R)$ acts simply-transitively on $\mathcal{E}(R)$, and such that $\mathcal{C}$-locally on $\mathrm{Spec}(R)$, we have $\mathcal{E}(R) \neq \emptyset$.

Let k be any field. For $R \in \mathrm{Aff}_k$, set $\mathbb{D}_R := \mathrm{Spec}\, R[\![t]\!]$, and $\mathbb{D}_R^* := \mathrm{Spec}\, R(\!(t)\!)$. Now assume $\mathcal{G} \to \mathbb{D}_k$ is an affine group scheme of finite type.

Definition 3.1.2. Let $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} : \mathrm{Aff}_k \to (\mathrm{Sets})$ be the functor sending to R to the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$ where

- $\mathcal{E} \to \mathbb{D}_R$ is a right étale $\mathcal{G} \times_{\mathbb{D}_k} \mathbb{D}_R$-torsor (call these "$\mathcal{G}$-torsors")
- $\alpha \in \mathcal{E}(\mathbb{D}_R^*)$, ie., an isomorphism of $\mathcal{G}$-torsors $\alpha : \mathcal{E}|_{\mathbb{D}_R^*} \xrightarrow{\sim} \mathcal{E}_0|_{\mathbb{D}_R^*}$, where $\mathcal{E}_0$ is the trivial torsor.

We declare $(\mathcal{E}, \alpha) \cong (\mathcal{E}', \alpha')$ if there is a morphism (necessarily an isomorphism) of $\mathcal{G}$-torsors $\pi : \mathcal{E} \to \mathcal{E}'$ such that $\alpha = \alpha' \circ \pi$.

Lemma 3.1.3. *The following properties hold:*

(0) *By effectivity of étale descent of affine schemes,* $\mathcal{E}$ *is represented by an affine scheme.*
(1) $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ *has a natural base point* $(\mathcal{E}_0, \mathrm{id})$*.*
(2) $L\mathcal{G}$ *acts on the left on* $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ *by* $g \in L\mathcal{G}(R) = \mathcal{G}(R(\!(t)\!))$ *sending* $(\mathcal{E}, \alpha)$ *to* $(\mathcal{E}, g \circ \alpha)$*.*
(3) $\mathcal{G} \mapsto \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ *is functorial: if* $\rho : \mathcal{G} \to \mathcal{H}$ *is a morphism, then let* $\rho_*\mathcal{E}$ *be the push-out torsor* $\mathcal{E} \times^{\mathcal{G}} \mathcal{H}$ *and* $\rho_*\alpha = (\alpha, 1) \in (\mathcal{E} \times^{\mathcal{G}} \mathcal{H})(\mathbb{D}_R^*)$*. Then get functorial map* $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} \to \mathrm{Gr}_{\mathcal{H}}^{\mathrm{tor}}$ *by* $(\mathcal{E}, \alpha) \mapsto (\rho_*\mathcal{E}, \rho_*\alpha)$*.*
(4) *For* $\mathcal{G} = \mathrm{GL}_{n,\mathcal{O}}$ *we have* $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} = \mathrm{Gr}_{\mathcal{G}}$*.*

Proof. We will only sketch (4). Any GL_n-torsor $\mathcal{E}$ gives rise to a vector bundle $V_{\mathcal{E}} := \mathcal{E} \times^{\mathrm{GL}_n} \mathcal{O}^n$. Any vector bundle $\mathcal{V}$ of rank n gives a GL_n-torsor $\underline{\mathrm{Isom}}(\mathcal{O}^n, \mathcal{V})$. Note $(\mathcal{E}, \alpha)$ gives an isomorphism $\mathcal{E}[t^{-1}] \xrightarrow{\sim} R[\![t]\!][1/t]^n = R(\!(t)\!)^n$, identifying $(\mathcal{E}, \alpha)$ with the lattice and its inclusion

$$\Lambda_{(\mathcal{E},\alpha)} := \mathrm{image}(\mathcal{E}) \subset R(\!(t)\!)^n.$$

Therefore we get an isomorphism $\mathrm{Gr}_{\mathrm{GL}_n} \cong \mathrm{Gr}_{\mathrm{GL}_n}^{\mathrm{tor}}$. □

Theorem 3.1.4. *Let* $\mathcal{G}$ *be a smooth affine group group scheme over* $k[\![t]\!]$*.*

(I) $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(k)$ *is represented by a separated ind-scheme of ind-finite type; in particular, it is an fpqc sheaf.*
(II) *If* $\mathcal{G}$ *is reductive,* $\mathrm{Gr}_{\mathcal{G}}$ *is ind-projective.*

The key ingredient is the following proposition.

Proposition 3.1.5. *Let* $\mathcal{G} \hookrightarrow \mathcal{H}$ *be a closed immersion of affine group schemes of finite type over* $\mathbb{D}_k$*, such that the fppf quotient* $\mathcal{H}/\mathcal{G}$ *is represented by a quasi-affine (resp., affine) scheme. Then* $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} \to \mathrm{Gr}_{\mathcal{H}}^{\mathrm{tor}}$ *is represented by a quasi-compact immersion (resp. closed immersion).*

Proof. We follow the proof in [Zhu17, Prop. 1.2.6]. To simplify notation, we write $\mathrm{Gr}_{\mathcal{G}}$ instead of $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ (later we will show these two functors agree; we are not using that fact at the moment however).

Take any morphism $(\mathcal{E}, \alpha) : \mathrm{Spec}(R) \to \mathrm{Gr}_{\mathcal{H}}$. We need to check that the morphism

$$\mathfrak{F} := \mathrm{Spec}(R) \times_{\mathrm{Gr}_{\mathcal{H}}} \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(R)$$

is represented by a locally closed immersion.

First, note that $\pi : \mathcal{E} \to \mathbb{D}_R$ and its section α over $\mathbb{D}_R^*$, induce a morphism of étale sheaves $\tilde{\pi} : \mathcal{E}/\mathcal{G} \to \mathbb{D}_R$ and a section $\tilde{\alpha}$ of $\tilde{\pi}$ over $\mathbb{D}_R^*$. By étale descent of (quasi-)affine morphisms [StaPro, Tags 0244, 0246], there exists a scheme W affine and finitely presented over $\mathbb{D}_R$ equipped with a quasi-compact open embedding $\mathcal{E}/\mathcal{G} \hookrightarrow W$ over $\mathbb{D}_R$. Further $\mathfrak{F}$ identifies with the presheaf

$$\mathfrak{F}(\operatorname{Spec}(R') \to \operatorname{Spec}(R)) = \{\text{sections } \beta \text{ of } \tilde{\pi} \text{ over } \mathbb{D}_{R'} \text{ such that } \beta|_{\mathbb{D}^*_{R'}} = \tilde{\alpha}|_{\mathbb{D}^*_{R'}}\}. \tag{3.1.1}$$

This is proved via the correspondence between sections of $\tilde{\pi}$ over $\mathbb{D}_R$ and isomorphism classes of pairs $(\mathcal{E}_{\mathcal{G}}, \varphi)$, where $\mathcal{E}_{\mathcal{G}}$ is a $\mathcal{G}$-torsor over $\mathbb{D}_R$ and φ is an isomorphism of $\mathcal{H}$-torsors $\mathcal{E}_{\mathcal{G}} \times^{\mathcal{G}} \mathcal{H} \xrightarrow{\sim} \mathcal{E}$. (Recall the definition of the fiber category over $\mathbb{D}_R$ of the quotient stack $[\mathcal{E}/\mathcal{G}]$. We give further details for this identification below, in subsection 3.2.)

To prove (3.1.1) is represented by a locally-closed subset of $\operatorname{Spec}(R)$, the main input is the following.

Lemma 3.1.6. *If $V \xrightarrow{p} \mathbb{D}_R$ is an affine scheme of finite presentation, and s is a section of p over $\mathbb{D}_R^*$, then the presheaf*

$$(R \to R') \mapsto \{\textit{section } s' \textit{ of } p \textit{ over } \mathbb{D}_{R'} \textit{ such that } s'|_{\mathbb{D}_{R'}} = s|_{\mathbb{D}^*_{R'}}\}$$

is represented by a closed subscheme of $\operatorname{Spec}(R)$.

Proof. For some integer N, there is a closed embedding $V \hookrightarrow \mathbb{A}^N_{\mathbb{D}_R}$ over $\mathbb{D}_R$. In coordinates,

$$s = (s_1(t), \dots, s_N(t)), \qquad\qquad s_i(t) = \sum_j s_{ij} t^j \in R(\!(t)\!).$$

The presheaf is represented by $\operatorname{Spec}(A) \subset \operatorname{Spec}(R)$ where

$$A = R/\langle s_{ij} = 0, \forall i, \forall j < 0\rangle.$$

That is, s extends to a section s' over R' if and only if all $s_{ij} = 0$ in R' for $j < 0$. (As $R[\![t]\!] \subset R(\!(t)\!)$, $(s_1'(t), \dots, s_N'(t))$ satisfies the equations s does.)

Now apply the lemma to $V = W$ and $s = \tilde{\alpha}$. We get a closed subset $\operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(R)$ and a universal section $s_A : \mathbb{D}_A \to W_{\mathbb{D}_A}$ such that

$$s_A|_{\mathbb{D}_A^*} = \tilde{\alpha}|_{\mathbb{D}_A^*}.$$

Base change along $\operatorname{Spec}(A) \to \mathbb{D}_A$ given by $t \mapsto 0$ gives $s_0 : \operatorname{Spec}(A) \to W_A = W \times_{\mathbb{D}_A} \operatorname{Spec}(A)$. Then $\mathfrak{F} = s_0^{-1}(\mathcal{E}/\mathcal{G} \times_{\mathbb{D}_R} \operatorname{Spec}(A))$, an open subset in $\operatorname{Spec}(A)$. □

Lemma 3.1.7. *Suppose $\mathcal{G}$ is a flat affine group scheme over $\mathbb{D}_k$ of finite type. Then:*

(a) *There exists a closed immersion of group schemes over $\mathbb{D}_k$*

$$\mathcal{G} \hookrightarrow \mathrm{GL}_n \times \mathrm{GL}_1$$

such that $(\mathrm{GL}_n \times \mathrm{GL}_1)/\mathcal{G}$ is quasi-affine.

(b) *If $\mathcal{G}$ is reductive, then $(\mathrm{GL}_n \times \mathrm{GL}_1)/\mathcal{G}$ is affine.*

Proof. Part (a) is proved in [PR08, Prop. 1.3]. For (b), use [Al14, Cor. 9.7.7]. □

Corollary 3.1.8. *For any flat affine group scheme of finite type $\mathcal{G} \to \mathbb{D}_k$, the presheaf $\mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}$ is represented by a separated ind-scheme of ind-finite type over k. If $\mathcal{G}$ is reductive, then $\mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}$ is ind-projective.*

Proof. Use Lemma 3.1.7, Proposition 3.1.5, Lemma 3.1.3, and Corollary 2.3.10. □

Now that we know $\mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}$ is an fpqc sheaf, following [Zhu17, Prop. 1.3.6], we have the following.

Lemma 3.1.9. *Suppose $\mathcal{G} \to \mathbb{D}_k$ is smooth and affine. Then the natural map $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}$ is an isomorphism.*

Proof. The map $L\mathcal{G}(R)/L^+\mathcal{G}(R) \to \mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}(R)$ sends g to $(\mathcal{E}_{0,R}, g)$, where $\mathcal{E}_{0,R}$ is the trivial torsor. We need to show that the induced map of sheaves $L\mathcal{G}/L^+\mathcal{G} \to \mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}$ is étale locally an epimorphism. That is, such $(\mathcal{E}, \alpha) \in \mathrm{Gr}^{\mathrm{tor}}_{\mathcal{G}}(R)$, we must find an étale cover $R \to R'$ such that $(\mathcal{E}_{R'_t}, \alpha_{R'_t})$ has $\mathcal{E}_{R'_t} \cong \mathcal{E}_{0,R'_t}$. Now $\mathcal{E} \times_{\mathbb{D}_R} \mathrm{Spec}(R)$ is a $\mathcal{G}_R := \mathcal{G}_{\mathbb{D}_R} \times_{\mathbb{D}_R} \mathrm{Spec}(R)$-torsor which has a section over some étale cover $R \to R'$ (because $\mathcal{G}_R$ is smooth over $\mathrm{Spec}(R)$). Then by the infinitesimal criterion for smoothness, we get a section $\mathrm{Spf}(R'[\![t]\!]) \to \mathcal{E}_{R'_t}$ and thus a section $\mathrm{Spec}(R'[\![t]\!]) \to \mathcal{E}_{R'_t}$ (since $\mathcal{E}$ is affine). □

Putting all the above ingredients together, we have proved Theorem 3.1.4. □

Remark 3.1.10. The above discussion applies to $\mathcal{G}$ **split**. It does not prove ind-projectivity in the case where $\mathcal{G}$ is a parahoric subgroup for a non-split group G over F. **In the final version of these notes, we shall prove that when $\mathcal{G}$ is any parahoric subgroup for any group G over F, the ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ is ind-proper, and hence automatically ind-projective**. We shall summarize the method that even applies to BD-affine Grassmannians, as explained in [HR20b, Thm. 4.16].

3.2. **Further details about the reduction of torsors.** In this section we give more explanations about the identification in (3.1.1). Let G be a group scheme over S which is faithfully flat and quasi-compact over S. Let Y be any S-scheme carrying a right G-action over S. Recall that the quotient stack $[Y/G]$ is the fibered category whose fiber over an S-scheme X is the category whose objects are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \\ X & \end{tikzcd}$$

where $P \to X$ is a right G-torsor, and $P \to Y$ is a G-equivariant morphism. The morphisms in the category are commutative diagrams

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Y \\ X' \arrow[r] & X & \end{tikzcd}$$

such that the composition $P' \to P \to Y$ is the given map $P' \to Y$ (all G-equivariant). Note that the square is automatically Cartesian. This is because the universal property gives a canonical morphism $P' \to X' \times_X P$, which is a morphism of G-torsors over X' and thus an isomorphism.

The discussion which follows in some sense explains why the above is the correct way to formulate the definition of quotient stack.

Let S by any scheme, and let $X \to S$ be any S-scheme. Let $G \subset H$ be a closed immersion of affine faithfully flat quasicompact group schemes over X. We fix throughout a scheme $E \to X$ with an G-action, **such that the quotient E/G exists as a scheme over X** and $E \to E/G$ is faithfully flat and quasi-compact. Then $E \to E/\mathrm{G}$ is an G-torsor (it becomes trivial after base-changing along $E \to E/G$). The structure morphism $E \to X$ induces the structure morphism $E/G \to X$.

Lemma 3.2.1. (a) *The sections $X \to E/G$ correspond bijectively to the isomorphism classes of G-torsors $E_G \to X$ endowed with G-equivariant morphisms $E_G \to E$, such that the following diagram is Cartesian (equivalently, commutative)*

$$\begin{tikzcd} E_G \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & E/G. \end{tikzcd}$$

Here if $E \to X$ is itself an H-torsor, then $E_G \to X$ gives a reduction to G structure on $E \to X$, as the induced map $E_G \times^G H \to E$ is an isomorphism of H-torsors over X.

(b) *Suppose $\alpha : X \to E$ is a section lifting the section $X \to E/G$. Then it induces a unique section $X \to E_G$ such that $X \to E_G \to E$ is the given section α.*

Proof. Part (a): Given $X \to E/G$, let $E_G := X \times_{E/G} E$, with the obvious G-equivariant projection morphism to E. The first projection gives a G-torsor $E_G \to X$. Conversely, suppose given a G-torsor $E_G \to X$ and a G-equivariant map $E_G \to E$. We get a commutative diagram

$$\begin{tikzcd} E_G \arrow[r] \arrow[d] & E \arrow[d] \\ E_G/G \arrow[r] & E/G. \end{tikzcd}$$

The canonical isomorphism $E/G \xrightarrow{\sim} X$ identifies the lower arrow as a section of $E/G \to X$. As $E_G \to X$ is a G-torsor, this diagram is Cartesian, so its isomorphism class is determined by the section $X \to E/G$.

Part (b): This follows, using the universal property of the fiber product $X \times_{E/G} E$, from Part (a). $\square$

Now, to justify (3.1.1) we apply Part (a) to $X = \mathbb{D}_{R'}$ and part (b) to $X = \mathbb{D}^*_{R'}$ with $E = \mathcal{E}(= \mathcal{E}|_{\mathbb{D}_{R'}}$ in the earlier notation), $\alpha = \alpha|_{\mathbb{D}^*_{R'}}$ and $\beta = \beta|_{\mathbb{D}_{R'}}$, and moreover $G := \mathcal{G}_{\mathbb{D}_{R'}}$ and $H := \mathcal{H}_{\mathbb{D}_{R'}}$ (in the earlier notation). Namely, we need to compare the Cartesian diagrams

$$\begin{tikzcd} \mathcal{F}^*_G \arrow[r] \arrow[d] & \mathcal{E}^* \arrow[d] \\ X^* \arrow[r, "\tilde{\alpha}"] \arrow[ur, "\alpha"] & \mathcal{E}^*/G \end{tikzcd}$$

with

$$\begin{tikzcd} \mathcal{E}_G \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X \arrow[r, "\beta"] & \mathcal{E}/G \end{tikzcd}$$

where the starred objects are over $X^* := \mathbb{D}^*_{R'}$ and the unstarred objects are over $X := \mathbb{D}_{R'}$, and $\mathcal{E}^* := \mathcal{E}|_{X^*}$. The data of the section β gives us the reduction $\mathcal{E}_G$ and its diagram, by Lemma 3.2.1(a). The assumption that $\beta|_{\mathbb{D}^*_{R'}} = \tilde{\alpha}$ means that $\mathcal{E}_G|_{\mathbb{D}^*_{R'}} \cong \mathcal{F}^*_G$, and then the existence of the section α extending $\tilde{\alpha}$ means, as in Lemma 3.2.1(b), that there is a section of $\mathcal{E}_G \to \mathbb{D}_{R'}$ over $\mathbb{D}^*_{R'}$ which is compatible with the section α via $\mathcal{E}_G|_{\mathbb{D}^*_{R'}} \to \mathcal{E}_{\mathbb{D}^*_{R'}}$. From this data we get the map $\mathrm{Spec}(R') \to \mathfrak{F} = \mathrm{Spec}(R) \times_{\mathrm{Gr}_{\mathcal{H}}} \mathrm{Gr}_{\mathcal{G}}$. $\square$

3.3. **Definition of (Middle) Perverse sheaves.**

Perverse sheaves are neither perverse nor sheaves – Robert MacPherson.

3.3.1. *Derived categories.* Assume k is an algebraically closed or finite field. Let X/k be a scheme of finite type over k, usually endowed with some stratification. Choose a prime number ℓ which lies in $k^\times$. Let Λ be an ℓ-torsion sheaf on X (e.g $\Lambda = \mathbb{Z}/\ell^N\mathbb{Z}$).

We have the bounded constructible derived category $D^b_c(X, \Lambda) := D^b_{c,et}(X, \Lambda)$. Thus we can define the 2-categories

$$D^b_c(X, \mathbb{Z}_\ell) := \varprojlim_n D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z}) \qquad , \qquad D^b_c(X, \mathbb{Q}_\ell) := D^b_c(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}.$$

Similarly we can replace $\mathbb{Q}_\ell$ and $\mathbb{Z}_\ell$ with finite integral extensions $E/\mathbb{Q}_\ell$ and $\mathcal{O}_E/\mathbb{Z}_\ell$. We set

$$D^b_c(X, \bar{\mathbb{Q}}_\ell) := \varinjlim_{E/\mathbb{Q}_\ell < \infty} D^b_c(X, E).$$

Remark 3.3.1. In the final version of these notes, I will discuss the version of Bhatt-Scholze here....

3.3.2. *Six functors and Verdier duality.* Let $f : X \to Y$ be a separated morphism of finite-type k-schemes. We have the functors

$$Rf_* : D^b_c(X, \bar{\mathbb{Q}}_\ell) \to D^b_c(Y, \bar{\mathbb{Q}}_\ell)$$
$$f^* : D^b_c(Y, \bar{\mathbb{Q}}_\ell) \to D^b_c(X, \bar{\mathbb{Q}}_\ell).$$

As f is *compactifiable* (i.e. there is a quasi-compact open immersion $j : X \hookrightarrow \bar{X}$ and a factorization

$$f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Y$$

with $\bar{f}$ proper), we may define

$$Rf_! := R\bar{f}_* \circ j_! : D^b_c(X, \bar{\mathbb{Q}}_\ell) \to D^b_c(Y, \bar{\mathbb{Q}}_\ell).$$

We have relative Hom sheaf functors $\mathcal{R}Hom(-,-)$ and derived tensor product functors $- \otimes^L -$. Further $f_!$ has a right adjoint $f^!$ with respect to these:

$$\mathrm{Hom}(M, f^! N) \xrightarrow{\sim} \mathrm{Hom}(Rf_! M, N)$$
$$Rf_* \mathcal{R}Hom(M, f^! N) \xrightarrow{\sim} \mathcal{R}Hom(Rf_! M, N)$$
$$\mathcal{R}Hom(M \otimes^L N, P) \xrightarrow{\sim} \mathcal{R}Hom(M, \mathcal{R}Hom(N, P)).$$

Definition 3.3.2 (Verdier duality)**.** Let $f : X \to \mathrm{Spec}(k) = S$ be the structure morphism. Define the *dualizing sheaf*

$$K_X := f^!(\bar{\mathbb{Q}}_{\ell,S})$$

For $\mathcal{F} \in D^b_c(X, \bar{\mathbb{Q}}_\ell)$, set $D_X(\mathcal{F}) := \mathcal{R}Hom(C, K_X)$, which lies in $D^b_c(X, \bar{\mathbb{Q}}_\ell)$.

We use our earlier choice of $q^{1/2} \in \bar{\mathbb{Q}}_\ell^\times$ to define the notion of **Tate twist** of $\mathcal{C}\big(\frac{d}{2}\big)$ for any $d \in \mathbb{Z}$.
Example If X is smooth and irreducible of dimension d, then $K_X = \bar{\mathbb{Q}}_\ell[2d](d)$.

Theorem 3.3.3. *We have the following properties of the Verdier duality operators D (interpret all below on the appropriate space):*

(a) $D \circ D = D$
(b) $D \circ f_! = Rf_* \circ D$
(c) $D \circ f^* = f^! \circ D$
(d) $Rf_!(M \otimes^L f^* N) = Rf_! M \otimes^L N$
(e) $D(\mathcal{F}[d](e)) = D(\mathcal{F})[-d](-e)$.

3.3.3. *Definition of (Middle) perverse t-structure.*

Definition 3.3.4. For X as above, define the (middle) perverse t-structure on $D^b_c(X, \bar{\mathbb{Q}}_\ell)$ by setting

$${}^pD^{\leq 0}(X, \bar{\mathbb{Q}}_\ell) = \{B \in D^b_c(X, \bar{\mathbb{Q}}_\ell) \,|\, \dim \mathrm{supp}(\mathcal{H}^i B) \leq -i, \ \forall i \in \mathbb{Z}\}$$
$${}^pD^{\geq 0}(X, \bar{\mathbb{Q}}_\ell) = \{D_X(B) \,|\, B \in {}^pD^{\leq 0}(X, \bar{\mathbb{Q}}_\ell)\}.$$

Theorem 3.3.5. *The category $P(X) = P(X, \bar{\mathbb{Q}}_\ell) := {}^pD^{\leq 0}(X, \bar{\mathbb{Q}}_\ell) \cap {}^pD^{\geq 0}(X, \bar{\mathbb{Q}}_\ell)$ is an abelian category, all of whose objects have finite length.*

Example: If X is smooth and irreducible of dimension d, then $\bar{\mathbb{Q}}_{\ell,X}[d]$ is perverse and simple. The perversity holds even if X is a local complete intersection of dimension d (see [KW, Lemma II.6.5]).

3.3.4. *Middle extension functor and simple perverse sheaves.* Let $j : U \hookrightarrow X$ be an open immersion with dense image, U smooth and irreducible of dimension d. Let $i : Y \hookrightarrow X$ be the closed complement.

Definition 3.3.6. Let $B \in P(U) \subset D^b_c(U, \bar{\mathbb{Q}}_\ell)$. There exists a unique extension $\overline{B} \in P(X)$ such that $\overline{B}$ has neight subobjects nor quotients of the form $i_*(A)$ for $A \in P(Y)$. Equivalently $j^*\overline{B} = B$, and ${}^pH^0(i^*\overline{B}) = 0 = {}^pH^0(i^!\overline{B})$.

We denote $j_{!*}(B) := \overline{B}$.

The fact that such extensions exist and are unique is proved in [BBD82, Cor. 1.4.25]. In fact $j_{!*} : P(U) \to P(X)$ is a functor: see [KW, III.5.2]. Since $Dj_{!*}B$ extends DB and satisfies the defining properties of $j_{!*}(DB)$, we see that

$$D \circ j_{!*} = j_{!*} \circ D.$$

Definition 3.3.7. If $\mathcal{L}$ is a locally constant $\bar{\mathbb{Q}}_\ell$-sheaf on U (sometimes called a *lisse* $\bar{\mathbb{Q}}_\ell$-sheaf), we define $\mathrm{IC}(U,\mathcal{L})$ as the unique prolongation of $\mathcal{L}[d]$ to a perverse sheaf on X with neither subobjects nor quotients of the form i_*A, where $A \in P(Y)$. We denote

$$\mathrm{IC}(U,\mathcal{L}) = j_{!*}(\mathcal{L}[d]).$$

Theorem 3.3.8. *A perverse sheaf $\mathcal{F}$ on X is simple if and only if it is of the form $i_{\bar{U},*}\mathrm{IC}(U,\mathcal{L})$ for $\mathcal{L}$ an* **irreducible** *lisse $\bar{\mathbb{Q}}_\ell$-sheaf on an irreducible smooth locally closed subvariety U with closure $\bar{U}$.*

Here the definitions are with respect to the embeddings $U \hookrightarrow \bar{U} \xrightarrow{i_{\bar{U}}} X$.

3.3.5. *A characterization of middle extension functor on self-dual perverse sheaves.* The next proposition follows from the proof of [BBD82, Prop. 2.1.17] (which is stated slightly differently).

Proposition 3.3.9. *If $U \overset{j}{\hookrightarrow} X$ is an open immersion of a union of strata, and if A is a self-dual perverse sheaf on U, then $j_{!*}A$ is the unique self-dual prolongation P in $D_c^b(X,\bar{\mathbb{Q}}_\ell)$ of A such that, if $i_S : S \hookrightarrow X$ is a stratum in $X - U$, then $H^i i_S^* P = 0$ when $i \geq -\dim\, S$.*

3.3.6. *Descent along smooth morphisms.* The following is [BBD82, Prop. 4.2.5].

Proposition 3.3.10. *Suppose $f : X \to Y$ is a smooth morphism of relative dimension d whose geometric fibers are connected and non-empty. Then the functor $f^*[d]$ is t-exact and induces a fully faithful functor $P(Y) \to P(X)$.*

The following gives a situation where this is essentially surjective, and hence an equivalence of categories. It is proved in [HN02, Lemma 21].

Lemma 3.3.11. *Suppose $f : X \to Y$ is a smooth surjective morphism of finite-type schemes over a field k, and suppose G_Y is a smooth connected Y-group scheme which acts trivially on Y and on X such that f is G_Y-equivarient and the action on each geometric fiber of f is transitive. Assume further that the stabilizer in G_Y of any geometric point of X is a connected essentially smooth subgroup. Then a G_Y-equivariant perverse sheaf $\mathcal{F}$ on X descends along f.*

3.4. **Properties of (Middle) Perverse sheaves.** We want to give another description of the full subcategory $P(X) \subset D_c^b(X,\bar{\mathbb{Q}}_\ell)$, for a finite type scheme X over a field k with $(\ell, \mathrm{char}(k)) = 1$. Consider the special case of a stratified scheme $X = \coprod_\beta X_\beta$, where each X_β is nonsingular and the boundary of any stratum is a union of other strata (N.B.: the former definition does not refer to any particular stratification of X). The following is an equivalent formulation.

Then we say $S \in D_c^b(X,\bar{\mathbb{Q}}_\ell)$ is **perverse** if

$$H^i S|X_\beta = 0, \;\text{ if } i > -\dim\, X_\beta$$
$$H^i DS|X_\beta = 0, \;\text{ if } i > -\dim\, X_\beta.$$

3.5. **(Semi-)Small Morphisms.**

3.5.1. *Locally trivial morphisms.* Let $X = \cup_\beta X_\beta$ and $Y = \cup_\alpha Y_\alpha$ be stratifications of algebraic varieties over a field k by locally closed subvarieties, having the property that the boundary of any stratum is a union of other strata.

Suppose we have a (surjective) morphism $f : X \to Y$. We suppose that f is proper and that each $f(X_\beta)$ is a union of strata Y_α. We say f is **locally trivial in the stratified sense**, if for every $y \in Y_\alpha$ there is a Zariski-open subset $V \subset Y_\alpha$ with $y \in V$, and a stratified variety F, such that there is an isomorphism of stratified varieties

$$f^{-1}(V) \cong F \times V \tag{3.5.1}$$

which commutes with the projections to V. In particular, for $y \in Y_\alpha \subset f(X_\beta)$,

$$\dim(f^{-1}(y) \cap X_\beta) + \dim\, Y_\alpha = \dim(f^{-1}(Y_\alpha) \cap X_\beta).$$

3.5.2. *Semi-small and small morphisms.* We say $f : X \to Y$ as above is **semi-small** if whenever $f(X_\beta) \supseteq Y_\alpha$, we have

$$\dim(f^{-1}(Y_\alpha) \cap X_\beta) \leq \frac{1}{2}\big(\dim\, X_\beta + \dim\, Y_\alpha\big), \text{ or, equivalently}$$
$$\dim(f^{-1}(y) \cap X_\beta) \leq \frac{1}{2}\big(\dim\, X_\beta - \dim\, Y_\alpha\big), \text{ for } y \in Y_\alpha.$$

We say f is **small** if is semi-small, and

- if X° and Y° are the open dense strata, then $X^\circ \supseteq f^{-1}(Y^\circ)$ and $f^{-1}(Y^\circ) \xrightarrow{f} Y^\circ$ is generically finite and étale; and
- whenever Y_α is not dense in $f(X_\beta)$, we have
$$\dim(f^{-1}(y) \cap X_\beta) < \frac{1}{2}\big(\dim\, X_\beta - \dim\, Y_\alpha\big).$$

Example: A finite surjective stratified map $f : X \to Y$ is small.

Proposition 3.5.1. *Let S be a perverse sheaf on X. If f is semi-small, then f_*S is perverse on Y.*

Proof. By definition, we know

$$H^iS|X_\beta = 0, \text{ if } i > -\dim\, X_\beta$$
$$H^iDS|X_\beta = 0, \text{ if } i > -\dim\, X_\beta.$$

We need to show

(i) $$(H^if_*S)|Y_\alpha = 0, \text{ if } i > -\dim\, Y_\alpha$$

(ii) $$(H^iDf_*S)|Y_\alpha = 0, \text{ if } i > -\dim\, Y_\alpha.$$

Let $y \in Y_\alpha$ and consider the spectral sequence $H^p(f^{-1}(y), H^qS) \Rightarrow H^{p+q}(f^{-1}(y), S)$. If $H^i(f^{-1}(y), S) = H^i_yf_*S \neq 0$, then $\exists\ p + q = i$ and X_β such that $H^p_c(f^{-1}(y) \cap X_\beta, H^qS) \neq 0$.

So $H^qS|X_\beta \neq 0$, hence $q \leq -\dim\, X_\beta$, and

$$p \leq 2(\dim(f^{-1}(y) \cap X_\beta)) \leq \dim\, X_\beta - \dim\, Y_\alpha \leq -q - \dim\, Y_\alpha$$

for any $y \in Y_\alpha \subset f(X_\beta)$ (the second $\leq$ coming from f semi-small).

Thus $i = p + q \leq -\dim\, Y_\alpha$ if $H^i_yf_*S \neq 0$ for some $y \in Y_\alpha$. This means (i) holds.

For (ii), note that $f_! = f_*$ (as f is proper), so that $Df_*S = f_*DS$. So (ii) follows as (i), using DS in place of S. □

Proposition 3.5.2. *Assume f is small and that X° and Y° are nonsingular open dense strata. Let $n = \dim\, X = \dim\, Y$. Let $\mathcal{L}$ be a local system on X° such that $\mathcal{L}[n]$ is Verdier self-dual. Let $\mathcal{L}' = (f|X^\circ)_*\mathcal{L}$ be the resulting local system on Y°. Let*

$$\mathrm{IC}(X, \mathcal{L}) = j_{X^\circ,!*}\mathcal{L}[n]$$
$$\mathrm{IC}(Y, \mathcal{L}') = j_{Y^\circ,!*}\mathcal{L}'[n],$$

where j_{X° is the open immersion $X^\circ \hookrightarrow X$. Then

$$f_*\mathrm{IC}(X, \mathcal{L}) = \mathrm{IC}(Y, \mathcal{L}').$$

Proof. We verify the conditions of Proposition 3.3.9.

- $f_*\mathrm{IC}(X, \mathcal{L})$ is a prolongation of $(f|X^\circ)_*\mathcal{L}[n]$: Since $j^*_{Y^\circ} = j^!_{Y^\circ}$, base change says
$$j^*_{Y_0}f_*\mathrm{IC}(X, \mathcal{L}) = (f|X^\circ)_*j^*_{X^\circ}\mathrm{IC}(X, \mathcal{L})$$
$$= (f|X^\circ)_*\mathcal{L}[n]$$
- $f_*\mathrm{IC}(X, \mathcal{L})$ is self-dual: since $f_! = f_*$, this follows since $\mathrm{IC}(X, \mathcal{L})$ is self-dual.
- Suppose $Y_\alpha \subset Y - Y^\circ$ and $i \geq -\dim\, Y_\alpha$. We need to show that
$$H^if_*\mathrm{IC}(X, \mathcal{L})|Y_\alpha = 0.$$

  First assume $H^if_*\mathrm{IC}(X, \mathcal{L})|Y_\alpha \neq 0$; we shall prove $i < -\dim\, Y_\alpha$. Let $y \in Y_\alpha$ be such that $H^i_yf_*\mathrm{IC}(X, \mathcal{L}) = H^i(f^{-1}(y), \mathrm{IC}(X, \mathcal{L})) \neq 0$.

By considering spectral sequences as before, $\exists\ p+q=i$ and X_β such that

$$H^p_c(f^{-1}(y)\cap X_\beta, H^q\mathrm{IC}(X,\mathcal{L}))\neq 0.$$

Note that $f(X_\beta)\supset Y_\alpha$.
Case 1: Y_α is dense in $f(X_\beta)$.

Since $Y_\alpha\subset Y-Y^\circ$, X_β is not X°: if $X_\beta = X^\circ$, then Y_α would be dense in $f(X_0)\supset Y_0$ and hence in Y, a contradiction.

Since $H^q\mathrm{IC}(X,\mathcal{L})|X_\beta = 0$ if $q > -\dim\, X_\beta$, or if $q\geq -\dim\, X_\beta$ and $X_\beta\neq X^\circ$, we see from $H^q(\mathrm{IC}(X,\mathcal{L}))|X_\beta\neq 0$ and $X_\beta\neq X^\circ$ that $q<-\dim\, X_\beta$. Also

$$p\leq 2\big(\dim(f^{-1}(y)\cap X_\beta)\big)\leq \dim\, X_\beta - \dim\, Y_\alpha < -q-\dim\, Y_\alpha.$$

(The middle $\leq$ is because f is semi-small.) Thus $i=p+q<-\dim\, Y_\alpha$, as desired.
Case 2: Y_α is not dense in $f(X_\beta)$.

Again $H^p_c(f^{-1}(y)\cap X_\beta, H^q\mathrm{IC}(X,\mathcal{L}))\neq 0$ implies that $H^q\mathrm{IC}(X,\mathcal{L})|X_\beta\neq 0$ and hence $q\leq -\dim\, X_\beta$. It follows that

$$p\leq 2\dim(f^{-1}(y)\cap X_\beta)<\dim\, X_\beta-\dim\, Y_\alpha\leq -q-\dim\, Y_\alpha.$$

(The $<$ is because f is small.) So again $i=p+q<-\dim\, Y_\alpha$.

□

4. Lecture 4

4.1. **Equivariant perverse sheaves.** Let X be a finite-type separated k-scheme, for k a field as above. Abbreviate $D(X)=D^b_c(X,\bar{\mathbb{Q}}_\ell)$ and $P(X)=P(X,\bar{\mathbb{Q}}_\ell)$. Let G be a **smooth and connected** affine k-group scheme acting on X on the left. We have the following morphisms

$$\begin{aligned} m &: G\times G\to G\\ a &: G\times X\to X\\ e &: X\to G\times X\\ \mathrm{pr}_2 &: G\times X\to X.\end{aligned}$$

Definition 4.1.1 (Equivariant perverse sheaf)**.** We say $K\in P(X)$ is G-*equivariant* if there is an isomorphism $\varphi : a^*K\xrightarrow{\sim}\mathrm{pr}_2^*K$ in $D(G\times X)$.

Remark 4.1.2. We can define the notion of a G-equivariant object in $D(X)$, but that definition involves a certain *ridigity* and a *cocycle* condition. More precisely, we would require $e^*\varphi=\mathrm{id}_K$, and $(m\times\mathrm{id}_X)^*\varphi = \mathrm{pr}_{23}^*\varphi\circ(\mathrm{id}_G\times a)^*\varphi$. The point here is that for *perverse* sheaves equivariant with respect to a *smooth connected* affine group scheme over k, these rigidity and cocycle conditions are automatic (really, the ridigity can be arranged and then the cocycle condition is automatic). This all follows from Lemma 3.3.10 applied to the morphism $\mathrm{pr}_3 : G\times G\times X\to X$. See [KW, pp. 187-188].

Exercise: Suppose G is a smooth connected k-group scheme. Let $K\in P_G(X)$. Let Q be any subquotient of K taken in the category $P(X)$. Show that $Q\in P_G(X)$.

4.1.1. *Perverse sheaves on orbit spaces.* Let G be a connected smooth group scheme over k of dimension d, and let $H\subset G$ be a smooth connected closed subgroup over k. Let $X=G/H$, a quotient which we assume exists in the category of separated k-schemes. Let $P_G(X)$ denote the category of G-equivariant perverse sheaves.

Proposition 4.1.3. *Any $K\in P_G(X)$ is of the form $K\cong\bar{\mathbb{Q}}^r_\ell[\dim X]$ for some integer $r\geq 0$.*

Proof. Applying Lemma 3.3.10 to the morphism $G\to G/H$ which has smooth and geometrically connected fibers, we see we may as well assume $H=e$ and $X=G$. Now we have $a=m : G\times G\to G$. Let $s : G\to\mathrm{Spec}(k)$ be the structure morphism, and write $e : \mathrm{Spec}(k)\to G$ for the zero section.

We need to show that $K\in P_G(G)$ implies that $K\cong\bar{\mathbb{Q}}^r_\ell[d]$ for some $r\geq 0$. Consider

$$i : G\xrightarrow{\sim} G\times e\hookrightarrow G\times G,$$

defined by $i(g) = (g, e)$. Using equivariance and $\mathrm{pr}_2 \circ i = e \circ s$, we have

$$K = \mathrm{id}_G^* K = i^* a^* K \cong i^* \mathrm{pr}_2^* K \cong s^*(K_0),$$

where K_0 is the stalk of K at e, a complex of $\bar{\mathbb{Q}}_\ell$-vector spaces. This shows that K is a *constant* complex of $\bar{\mathbb{Q}}_\ell$-vector spaces. On the other hand, there exists an essentially smooth irreducible open $U \overset{j}{\hookrightarrow} G$ such that $j^* K = \mathcal{L}[d]$, for $\mathcal{L}$ a local system on U. We conclude that $K = \bar{\mathbb{Q}}_\ell^r[d]$. □

Remark 4.1.4. Add remark about what happens when H is not connected, but H/H° is étale.

4.1.2. *$L^+\mathcal{G}$-actions on Schubert varieties.* For $\mu \in X_*(T)_I^+$, we have the Schubert variety $\mathrm{Gr}_{G,\leq\mu}$. Recall that $\mathrm{Gr}_\mathcal{G}$ carries a left $L^+\mathcal{G}$-action, given by $g \cdot (\mathcal{E}, \alpha) = (\mathcal{E}, g \circ \alpha)$. Let $e_0 = (\mathcal{E}_0, \mathrm{id})$ be the base point in $\mathrm{Gr}_\mathcal{G}$.

Definition 4.1.5. We define $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ to be the scheme-theoretic image of the morphism

$$L^+\mathcal{G} \to \mathrm{Gr}_\mathcal{G}, \qquad g \mapsto g t^\lambda e_0.$$

We define $\mathrm{Gr}_{\mathcal{G},\mu}$ to be the orbit of the $L^+\mathcal{G}$-action through the point $t^\mu e_0$.

Notation: When $\mathcal{G}$ is understood, we sometimes abbreviate by writing $\mathrm{Gr}_{\mathcal{G},\mu} = \mathrm{Gr}_\mu = \mathcal{O}_\mu$, and $\mathbf{Gr}_{\mathcal{G},\leq\mu} = \mathrm{Gr}_{\leq\mu} = \overline{\mathcal{O}}_\mu$.

Lemma 4.1.6. *The following facts hold:*

(a) *We have $\mathrm{Gr}_{\mathcal{G},\leq\mu} = \coprod_{\lambda\preceq\mu} \mathrm{Gr}_{\mathcal{G},\lambda}$ a stratification by reduced finite-type locally closed subschemes, where $\lambda \in X_*(T)_I^+$ and $\lambda \preceq \mu$ means that $\mu - \lambda$ is a sum of positive coroots in $\Sigma^\vee$.*

(a) *The scheme $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ is an irreducible projective reduced subscheme of $\mathrm{Gr}_\mathcal{G}$ of dimension $\langle 2\rho, \mu\rangle$, for 2ρ the sum of the B-positive roots in Φ.*

(b) *The Cartan decomposition gives an exhaustive family of underlying reduced spaces*

$$\mathrm{Gr}_{\mathcal{G},\mathrm{red}} = \operatorname*{colim}_\mu \mathrm{Gr}_{\mathcal{G},\leq\mu}.$$

[REFS: LATER.]

For the next part of the discussion, we need some notation. **Temporarily for this section when we talk about the big open cell below we assume G is split, for simplicity. In the final version, this assumption will be removed.**

Notation:

$$L^{\geq 0}\mathcal{G} := L^+\mathcal{G}$$
$$L^{>0}\mathcal{G} := \ker[L^{\geq 0}\mathcal{G} \to \mathcal{G}]\ , \quad t \mapsto 0$$
$$L^{\geq\lambda}\mathcal{G} := t^\lambda\ L^+\mathcal{G}\ t^{-\lambda}.$$

Proposition 4.1.7. *The action of $L^+\mathcal{G}$ on $\mathrm{Gr}_{\mathcal{G},\lambda}$ is transitive, and the stabilizer of $t^\lambda e_0$ is a connected smooth group isomorphic to $P_\lambda \ltimes \left(L^{>0}\mathcal{G} \cap L^{\geq\lambda}\mathcal{G}\right)$, where P_λ is the parabolic subgroup corresponding to λ.*

Proof. Proof will appear in the final version of the notes. Now we just remark that the stabilizer group was only recently proved to be smooth and connected, in work of Richarz-Scholbach on motivic geometric Satake (we have to adapt their proof from the split case). □

Corollary 4.1.8. *The only simple $L^+\mathcal{G}$-equivariant lisse $\bar{\mathbb{Q}}_\ell$-sheaves on $\mathrm{Gr}_{\mathcal{G},\lambda}$ are of the form $\bar{\mathbb{Q}}_\ell$. Therefore the only simple $L^+\mathcal{G}$-equivariant perverse sheaves on $\mathrm{Gr}_\mathcal{G}$ are of the form*

$$\mathrm{IC}_\mu := i^\mu_* j^\mu_{!*} \bar{\mathbb{Q}}_\ell[\langle 2\rho, \mu\rangle]$$

where $\mu \in X_*(T)_I^+$, $j^\mu : \mathrm{Gr}_{\mathcal{G},\mu} \hookrightarrow \mathrm{Gr}_{\mathcal{G},\leq\mu}$ is the open embedding, and $i^\mu : \mathrm{Gr}_{\mathcal{G},\leq\mu} \hookrightarrow \mathrm{Gr}_\mathcal{G}$ is the closed embedding.

4.2. **Semisimplicity of the category.** Our goal is to prove:

Theorem 4.2.1. *The category $P_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}})$ is semisimple, with simple objects the intersection complexes IC_μ.*

We continue to abbreviate: $D(X) := D^b_c(X, \bar{\mathbb{Q}}_\ell)$, and $P(X) := P(X, \bar{\mathbb{Q}}_\ell)$.

It is enough to show the following: given $\lambda, \mu \in X_*(T)^+_I$, we have

$$\mathrm{Hom}_{D(\mathrm{Gr}_{\mathcal{G}})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) = 0. \tag{4.2.1}$$

Indeed, if we have an exact sequence in $P(\mathrm{Gr}_{\mathcal{G}})$

$$0 \to \mathrm{IC}_\mu \to \mathcal{F} \to \mathrm{IC}_\lambda \to 0 \tag{4.2.2}$$

this gives a distinguished triangle in $D(\mathrm{Gr}_{\mathcal{G}})$ and hence an exact sequence

$$\mathrm{Hom}_{D(\mathrm{Gr}_{\mathcal{G}})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \to \mathrm{Hom}_{D(\mathrm{Gr}_{\mathcal{G}})}(\mathrm{IC}_\lambda, \mathcal{F}) \to \mathrm{Hom}_{D(\mathrm{Gr}_{\mathcal{G}})}(\mathrm{IC}_\lambda, \mathrm{IC}_\lambda) \to \mathrm{Hom}_{D(\mathrm{Gr}_{\mathcal{G}})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) \to \cdots$$

and the vanishing in (4.2.1) would prove that the element $\mathrm{id} \in \mathrm{Hom}_{D(\mathrm{Gr}_{\mathcal{G}})}(\mathrm{IC}_\lambda, \mathrm{IC}_\lambda)$ lifts and therefore the sequence (4.2.2) splits. Then one proves in a similar way that every object is semisimple, by induction on the length.

Before starting the proof, let us state a technical lemma.

Lemma 4.2.2. *If $i : \overline{\mathcal{O}}_\lambda \hookrightarrow \overline{\mathcal{O}}_\mu$ is the closed embedding of Schubert varieties in $\mathrm{Gr}_{\mathcal{G}}$ corresponding to $\lambda, \mu \in X_*(T)^+_I$ with $\lambda \prec \mu$, then $i^*\mathrm{IC}_\mu$ lies in perverse degrees $p \leq -2$.*

We shall assume this for now, and use it to prove the vanishing (4.2.1). We shall give the proof of Lemma 4.2.2 in section 4.3.

4.2.1. *Proof of (4.2.1) assuming Lemma 4.2.2.*
Case 1: $\lambda = \mu$. Consider the open embedding $j : \mathcal{O}_\mu \hookrightarrow \overline{\mathcal{O}}_\mu$ and its complementary closed embedding $i : \overline{\mathcal{O}}_\mu \backslash \mathcal{O}_\mu \hookrightarrow \overline{\mathcal{O}}_\mu$. The distinguished triangle $i_! i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+}$ yields the exact sequence

$$\mathrm{Hom}(\mathrm{IC}_\mu, i_! i^! \mathrm{IC}_\mu[1]) \to \mathrm{Hom}(\mathrm{IC}_\mu, \mathrm{IC}_\mu[1]) \to \mathrm{Hom}(\mathrm{IC}_\mu, j_* j^* \mathrm{IC}_\mu[1]).$$

We need to prove the outer terms vanish.
Claim A: The right term vanishes. Recall $j^*\mathrm{IC}_\mu = \bar{\mathbb{Q}}_\ell[d_\mu]$ where $d_\mu = \langle 2\rho, \mu \rangle$. Then using adjunctions the right term identifies with

$$\mathrm{Hom}_{D(\mathcal{O}_\mu)}(j^*\mathrm{IC}_\mu, j^*\mathrm{IC}_\mu[1]) = \mathrm{Hom}_{D(\mathcal{O}_\mu)}(\bar{\mathbb{Q}}_\ell, \bar{\mathbb{Q}}_\ell[1]) = \mathrm{Ext}^1_{\mathrm{Sh}(\mathcal{O}_\mu)}(\bar{\mathbb{Q}}_\ell, \bar{\mathbb{Q}}_\ell) = H^1_{et}(\mathcal{O}_\mu, \bar{\mathbb{Q}}_\ell).$$

Here we used the fact that the usual category of $\bar{\mathbb{Q}}_\ell$-sheaves $\mathrm{Sh}(\mathcal{O}_\mu)$ is the heart of the usual t-structure on $D(\mathcal{O}_\mu)$. Further, for the final equality, we used that both sides result from forming the right derived functors of the same functor from the category of $\bar{\mathbb{Q}}_\ell$-sheaves to the category of abelian groups:

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{O}_\mu)}(\bar{\mathbb{Q}}_\ell, -) = \Gamma(\mathcal{O}_\mu, -) \ : \ \mathrm{Sh}(\mathcal{O}_\mu) \to (\mathrm{Ab}).$$

So it suffices to show that $H^1(\mathcal{O}_\mu, \bar{\mathbb{Q}}_\ell) = 0$. But from our earlier discuss of stabilizers (**Note: currently limited to the split case**) we have a locally trivial fibration

$$\mathcal{O}_\mu \to L^{\geq 0}\mathcal{G}/(L^{>0}\mathcal{G} \rtimes P_{\mu,k}) = G_k/P_{\mu,k}$$

with target the classical Grassmanian variety over k and with typical fiber isomorphic to

$$L^{>0}\mathcal{G}/(L^{>0}\mathcal{G} \cap L^{\geq \mu}\mathcal{G}).$$

As the fiber is paved by affine spaces, it has vanishing higher cohomology groups, and we have

$$H^1(\mathcal{O}_\mu, \bar{\mathbb{Q}}_\ell) = H^1(G_k/P_{\mu,k}, \bar{\mathbb{Q}}_\ell).$$

HERE we use a combination of the Leray spectral sequence, and the local triviality to have access to Künneth theorem as in [StaPro, Lem. 59.97.9]**. Details in the final version of the notes.** On the other hand the group $H^1(G_k/P_{\mu,k}, \bar{\mathbb{Q}}_\ell)$ vanishes, like all odd cohomology groups of any Schubert variety such as $G_k/P_{\mu,k}$, as we shall see later (see Remark 4.3.2).

Claim B: The left term vanishes. It is a standard fact that $i^*\mathrm{IC}_\mu$ lives in perverse degree $p \leq -1$ (see [KW, III.5.1]). Applying Verdier duality, we deduce that $i^!\mathrm{IC}_\mu$ lives in perverse degress $p \geq 1$. Then we have

$$\mathrm{Hom}(\mathrm{IC}_\mu, i_! i^! \mathrm{IC}_\mu[1]) = \mathrm{Hom}(i^*\mathrm{IC}_\mu, i^!\mathrm{IC}_\mu[1]) = 0.$$

This finishes Case 1.

Case 2: $\lambda \neq \mu$ and $\lambda \prec \mu$ or $\mu \prec \lambda$.

If $\lambda \prec \mu$, consider $i : \overline{\mathcal{O}}_\lambda \hookrightarrow \overline{\mathcal{O}}_\mu$. We have

$$\mathrm{Hom}(i_*\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) = \mathrm{Hom}(\mathrm{IC}_\lambda, i^!\mathrm{IC}_\mu[1]) = 0,$$

where for the last equality we use that IC_λ lives in perverse degrees $p \leq 0$, and (thanks to Lemma 4.2.2 and Verdier duality) $i^!\mathrm{IC}_\mu[1]$ lives in perverse degrees $p \geq 1$.

If $\mu \prec \lambda$, consider $i : \overline{\mathcal{O}}_\mu \hookrightarrow \overline{\mathcal{O}}_\lambda$. Then

$$\mathrm{Hom}(\mathrm{IC}_\lambda, i_*\mathrm{IC}_\mu[1]) = \mathrm{Hom}(i^*\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) = 0$$

using that $i^*\mathrm{IC}_\lambda$ lives in $p \leq -2$ (Lemma 4.2.2) and that $\mathrm{IC}_\mu[1]$ lives in $p \geq -1$. This finishes Case 2.

Case 3: $\lambda \not\preceq \mu$ and $\mu \not\preceq \lambda$.

Without loss of generality we may assume λ and μ are such that $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ and $\mathrm{Gr}_{\mathcal{G},\leq\lambda}$ lie in the same connected component in $\mathrm{Gr}_{\mathcal{G}}$. This means that $\mu - \lambda \in \mathbb{Z}[\Sigma^\vee]$: PROBABLY cite [PR08, Thm. 0.1]. This means we can choose $\nu \in X_*(T)_I^+$ with $\lambda \prec \nu$ and $\mu \prec \nu$. Consider the Cartesian diagram of closed immersions

$$\begin{tikzcd}
\overline{\mathcal{O}}_\lambda \times_{\overline{\mathcal{O}}_\nu} \overline{\mathcal{O}}_\mu \arrow[r, "\iota_1"] \arrow[d, "\iota_2"'] & \overline{\mathcal{O}}_\mu \arrow[d, "i_2"] \\
\overline{\mathcal{O}}_\lambda \arrow[r, "i_1"] & \overline{\mathcal{O}}_\nu .
\end{tikzcd}$$

We have

$$\begin{aligned}
\mathrm{Hom}(i_{1,*}\mathrm{IC}_\lambda, i_{2,*}\mathrm{IC}_\mu[1]) &= \mathrm{Hom}(i_2^* i_{1,*}\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) \\
&= \mathrm{Hom}(\iota_{1,*}\iota_2^*\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) \\
&= \mathrm{Hom}(\iota_2^*\mathrm{IC}_\lambda, \iota_1^!\mathrm{IC}_\mu[1]) = 0.
\end{aligned}$$

The argument for the vanishing is exactly the same as for Claim B above. □

4.3. **Proof of Lemma 4.2.2.** Recall that ${}^p\mathrm{H}^j(i^*\mathrm{IC}_\mu) = 0$ for $j > -1$. Therefore we just need to show that ${}^p\mathrm{H}^{-1}(i^*\mathrm{IC}_\mu) = 0$. We shall show more generally that

$$i^*\mathrm{IC}_\mu \in {}^pD^{\leq j} \;\Rightarrow\; i^*\mathrm{IC}_\mu \in {}^pD^{\leq j-1},$$

for all **odd** j. This comes down to the facts that

(I) the stalks of $\mathcal{H}^i(\mathrm{IC}(\mathrm{Gr}_{\leq\mu}))$ vanish unless $i \equiv d_\mu \pmod 2$, where $d_\mu := \dim \mathrm{Gr}_\mu$; and

(II) $\dim \mathrm{Gr}_{\leq\lambda}$ and $\dim \mathrm{Gr}_{\leq\mu}$ have the same parity whenever $\lambda \preceq \mu$ in the usual partial order on dominant cocharacters $X_*(T)_I^+$.

First we prove these facts. In what follows let $\mathrm{IC}_\mu = \mathrm{IC}(\mathrm{Gr}_{\leq\mu})$, and similarly write $\mathrm{IC}_w = \mathrm{IC}(\mathrm{Fl}_{\leq w})$, the analogue for the affine flag variety (**in final version of notes, we will recall that** $\mathrm{Fl} = \mathrm{Gr}_{\mathcal{G}_{\mathbf{a}}}$**, hence has already been shown to be representable, etc**). Statement (II) is obvious from the formula $\dim(\mathrm{Gr}_{\leq\mu}) = \langle 2\rho, \mu\rangle$ and the fact that $\langle 2\rho, \alpha^\vee\rangle$ is even for any $\alpha^\vee \in \Sigma^\vee$ (**we will add more details in the non-split case**).

Let us explain why (I) holds. We have a smooth surjective morphism $p : \mathrm{Fl} \to \mathrm{Gr}$ with connected geometric fibers isomorphic to the finite flag variety G/B over k. Moreover, if $w_\mu := t_\mu w_0$, then $p^{-1}(\mathrm{Gr}_{\leq\mu}) = \mathrm{Fl}_{\leq w_\mu}$. Let $d = \dim G/B$ be the relative dimension of p. By Lemma 3.3.10, $p^*[d]$ gives a fully faithful embedding taking $L^+\mathcal{G}$-equivariant perverse sheaves on $\mathrm{Gr}_{\leq\mu}$ to those on $\mathrm{Fl}_{\leq w_\mu}$.

Note also that [BBD82, 4.2.5] proves that $p^*[d]\mathrm{IC}_\mu = \mathrm{IC}_{w_\mu}$. To understand the parity vanishing at the stalk at $y \in \mathrm{Gr}_{\leq\mu}$, choose $x \in \mathrm{Fl}_{\leq w_\mu}$ with $p(x) = y$. Then we have

$$\mathcal{H}^{i-d}_x \mathrm{IC}_{w_\mu} = \mathcal{H}^i_y \mathrm{IC}_\mu.$$

Let $\pi : D(\dot{w}_\mu) \to \mathrm{Fl}_{\leq w_\mu}$ be the Demazure resolution corresponding to a choice $\dot{w}_\mu$ of reduced expression for w_μ in the quasi-Coxeter group W. The Decomposition Theorem [BBD82] implies that IC_{w_μ} is a direct summand of $\pi_*(\mathrm{IC}(D(\dot{w}_\mu)) = \pi_*(\bar{\mathbb{Q}}_\ell[d + d_\mu])$. So from the paving by affine spaces of the fibers of (see [Hai05], [dCHL18], or [Hai23] – **need additional refs in non-split case**), the above stalk cohomology vanishes unless $i - d - (d + d_\mu)$ is even. We conclude that

$$\mathcal{H}^i_y \mathrm{IC}_\mu = 0 \ \text{ unless } i - d_\mu \text{ is even.}$$

This proves (I).

Using (I) and (II), the desired vanishing follows immediately from the lemma below, taking $\mathcal{K} = i^*\mathrm{IC}_\mu$.

Lemma 4.3.1. *Let $\mathcal{K} \in D^b_c(\mathrm{Gr}_{\leq\mu}, \overline{\mathbb{Q}}_\ell)$ be such that*

(a) *$\mathcal{H}^i\mathcal{K}$ vanishes unless $i - d_\mu$ is even; and*
(b) *for any i, $\dim \operatorname{supp} \mathcal{H}^i\mathcal{K} \equiv d_\mu \pmod 2$.*

*Then for any j **odd** we have*

$$\mathcal{K} \in {}^pD^{\leq j} \Rightarrow \mathcal{K} \in {}^pD^{\leq j-1},$$

or, equivalently (by [KS90, Prop. 10.1.10]) we have ${}^p\mathrm{H}^j(\mathcal{K}) = 0$.

Proof. Since $\mathcal{K} \in {}^pD^{\leq j}$, we have $\mathcal{K}[j] \in {}^pD^{\leq 0}$, and so for all i

$$\dim \operatorname{supp} \mathcal{H}^i\mathcal{K}[j] \leq -i.$$

This means

$$\dim \operatorname{supp} \mathcal{H}^i\mathcal{K} \leq -i + j.$$

By (a), without loss of generality $i = 2m + d_\mu$. So

$$\dim \operatorname{supp} \mathcal{H}^i\mathcal{K} \leq -2m - d_\mu + j.$$

By (b), the left hand side is congruent to d_μ modulo 2, and the right hand side is not. Therefore we must have

$$\dim \operatorname{supp} \mathcal{H}^i\mathcal{K} \leq -2m - d_\mu + j - 1.$$

Now reversing the steps above, this means that $\mathcal{K} \in {}^pD^{\leq j-1}$, and the lemma is proved. □

Remark 4.3.2. A more general version of the paving of fibers by affine spaces used above can be used to prove that every affine Schubert variety has parity vanishing of its intersection cohomology complexes and its intersection cohomology. See [dCHL18] – **in the split case**.

5. Lecture 5

5.1. **Convolution diagram. For simplicity of notation only, for now assume G split, and write $\mathcal{G} = G$, etc.**

Definition 5.1.1. The *convolution diagram* is the digram of morphisms of ind-schemes over k

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times LG \xrightarrow{q} \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G.$$

Here p is the product of the two obvious projection morphisms, that is, the quotient by the obvious right action α_1. We define the twisted product as the étale sheaf quotient

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G := LG \times^{L^+G} LG/L^+G,$$

for the right action α_2 of $L^+G \times L^+G$ on $LG \times LG$ given by $(g_1, g_2) \cdot (h_1, h_2) = (g_1h_1, h_1^{-1}g_2h_2)$. Then q is the quotient morphisms, and m is the "multiplication" morphism.

There is a also a "finite-dimensional version", meaning we mod out everywhere by the principal congruence subgroups $L^{\geq n}G \subset L^+G$, and thereby achieve finite dimensional varieties. To do this correctly, one fixes any dominant λ and μ, any integers $n \gg m \gg 0$ and write $L^+G/L^{\geq n}\mathcal{G} =: G_n$, a finite-type smooth connected k-group scheme **(at least in the split case – we have to define the jet scheme differently in general)**.

The finite-type convolution diagram takes the following form:

$$\mathrm{Gr}_{\leq\mu} \times \mathrm{Gr}_{\leq\lambda} \xleftarrow{p_{n,m}} \overline{L^+\mathcal{G}\, t^\mu\, L^+\mathcal{G}}/L^{\geq n}\mathcal{G} \times \overline{L^+\mathcal{G}\, t^\lambda\, L^+\mathcal{G}}/L^{\geq m}\mathcal{G} \xrightarrow{q_{n,m}} \mathrm{Gr}_{\leq\mu}\widetilde{\times}\mathrm{Gr}_{\leq\lambda} \xrightarrow{m} \mathrm{Gr}_{\leq\mu+\lambda}.$$

The reason we need to consider two integers $n > m > 0$ is that having fixed a large m, we need a possibly larger n such that the usual left multiplication action of the subgroup $L^{\geq n}\mathcal{G}$ is trivial on $\overline{L^+\mathcal{G}\, t^\lambda\, L^+\mathcal{G}}/L^{\geq m}\mathcal{G}$.

Here

$$p_{n,m} = \text{quotient by the usual right action of } G_n \times G_m$$
$$q_{n,m} = \text{quotient by the twisted right action } (x,y)\cdot(h_n,h_m) = (xh_n, h_n^{-1}yh_m).$$

Theorem 5.1.2. *The following properties hold:*

(a) *The morphisms p, q are smooth of the same relative dimension (meaning, $p_{n,m}$ and $q_{n,m}$ are).*
(b) *The morphism m is locally trivial in the stratified sense and is semismall.*

Proof. (Sketch.)

Part (a): The smoothness is clear for p. For q, it follows from the theory of the big cell ($L^{<0}G \times L^+G \to LG$ is an open immersion) that the target of q is étale-locally (even Zariski-locally, in the split case) isomorphic to the usual product and q is locally in the same sense isomorphic to p; hence q is also smooth, and has the same relative dimension as q.

Part (b): For local triviality, see [Hai06, Lem. 2.1]. Here is the precise statement for the finite-dimension version of m: The scheme $\mathrm{Gr}_{\leq\mu}\widetilde{\times}\mathrm{Gr}_{\leq\lambda}$ is stratified by locally closed twisted products

$$\mathrm{Gr}_{\mu'}\widetilde{\times}\mathrm{Gr}_{\lambda'} = \{\mathcal{E}_0 \overset{\mu'}{-} \mathcal{E}_1 \overset{\lambda'}{-} \mathcal{E}_2\}$$

where $\mu' \preceq \mu$ and $\lambda' \preceq \lambda$, and where $\mathcal{E}_1 \overset{\lambda'}{-} \mathcal{E}_2$ denotes the relative position of the two L^+G-torsors (defined using the Cartan decomposition). In this description, $m(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{E}_2$. We also have

$$\mathrm{Gr}_{\leq\mu+\lambda} = \coprod_{\nu\preceq\mu+\lambda} \mathrm{Gr}_\nu.$$

By L^+G-equivariance, $m(\mathrm{Gr}_{\mu'}\widetilde{\times}\mathrm{Gr}_{\lambda'}) = \coprod_{\text{certain } \nu} \mathrm{Gr}_\nu$.

The local triviality of m means the following: if $y \in \mathrm{Gr}_\nu$, then there exists an open subset V with $y \in Y \subset \mathrm{Gr}_\nu$ such that

$$\big(m|_{\mathrm{Gr}_{\mu'}\widetilde{\times}\mathrm{Gr}_{\lambda'}}\big)^{-1}(V) \cong \big(m^{-1}(y) \cap (\mathrm{Gr}_{\mu'}\widetilde{\times}\mathrm{Gr}_{\lambda'}\big) \times V.$$

As for semi-smallness of m this is proved in [MV07, ??] and [NP01, ??]. We shall also sketch a purely combinatorial proof **in the final version of the notes we will handle the non-split case**. $\square$

5.2. **Descent to the twisted sheaf and definition of convolution.** Now supppose $\mathcal{F}_1, \mathcal{F}_2 \in P_{L^+G}(\mathrm{Gr}_G)$. Then $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_1)$ is perverse (up to shift) and α_1-equivariant. This sheaf is automatically α_2-equivariant (**in final version, this will be in the Appendices**). The action α_2 satisfies the axioms of Lemma 3.3.11, and as q has the same relative dimension as p, there is a unique perverse sheaf $\mathcal{F}_1\widetilde{\boxtimes}\mathcal{F}_2$ on $\mathrm{Gr}_G\widetilde{\times}\mathrm{Gr}_G$ such that

$$p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_1) \cong q^*(\mathcal{F}_1\widetilde{\boxtimes}\mathcal{F}_2).$$

Definition 5.2.1. We set $\mathcal{F}_1 \star \mathcal{F}_2 = Rm_!(\mathcal{F}_1\widetilde{\boxtimes}\mathcal{F}_2)$.

We note that $\mathcal{F}_1 \star \mathcal{F}_2$ is perverse due to the semismallness of m (use Proposition 3.5.1). It is also L^+G-equivariant.

The final version of these notes will have full proofs of the following facts in the non-split cases:

- **semismallness of** m
- α_2**-equivariance above (see the Appendices)**

5.3. **Some explanations.**

5.3.1. *Convolution morphisms are semi-small: statement.* We can consider r-fold convolution morphisms, for any $r \geq 2$. Let $\mu_\bullet = (\mu_1, \dots, \mu_k)$ be dominant cocharacters and recall we can describe the convolution diagram total space as the projective variety

$$\overline{\mathcal{O}}_{\mu_\bullet} = p^{-1}(\overline{\mathcal{O}}_{\mu_1}) \times^{L^+G} \cdots \times^{L^+G} p^{-1}(\overline{\mathcal{O}}_{\mu_{k-1}}) \times^{L^+G} \overline{\mathcal{O}}_{\mu_k},$$

inside $LG \times^{L^+G} \cdots \times^{L^+G} \mathrm{Gr}_G$, where $p : LG \to \mathrm{Gr}_G$ is the projection map.

Let $|\mu_\bullet| = \mu_1 + \cdots + \mu_k$. Then the k-fold convolution factors as

$$m_{\mu_\bullet} : \overline{\mathcal{O}}_{\mu_\bullet} \to \overline{\mathcal{O}}_{|\mu_\bullet|}.$$

Lemma 5.3.1. *For $\lambda \in X_*(T)_I^+$ with $\lambda \preceq |\mu_\bullet|$, and $x \in \mathcal{O}_\lambda$, we have*

$$\dim(m_{\mu_\bullet}^{-1}(x)) \leq \langle \rho, |\mu_\bullet| - \lambda \rangle$$

i.e., the convolution morphism is semi-small.

Proof. This is essentially a consequence of the dimension estimate in Proposition 5.3.2, see the proof in [NP01, Lemme 9.3], as we explain in more detail below. □

5.3.2. *A dimension bound for LU-orbits.* Again abbreviate $\mathrm{Gr}_{G,\leq\mu} = \overline{\mathcal{O}}_\mu$. We also define for any $\nu \in X_*(T)_I$ the locally closed sub-ind-scheme

$$S_\nu = LUt^\mu e_0.$$

in the final version of the notes we will prove this is actually a locally closed sub-ind-scheme. This will be done using general facts about contracting sets with respect to a $\mathbb{G}_m$**-orbit, see** [HR21]**...**

We have

$$\overline{\mathcal{O}}_\mu = \coprod_{\lambda \preceq \mu} \mathcal{O}_\lambda \qquad \text{(Cartan stratification)}$$

$$\overline{\mathcal{O}}_\mu = \coprod_{\nu \in X_*(T)_I} S_\nu \cap \overline{\mathcal{O}}_\mu \qquad \text{(Iwasawa stratification)}$$

Proposition 5.3.2. (split version) *For $\nu \in X_*(T)$ and $\mu \in X_*(T)^+$, the intersection $S_\nu \cap \overline{\mathcal{O}}_\mu$ is non-empty if and only if $\nu \in \Omega(\mu)$ ($:=$ weight space for V_μ), and in this case is pure of dimension $\langle \rho, \mu + \nu \rangle$, with number of irreducible components $m_\mu(\nu)$, the dimension of the ν-weight space in V_μ.*

Proof. **This statement and proof is currently just for the split case.** The schemes are all defined over $\mathbb{Z}$ and all associated data are defined over a finitely generated $\mathbb{Z}$-algebra. By generic flatness, we reduce to the case where $k = \bar{\mathbb{F}}_p$. The dimension estimate

$$\dim(S_\nu \cap \mathcal{O}_\mu) \leq \langle \rho, \mu + \nu \rangle.$$

(which is all that is needed for semi-smallness of convolution morphisms – see below) is proved by a combinatorial argument using Macdonald's formula (alternatively the Lusztig-Kato formula), see below. After that, **at least in the split case**, one proof of the equidimensionality is given in [GHKR06, 2.17.4], making use of the purity of the cohomology $R\Gamma_c(S_\nu, \mathrm{IC}_\mu)$ proved in [NP01, Theorem 3.1]. □

In this course, we do not really need the equidimensionality and the purity. We need only the dimension bound

$$\dim(S_\nu \cap \mathcal{O}_\mu) \leq \langle \rho, \mu + \nu \rangle. \tag{5.3.1}$$

5.3.3. *Why the dimension bound (5.3.1) implies semi-smallness of m.* For proofs of this in the literature, see [MV07, proof of Lem. 4.4], and [NP01, Proof of Lem. 9.3]. We shall explain something along the lines of [NP01] but we will not directly get the purity result they prove, only the semi-smallness bound.

Let $(\mu_1, \dots, \mu_r)$ be an r-tuple in $(X_*(T)^+_I)^r$, let $\lambda \in X_*(T)^+_I$, and suppose $\lambda \preceq \mu := |\mu_\bullet| := \sum_i \mu_i$. We consider the convolution morphism

$$m_{\mu_\bullet} : \overline{\mathcal{O}}_{\mu_1} \widetilde{\times} \overline{\mathcal{O}}_{\mu_2} \widetilde{\times} \cdots \widetilde{\times} \overline{\mathcal{O}}_{\mu_r} \longrightarrow \overline{\mathcal{O}}_{\mu}.$$

By the local triviality of m GIVE CROSS-REF], the semi-smallness of $m_{\mu_\bullet}$ is equivalent to: for any λ as above,

$$\dim m_{\mu_\bullet}^{-1}(\mathcal{O}_\lambda) \leq \langle \rho, \mu + \lambda \rangle.$$

We claim that $S_\lambda \cap \mathcal{O}_\lambda$ is open dense in $\mathcal{O}_\lambda$.

The proof which follows ultimately relies on negative loop groups so works for the split case – final version will have a different proof. For example, we could also argue that $\mathcal{O}_\lambda$ is stratified by finitely many locally closed subsets of the form $S_\nu \cap \mathcal{O}_\lambda$, and by the dimension calculation below, $S_\lambda \cap \mathcal{O}_\lambda$ is the only one of full dimension in $\mathcal{O}_\lambda$, hence it must be dense and open, and the union of all the others is closed.

Indeed, since λ is *dominant*, $J^{<\lambda} := (t^\lambda L^{<0}G\, t^{-\lambda}) \cap L^{\geq 0}G$ maps isomorphically via $j \mapsto jt^\lambda *$ onto the open

$$t^\lambda L^{<0}G * \cap \overline{\mathcal{O}}_\lambda \subset \overline{\mathcal{O}}_\lambda.$$

Also, $J^\lambda \cap LU \xrightarrow{\sim} S_\lambda \cap \overline{\mathcal{O}}_\lambda$ via $j \mapsto jt^\lambda *$. (Use $S_\lambda \cap \overline{\mathcal{O}}_\lambda \subseteq t^\lambda L^{<0}G * \cap \overline{\mathcal{O}}_\lambda$, as $LU = (t^\lambda L^{<0}G\, t^{-\lambda} \cap LU) \cdot (t^\lambda L^{\geq 0}G\, t^{-\lambda} \cap LU)$.)

Now observe that claim follows, as

$$J^{<\lambda} \cap L^{\geq 0}U = (t^\lambda L^{<0}Gt^{-\lambda} \cap (U\overline{B} \rtimes L^{>0}G)$$

is open dense in

$$(t^\lambda L^{<0}Gt^{-\lambda}) \cap L^{\geq 0}G = J^{<\lambda}.$$

The claim is proved, so we just need to show that

$$\dim m_{\mu_\bullet}^{-1}(S_\lambda \cap \mathcal{O}_\lambda) \leq \langle \rho, \mu + \lambda \rangle. \tag{5.3.2}$$

We let $\nu_\bullet = (\nu_1, \dots, \nu_r) \in X_*(T)^r$ be such that $\nu_i \in \Omega(\mu_i)$ for all i. Then the key observation is that for any $\lambda \leq \mu = |\mu_\bullet|$, we have

$$m_{\mu_\bullet}^{-1}(S_\lambda \cap \overline{\mathcal{O}}_\mu) \cong \bigcup_{|\nu_\bullet| = \lambda} (S_{\nu_1} \cap \overline{\mathcal{O}}_{\mu_1}) \widetilde{\times} \cdots \widetilde{\times} (S_{\nu_r} \cap \overline{\mathcal{O}}_{\mu_r}).$$

Thus the dimension of the LHS is $\max_{\nu_\bullet} \sum_i \dim(S_{\nu_i} \cap \overline{\mathcal{O}}_{\mu_i})$, which by repeated application of (5.3.1) gives (5.3.2). **Definition of this twisted product as well as computation of its dimension relies on étale-local sections of $LG \to LG/L^+\mathcal{G}$ – more details in final version.**

5.3.4. *Macdonald's formula implies the Mirkovic-Vilonen dimension formula.* **The argument below is for the split case. The non-split version of Macdonald's formula exists and in the final version of the notes we will give the proof in general.**

Let $m_\mu(\nu)$ be the multiplicity of ν in the highest weight representation $V_\mu \in \mathrm{Rep}(\widehat{G})$. The variety $S_\nu \cap \overline{\mathcal{O}}_\mu$ is is defined over the finite field $\mathbb{F}_q$ for all $q = p^n$. Write $U = U(\mathbb{F}_q((t)))$ and $K = G(\mathbb{F}_q[[t]])$, and $G = G(\mathbb{F}_q((t)))$. Write $B = TU$ similarly, and $T_\mathcal{O}$ for $T(\mathbb{F}_q[[t]])$. We think of q as varying. For $\lambda \in X_*(T)$ write $e_\lambda = t^\lambda e_0$. Write $f_\mu := 1_{Kt^\mu K}$, and let $f_\mu^\vee$ denote its Satake transform.

Proposition 5.3.3. (split version) *The dimension of $S_\lambda \cap \overline{\mathcal{O}}_\mu$ is at most $\langle \rho, \mu + \lambda \rangle$.*

Proof. As stated above, by defining objects over $\mathbb{Z}$ and using a generic flatness argument, it suffices to prove the theorem for $k = \bar{\mathbb{F}}_q$. Then we may define all the objects over a fixed finite field and look at how the number of points grows over larger finite fields. Over the field $\bar{\mathbb{F}}_q$ we will show the dimension is exactly $\langle \rho, \mu + \lambda \rangle$ and the number of top dimensional irreducible components is $m_\mu(\lambda)$.

By the Weil conjectures, it is enough to show that

$$\lim_{q\to\infty} \frac{\#(Ut^\lambda K/K \cap Kt^\mu K/K)}{q^{\langle\rho,\mu+\lambda\rangle}} = m_\mu(\lambda).$$

The following lemma was taken from [HKM, Lem. 10.2].

Lemma 5.3.4. *We have*

$$\#\big(Ut^\lambda K/K \cap Kt^\mu K/K\big) = q^{\langle\rho,\lambda\rangle} f_\mu^\vee(t^\lambda).$$

Proof. The Iwasawa decomposition $G = KUT$ gives rise to an integration formula, relating integration over G to an iterated integral over the subgroups K, U, and T, where if Γ is any of these unimodular groups, we equip Γ with the Haar measure which gives $\Gamma \cap K$ volume 1. For a subset $S \subset G$, write 1_S for the characteristic function of S. Using the substitution $y = kut$ in forming the iterated integral, the left hand side above can be written as

$$\begin{aligned}
\int_G 1_{UK}(t^{-\lambda}y)1_{Kt^\mu K}(y)dy &= \int_G 1_{UK}(t^{-\lambda}y^{-1})1_{Kt^\mu K}(y^{-1})\,dy\\
&= \int_T\int_U\int_K 1_{UK}(t^{-\lambda}t^{-1}u^{-1}k^{-1})1_{Kt^\mu K}(t^{-1}u^{-1}k^{-1})\,dk\,du\,dt\\
&= \int_T\int_U 1_{UK}(t^{-1}u^{-1})1_{Kt^\mu K}(t^\lambda t^{-1}u^{-1})\,du\,dt\\
&= \int_T\int_U 1_{UK}(t)1_{Kt^\mu K}(t^\lambda tu)\,du\,dt\\
&= \int_U 1_{Kt^\mu K}(t^\lambda u)\,du\\
&= \delta_B^{-1/2}(t^\lambda)f_\mu^\vee(t^\lambda),
\end{aligned}$$

which implies the lemma since $\delta_B^{1/2}(t^\lambda) = q^{-\langle\rho,\lambda\rangle}$. □

By Macdonald's formula (see [HKP, Thm. 5.6.1] – **in the split case**), the quantity in Lemma 5.3.4 is the coefficient of t^λ in

$$\frac{q^{\langle\rho,\mu+\lambda\rangle}}{W_\mu(q^{-1})}\sum_{w\in W_0} w\Big(\prod_{\alpha>0}\frac{1-q^{-1}t^{-\alpha^\vee}}{1-t^{-\alpha^\vee}}\Big)\cdot t^{w\mu}.$$

Divide this by $q^{\langle\rho,\mu+\lambda\rangle}$ and take the limit as $q\to\infty$. The Weyl character formula implies we get $m_\mu(\lambda)$. This completes the proof.

In the final version of these notes, this argument will be given for all quasi-split groups. □

6. Lecture 6

Our goal is to show that $(P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}), \star)$ is a neutral Tannakian category. Our reference for the definitions and results below is [DM82].

6.1. Preliminaries and definition of neutral Tannakian categories.

6.1.1. *Symmetric monoidal categories.* Let $(\mathcal{C}, \otimes, I)$ be a category $\mathcal{C}$ endowed with a functor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and an "identity" object I. The identity object comes with "right unit" $r_A : A\otimes I \xrightarrow{\sim} A$ and "left unit" $l_A : I\otimes A \xrightarrow{\sim} A$ isomorphisms, which are natural in A.

Definition 6.1.1. We say $(\mathcal{C}, \otimes, I)$ is a *symmetric monoidal category* (or a *tensor category*) if there exist isomorphisms $s_{AB} : A\otimes B \xrightarrow{\sim} B\otimes A$, for $A, B \in \mathrm{ob}(\mathcal{C})$, which are natural in A, B and are such that the following properties hold:

- the following diagram commutes

$$\begin{tikzcd} A \otimes I \arrow[rr, "s_{AI}"] \arrow[dr, "r_A"'] & & I \otimes A \arrow[dl, "l_A"] \\ & A & \end{tikzcd},$$

- there is an *associativity constraint* $\alpha_{ABC} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ satisfying the **pentagon axiom**, i.e., identifying the two obvious isomorphisms

$$A \otimes (B \otimes (C \otimes D)) \xrightarrow{\sim} ((A \otimes B) \otimes C) \otimes D$$

and which further is compatible with the s_{AB} in that it satisfies the **hexagon axiom**, i.e., the following diagram commutes:

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "s_{AB} \otimes 1_C", "\sim"'] \arrow[d, "\alpha_{ABC}"', "\wr"] & (B \otimes A) \otimes C \arrow[d, "\alpha_{BAC}", "\wr"'] \\ A \otimes (B \otimes C) \arrow[d, "s_{A,B\otimes C}"', "\wr"] & B \otimes (A \otimes C) \arrow[d, "\mathrm{id}_B \otimes s_{AC}", "\wr"'] \\ (B \otimes C) \otimes A \arrow[r, "\alpha_{BCA}", "\sim"'] & B \otimes (C \otimes A) \end{tikzcd}$$

- the composition $A \otimes B \stackrel{s_{AB}}{\to} B \otimes A \stackrel{s_{BA}}{\to} A \otimes B$ is the identity $1_{A \otimes B}$.

6.1.2. *Internal Hom and dual objects.* Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category, as above (except we now denote the identity object by 1). If $T \mapsto \mathrm{Hom}(T \otimes X, Y)$ is representable, we denote the representing object by $\underline{\mathrm{Hom}}(X, Y)$ and call it the "internal hom". In other words we have an identification

$$\mathrm{Hom}(T \otimes X, Y) = \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y)). \tag{6.1.1}$$

When it exists, we denote

$$X^\vee = \underline{\mathrm{Hom}}(X, 1). \tag{6.1.2}$$

Then there exists a unique morphism $\mathrm{ev}_X : X^\vee \otimes X \to 1$ corresponding to $\mathrm{id}_{X^\vee}$ under the isomorphism

$$\mathrm{Hom}(T, X^\vee) \xrightarrow{\sim} \mathrm{Hom}(T \otimes X, 1).$$

Under this isomorphism $X \otimes X^\vee \stackrel{s_{X,X^\vee}}{\to} X^\vee \otimes X \stackrel{\mathrm{ev}_X}{\to} 1$ corresponds to a morphism $i_X : X \to X^{\vee\vee}$.

Definition 6.1.2. An object X is called *reflexive* if i_X is an isomorphism.

Remark 6.1.3. We can make the correspondence between $f \in \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y))$ and $g : T \otimes X \to Y$ explicit: given f the morphism g is the unique one making the following diagram commute:

$$\begin{tikzcd} T \otimes X \arrow[d, "f \otimes 1_X"'] \arrow[dr, "g"] & \\ \underline{\mathrm{Hom}}(X, Y) \otimes X \arrow[r, "\mathrm{ev}_{X,Y}"'] & Y \end{tikzcd}$$

where $\mathrm{ev}_{X,Y}$ is the morphism corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ under (6.1.1).

For finite families $(X_i), (Y_i)$, $i \in I$, there is a morphism

$$\bigotimes_i \underline{\mathrm{Hom}}(X_i, Y_i) \longrightarrow \underline{\mathrm{Hom}}(\otimes_i X_i, \otimes_i Y_i).$$

corresponding to

$$\bigotimes_i \underline{\mathrm{Hom}}(X_i, Y_i) \otimes \bigotimes_i X_i \xrightarrow{s,\alpha} \bigotimes_i \big(\underline{\mathrm{Hom}}(X_i, Y_i) \otimes X_i\big) \xrightarrow{\otimes \mathrm{ev}} \bigotimes_i Y_i.$$

In particular we have morphisms

$$\begin{aligned} \otimes_i X_i^\vee &\longrightarrow \big(\otimes_i X_i\big)^\vee \\ X^\vee \otimes Y &\longrightarrow \underline{\mathrm{Hom}}(X,Y). \end{aligned}$$

6.1.3. *Rigid Tensor Categories.*

Definition 6.1.4. A *rigid tensor category* $(\mathcal{C}, \otimes, 1)$ is a symmetric monoidal category such that

(a) $\underline{\mathrm{Hom}}(X,Y)$ exists for all $X, Y \in \mathrm{ob}(\mathcal{C})$
(b) $\underline{\mathrm{Hom}}(X_1,Y_1) \otimes \underline{\mathrm{Hom}}(X_2,Y_2) \overset{\sim}{\longrightarrow} \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$. for all $X_1, X_2, Y_1, Y_2 \in \mathrm{ob}(\mathcal{C})$
(c) All objects in $\mathcal{C}$ are reflexive.

The following is a consequence of the definitions.

Lemma 6.1.5. *In an rigid tensor category the canonical morphisms above are isomorphisms, i.e.,*

$$\bigotimes_i X_i^\vee \overset{\sim}{\longrightarrow} \Big(\bigotimes_i X_i\Big)^\vee,$$

and

$$\begin{aligned} X^\vee \otimes Y &\overset{\sim}{\longrightarrow} X^\vee \otimes Y^{\vee\vee} \\ &\overset{\sim}{\longrightarrow} \underline{\mathrm{Hom}}(X \otimes Y^\vee, 1) \\ &\cong \underline{\mathrm{Hom}}(X, Y^{\vee\vee}) \cong \underline{\mathrm{Hom}}(X,Y). \end{aligned}$$

More generally, $\forall X, Y, Z$ *we have canonical natural isomorphisms*

$$\underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y,Z)) \cong \underline{\mathrm{Hom}}(X \otimes Y, Z).$$

For the last statement, observe that both sides represent isomorphic functors: for every object T, we have

$$\mathrm{Hom}(T, \mathrm{LHS}) = \mathrm{Hom}(T \otimes X, \underline{\mathrm{Hom}}(Y,Z)) \cong \mathrm{Hom}(T \otimes X \otimes Y, Z) \cong \mathrm{Hom}(T, \mathrm{RHS}).$$

In the final version of the notes, more properties of rigid tensor categories will be inserted here...

Remark 6.1.6. Let K be any field. Then $\mathcal{C}$ is often a K-linear abelian category, that is, the hom sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ have the structure of K-vector spaces in such a way that the composition laws are K-bilinear.

We have the following criterion for a K-linear abelian category to be a rigid tensor category.

Proposition 6.1.7. [DM82, Prop. 1.20] *Suppose K is a field and $\mathcal{C}$ is a K-linear abelian category, and suppose $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a K-bilinear functor. Suppose that $F : \mathcal{C} \to \mathrm{Vect}_K$ is a faithful, exact, and K-linear functor. Suppose that $\forall X, Y, Z$ we are given functorial isomorphisms ϕ_{XYZ} : $(X \otimes Y) \otimes Z \overset{\sim}{\to} X \otimes (Y \otimes Z)$ and $\psi_{XY} : X \otimes Y \overset{\sim}{\to} Y \times X$ such that*

(a) $F \circ \otimes = \otimes \circ (F \times F)$
(b) $F(\phi_{XYZ})$ *is the usual associativity isomorphism in* Vect_K
(c) $F(\psi_{XY})$ *is the usual commutative isomorphism in* Vect_K
(d) *there is an identity object* $U \in \mathrm{ob}(\mathcal{C})$ *and an isomorphism* $K \overset{\sim}{\to} \mathrm{End}(U)$ *and* $\dim_K F(U) = 1$
(e) *If* $\dim_K F(L) = 1$*, then L is invertible in $\mathcal{C}$, i.e.,there exists another object L^{-1} and an isomorphism $L \otimes L^{-1} \overset{\sim}{\to} U$.*

Then $(\mathcal{C}, \otimes, U, \phi, \psi)$ *is a rigid tensor category.*

6.1.4. *Tensor functors and morphisms between them.*

Definition 6.1.8. A $\otimes$-*functor* $(\mathcal{C},\otimes) \to (\mathcal{C}',\otimes')$ of symmetric monoidal categories is a pair (F,c) consisting of a functor $F : \mathcal{C} \to \mathcal{C}'$ and the data of functorial isomorphisms

$$c_{XY} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$$

which are compatible with the commutativity and associativity constraints, and with unit objects. That is, they satisfy:

(a) $\forall X, Y, Z \in \mathrm{ob}(\mathcal{C})$, the following commutes

$$\begin{tikzcd} FX \otimes (FY \otimes FZ) \arrow[r, "\mathrm{id}\otimes c"] \arrow[d, "\wr"'] & FX \otimes F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "\wr"] \\ (FX \otimes FY) \otimes FZ \arrow[r, "c\otimes\mathrm{id}"] & F(X \otimes Y) \otimes FZ \arrow[r, "c"] & F\big((X \otimes Y) \otimes Z\big). \end{tikzcd}$$

(b) $\forall X, Y$, the following commutes

$$\begin{tikzcd} FX \otimes FY \arrow[r, "c"] \arrow[d, "s'"'] & F(X \otimes Y) \arrow[d, "F(s)"] \\ FY \otimes FX \arrow[r, "c"] & F(Y \otimes X) \end{tikzcd}$$

(c) if (U,u) is an identity for $(\mathcal{C},\otimes)$ in the sense of [DM82, §1], then (FU, Fu) is an identity object in $(\mathcal{C}',\otimes')$.

Remark 6.1.9. We can clearly upgrade the definition so that c consists of isomorphisms indexed by finite families

$$c : \bigotimes_i F(X_i) \xrightarrow{\sim} F\big(\bigotimes_i X_i\big)$$

satisfying the obvious compatibility with the symmetric monoidal structures.

Definition 6.1.10. Let (F,c) and (G,d) be two $\otimes$-functors $(\mathcal{C},\otimes) \to (\mathcal{C}',\otimes')$ between two symmetric monoidal categories. A *morphism of $\otimes$-functors* $(F,c) \to (G,d)$ is a natural transformation of functors $\lambda : F \to G$ **such that the following diagrams commute** for every finite I-family of objects $(X_i)_{i\in I}$

$$\begin{tikzcd} \bigotimes_i F(X_i) \arrow[r, "c", "\sim"'] \arrow[d, "\otimes_i \lambda_{X_i}"'] & F(\bigotimes_i X_i) \arrow[d, "\lambda_{\otimes_i X_i}"] \\ \bigotimes_i G(X_i) \arrow[r, "d", "\sim"'] & G(\bigotimes_i X_i). \end{tikzcd}$$

If $I = \emptyset$, this diagram should be interpreted as the commutative diagram

$$\begin{tikzcd} 1' \arrow[r, "\sim"] \arrow[d, equal] & F(1) \arrow[d, "\lambda_1"] \\ 1' \arrow[r, "\sim"] & G(1). \end{tikzcd}$$

Definition 6.1.11. Denote by $\mathrm{Hom}^{\otimes}(F,G)$ the class of $\otimes$-morphisms $(F,c) \to (G,d)$.

Proposition 6.1.12. *Suppose $(F,c) : (\mathcal{C},\otimes) \to (\mathcal{C}',\otimes')$ is a $\otimes$-functor of* **rigid** *tensor categories. Then the canonical map $F(\mathrm{ev}_{XY})$ gives an isomorphism*

$$F_{XY} : F(\underline{\mathrm{Hom}}(X,Y)) \xrightarrow{\sim} \underline{\mathrm{Hom}}(FX, FY).$$

Proof. **Will appear in final version of notes.** $\square$

Proposition 6.1.13. *If $(\mathcal{C},\otimes)$ and $(\mathcal{C}',\otimes')$ are rigid, then any $\lambda \in \mathrm{Hom}^{\otimes}(F,G)$ is an isomorphism.*

Proof. **Will appear in final version of notes.** ☐

Proposition 6.1.14. *Assume $(\mathcal{C}, \otimes)$ is abelian and rigid. Then $\otimes$ is biadditive and commutes with direct and inverse limits in each variable. In particular, it is exact in each variable.*

Proof. For each object Y, the functor $X \to X \otimes Y$ has a right adjoint (namely $Z \to \underline{\mathrm{Hom}}(Y, Z)$), so it is additive and preserves colimits. On the other hand, one can show that $Z \to \underline{\mathrm{Hom}}(Y, Z) = Y^\vee \otimes Z$ is also left adjoint to $X \to X \otimes Y \cong \underline{\mathrm{Hom}}(Y^\vee, X)$. ☐

6.1.5. *Neutral Tannakian categories.*

Definition 6.1.15. A *neutral Tannakian category* over K is a rigid K-linear abelian tensor category such that $K = \mathrm{End}(1)$, for which there is an exact faithful K-linear tensor functor

$$\omega : \mathcal{C} \longrightarrow \mathrm{Vect}_K.$$

Example 6.1.16. The most important example of a neutral Tannakian category is one of the form $(\mathrm{Rep}_K(G^\vee), \otimes)$, where $G^\vee$ is an affine group scheme over K.

When working over a field K, we may drop the hypothesis "faithful" in the above definition, as it is automatic.

Proposition 6.1.17. *Suppose $\mathcal{C}, \mathcal{C}$ are rigid tensor categories,* $\mathrm{End}(1)$ *is a field, and* $1' \neq 0$*. Then every exact $\otimes$-functor $F : \mathcal{C} \to \mathcal{C}'$ is faithful.*

Proof. Note that $C \neq 0 \iff C^\vee \otimes C \xrightarrow{\mathrm{ev}_C} 1$ is surjective (use Lemma 6.1.5). Hence $C \neq 0$ implies $F(C) \neq 0$. In other words, F is *conservative*.

Now $f, g : C_1 \to C_2$ in $\mathcal{C}$ have $f \neq g$ if and only if $C_1/\mathrm{eq}(f, g) \neq 0$. But

$$F(C_1/\mathrm{eq}(f, g)) = FC_1/\mathrm{eq}(Ff, Fg) \tag{6.1.3}$$

as F is exact. This implies the result. ☐

Corollary 6.1.18 (of proof)**.** *If F is exact and conservative, then F is faithful.*

Remark 6.1.19. For the Corollary, it seems essential that we are working over a field and not a more general ring. Indeed, suppose $K = \mathbb{Z}_p$, $G^\vee$ is an affine $\mathbb{Z}_p$-group scheme, and $\mathcal{C} = \mathrm{Rep}_{\mathbb{Z}_p}(G^\vee)$ is the exact category of representations of $G^\vee$ on the category $\mathrm{Proj.ft}_{\mathbb{Z}_p}$ of finite-type projective $\mathbb{Z}_p$-modules. Let $F : \mathrm{Rep}_{\mathbb{Z}_p}(G^\vee) \to \mathrm{Proj.ft}_{\mathbb{Z}_p}$ be any exact and conservative functor. These conditions need not imply that F is faithful. The problem we encounter is that these conditions are not strong enough to force F to preserve kernels. The above proof breaks down because $C_2/\mathrm{im}(f - g)$ could have $\mathbb{Z}_p$-torsion, hence there is no reason that $F(\mathrm{im}(f - g))$ should inject into $F(C_2)$, and hence no reason that $F(\ker(f - g)) \subset \ker(Ff - Fg)$ should be an equality, hence no reason that the natural surjection $F\big(C_1/\mathrm{eq}(f, g)\big) \to F(C_1)/\mathrm{eq}(Ff, Fg)$ should be an isomorphism. The assumption that F is conservative implies the source is non-zero when $f \neq g$, but the target could well be zero, that is, we could have $Ff = Fg$.

The following is one of the main theorems about Tannakian categories. It tells us there are no other examples of neutral Tannakian categories besides those listed in the Example above. Our reference is [DM82, Thm. 2.11].

Theorem 6.1.20. *Let $(\mathcal{C}, \otimes, 1)$ be a neutral Tannakian category over a field K with fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}_K$. Then:*

(a) *the functor of K-algebras $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ (cf.* [DM82, Prop. 2.8]*) is represented by an affine K-group scheme $G^\vee$.*
(b) *$\mathcal{C} \to \mathrm{Rep}_K(G^\vee)$ defined by ω is an isomorphism of tensor categories.*

We term $G^\vee$ the *Tannakian group* of $\mathcal{C}$.

6.2. **Properties of $G^\vee$ and $\mathrm{Rep}_K(G^\vee)$.** In this section we see how properties of $G^\vee$ can be read off from properties of $(\mathrm{Rep}_K(G^\vee), \otimes)$. A reference for the following facts is [DM82, §2]. We assume throughout that K is an algebraically closed field.

(i) The group $G^\vee$ is **algebraic** (i.e. finite-type over K) if and only if $\mathcal{C} = \mathrm{Rep}_K(G^\vee)$ has a tensor generator, namely an object X such that every object Y is isomorphic to a subquotient of $P(X, X^\vee)$ for some $P \in \mathbb{N}[r, s]$. Here we interpret $U^n + V^m := U^{\otimes n} \oplus V^{\otimes m}$, and $U^0 := 1$.

(ii) Assume $\mathrm{char}(K) = 0$. Then $G^\vee$ is **connected** if and only if for every nontrivial representation X the subcategory

$$\langle X \rangle := \{\text{all subquotients of } X^{\oplus n} \mid n \in \mathbb{N}\}$$

is *not* $\otimes$-stable.

(iii) Assume $\mathrm{char}(K) = 0$. Then $G^\vee$ is **pro-reductive** if and only if $\mathrm{Rep}_K(G^\vee)$ is semisimple.

6.3. **Sketch of proof that our category is Tannakian.**

6.3.1. *First steps.*

- Let $\mathcal{C} = (P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}), \star)$. We wish to show this is a neutral Tannakian category.
- We define $\omega : \mathcal{C} \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ by

$$\mathcal{F} \mapsto \bigoplus_{i \in \mathbb{Z}} R^i\Gamma(\mathrm{Gr}_\mathcal{G}, \mathcal{F}).$$

- $\mathcal{C}$ has an identity object $\mathbf{1} := \bar{\mathbb{Q}}_{\ell,*}$, the constant sheaf supported on the base point $*$. Clearly $\mathbf{1} \neq 0$ and $\omega(\mathbf{1}) = \overline{\mathbb{Q}}_\ell$, the identity object in $\mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$.
- We want to show that $\mathcal{C}, \mathbf{1}, \omega$ satisfies the Deligne-Milne axioms for a category to be rigid.
- We will use ULA sheaves and a global convolution diagram of BD-Grassmannians to show that $\star$ has commutativity and associativity isomorphisms.
- The ULA/global method will show that ω is symmetric monoidal, ie., it takes the tensor product and the commutativity and associativity isomorphisms to the natural ones in $\mathrm{Vect}_{\bar{\mathbb{Q}}_\ell}$.
- We postpone this and assume it for now.
- We will next check the remaining conditions in Deligne-Milne Proposition 1.20 (Proposition 6.1.7).

6.3.2. *Checking the Deligne-Milne axioms.*

- It is clear that $\star : P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}) \times P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}) \to P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G})$ is $\bar{\mathbb{Q}}_\ell$-bilinear.
- $\omega : P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}) \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ is exact since it is clearly additive and $\bar{\mathbb{Q}}_\ell$-linear, and since $P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G})$ is semisimple, so that every short exact sequence splits (any additive functor preserves split exact sequences).
- ω is faithful: we proved above that any exact conservative functor is faithful (conservative: $C \neq 0 \Rightarrow F(C) \neq 0$).
- But ω **is** conservative: the IC-complex of any Schubert variety $\overline{\mathcal{O}}_\mu$ has non-vanishing global cohomology (e.g. by a theorem of Kazhdan-Lusztig relating the latter to KL-polynomials).
- A different (more elementary) proof is given in [dCHL18] − **for split groups**.
- Finally, suppose $\mathcal{A} \in P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G})$ has $\dim \omega(\mathcal{A}) = 1$. Then $\mathcal{A}$ is a single IC_μ.
- By Poincaré duality for $\omega(\mathrm{IC}_\mu) = \mathrm{IH}^\bullet(\overline{\mathcal{O}}_\mu, \bar{\mathbb{Q}}_\ell)$ (which forces cohomology in top and bottom degrees) we must have $\overline{\mathcal{O}}_\mu = t^\mu *$, that is, μ is a central cocharacter.
- Then $\mathrm{IC}_{\pm\mu} = \bar{\mathbb{Q}}_{\ell, t^{\pm\mu}*}$ satisfy

$$\mathrm{IC}_\mu \star \mathrm{IC}_{-\mu} = \mathrm{IC}_0,$$

which follows from the definition of $\star$ in this case.
- We conclude that $(P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}), \star)$ is a rigid tensor category by Proposition 6.1.7. □

Thus, assuming the ULA ingredients, we have proved that there is a unique (up to isomorphism) affine group scheme $G^\vee$ over $\bar{\mathbb{Q}}_\ell$, such that there is an isomorphism of rigid tensor categories

$$(P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}), \star) \;\cong\; (\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G^\vee), \otimes). \tag{6.3.1}$$

We next consider what we can say about the group $G^\vee$ as this point.

6.3.3. *Finite type.* To check that $G^\vee$ is algebraic, choose a finite set $\lambda_1, \dots, \lambda_n$ of generators for the monoid of dominant coweights $X_*(T)_I^+$. So any $\lambda = k_1\lambda_1 + \cdots + k_n\lambda_n$ for some $k_i \in \mathbb{Z}_{\geq 0}$. One can see that IC_λ appears as a direct summand of

$$(\mathrm{IC}_{\lambda_1})^{\star k_1} \star \cdots \star (\mathrm{IC}_{\lambda_n})^{\star k_n}.$$

(See e.g. [dCHL18] **at least for the split case**). Thus $\mathrm{IC}_{\lambda_1} \oplus \cdots \oplus \mathrm{IC}_{\lambda_n}$ is a tensor generator, which proves $G^\vee$ is algebraic.

6.3.4. *Reductivity.* The group $G^\vee$ is reductive because the category is semisimple by Proposition 4.2.1.

6.3.5. *Connectedness.* First suppose that $G/k(\!(t)\!)$ is split. Then is enough to show that if $\mathcal{F} \in P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G})$ is a non-zero object and not a sum of copies of the unit object 1, then $\langle \mathcal{F} \rangle$ is not closed under $\otimes$. This follows easily from the observation that $\mathcal{F}$ must have a summand of the form IC_μ where $\mu \neq 0$ and this μ has infinite order in the additive group $X_*(T)$.

In the non-split case, the same argument does not work, as then $X_*(T)_I^+$ can indeed have torsion elements and in fact these parametrize the connected components of the eventual Tannakian group $\widehat{G}^{I,\circ}$, as $\pi_0(\widehat{T}^{I,\circ}) \xrightarrow{\sim} \pi_0(\widehat{G}^{I,\circ})$ by e.g. [Hai15, Prop. 4.1].

6.3.6. *Aside: What are the dual and internal Hom objects?*

- Assuming the ULA ingredients, we have proved $(P_{L^+G}\mathrm{Gr}_G), \star)$ is Tannakian, in particular, it is a rigid tensor category.
- In particular, each object X has a dual $X^\vee$ and is reflexive. What is $X^\vee$?
- We claim that $\mathrm{IC}_\mu^\vee = \mathrm{IC}_{\mu^*}$, where $\mu^* := -w_0(\mu)$ (here w_0 is the longest element in the finite Weyl group).
- I do not know a direct proof, but here is an argument using the geometric Satake correspondence.
- Under the correspondence we will prove $(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\widehat{G}), \otimes) \cong (P_{L^+\mathcal{G}}(\mathrm{Gr}_\mathcal{G}), \star)$, the highest weight representation V_μ of $\widehat{G}$ corresponds to IC_μ **(up to some normalization)**.
- The identity characterizing $\underline{\mathrm{Hom}}(\mathrm{IC}_\mu, \mathrm{IC}_\lambda)$ we wish to prove for dominant λ, μ, ν follows from the identity

  $$\mathrm{Hom}(V_\nu, V_{\mu^*} \otimes V_\lambda) = \mathrm{Hom}(V_\nu \otimes V_\mu, V_\lambda),$$

  which follows from the definitions using the well-known fact that V_{μ^*} is the $\widehat{G}$-dual of V_μ.

6.4. **Our approach to identifying the Tannakian group $G^\vee$.** We will follow the approach of Timo Richarz in his paper [Ri14].

The basic idea is to use a result of Kazhdan-Larsen-Varshavsky which shows how to recover a root system from its Grothendieck semiring.

Useful References:

(1) [Hai03] T. Haines, *Structure constants for Hecke and representation rings* , IMRN, no. **39**, (2003), 21030-2119.
(2) [KLV] D. Kazhdan, M. Larsen, Y. Varshavsky, *The Tannakian formalism and the Langlands conjectures*, Alg. Numb. Th. **8**, (2014), no. 1, 243-256. [**?**, NP01]. C. Ngô, P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shaliko géométrique*, J. Alg. Geom. **10** (2001), no. 3, 515-547.
(3) [Ri14] T. Richarz, *A new approach to the geometric Satake equivalence*, Documenta Math. **19** (2014), 209-246.
(4) [Kum90] S. Kumar, *Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture*, Invent. Math. **93**, (1988), 117-130.

6.4.1. *Preliminaries.* **The following will only discuss the split case. The general case is almost the same and will appear in the final version.**

In what follows, we will abbreviate $X^\vee = X_*(T) = X^*(\widehat{T})$, $X^\vee_+ = X_*(T)^+ = X^*(\widehat{T})^+$, and also we set $X = X^*(T)$ and $X_+ = X^*(T)^+$. We let $2\rho = \sum_{\alpha>0}\alpha$, the sum of B-positive roots in the absolute root system $\Phi = \Phi(G,T)$.

Reminders:

- Let μ, λ generally denote *dominant* cocharacters: $\mu, \lambda \in X^\vee_+$. Let ν generally denote any cocharacter: $\nu \in X^\vee$
- $\overline{\mathcal{O}}_\mu$ is the Zariski-closure $\mathrm{Gr}_{G,\leq\mu}$ of the $L^+(G)$-orbit of $t^\mu * \in \mathrm{Gr}_G$. (Recall $\dim \bar{\mathcal{O}}_\mu = \langle 2\rho, \mu\rangle$.) Let $S_\nu := LUt^\nu *$
- Q (resp. $Q^\vee$) dnotes the $\mathbb{Z}$-lattice in $X^*(T)$ (resp. $X^\vee$) spanned by the roots (resp. coroots). Write $\nu \leq \nu'$ whenever $\nu' - \nu$ is a sum of positive coroots.
- $\Omega(\mu) := \{\nu \in X^\vee \mid w\nu \leq \mu, \ \ \forall w \in W\}$
- Recall the twisted product $\bar{\mathcal{O}}_{\mu_\bullet} = \bar{\mathcal{O}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \bar{\mathcal{O}}_{\mu_r}$ and morphism $m_{\mu_\bullet} : \bar{\mathcal{O}}_{\mu_\bullet} \to \bar{\mathcal{O}}_{|\mu_\bullet|}$ associated to $\mu_\bullet = (\mu_1, \dots, \mu_r)$ and $|\mu_\bullet| := \sum_i \mu_i$.
- The morphism $m_{\mu_\bullet}$ is *semi-small*, which translates to the following: for any $\lambda \in X^\vee_+$ with $\lambda \leq |\mu_\bullet|$, and $x \in \mathcal{O}_\lambda(k)$, we have
$$\dim m^{-1}_{\mu_\bullet}(x) \leq \langle \rho, |\mu_\bullet| - \lambda\rangle.$$
(Recall we proved semi-smallness earlier, relying on Macdonald's formula for Satake transforms of basis elements in the Hecke algebra.)

6.4.2. *Result on semismallness, fibers, and cohomology.* By semi-simplicity and perverse-presernation of $\star$, we can write
$$\mathrm{IC}_{\mu_1} \star \cdots \star \mathrm{IC}_{\mu_r} = \bigoplus_{\lambda \leq |\mu_\bullet|} V^\lambda_{\mu_\bullet} \otimes \mathrm{IC}_\lambda,$$
for certain *multiplicity* vector spaces $V^\lambda_{\mu_\bullet}$ over $\bar{\mathbb{Q}}_\ell$. The following two results are well-known, and the author gives citations to [Hai03] only for convenience.

Lemma 6.4.1. ([Hai03, Prop. 3.1]) *For any $\lambda \leq |\mu_\bullet|$ and $y \in \mathcal{O}_\lambda(k)$, the vector space $V^\lambda_{\mu_\bullet}$ has a canonical basis indexed by the set of irreducible components of $m^{-1}_{\mu_\bullet}(y)$ having the maximal possible dimension of $\langle \rho, |\mu_\bullet| - \lambda\rangle$.*

Proof. (Sketch). By the characterization of intersection complexes in Proposition 3.3.9, we see that
$$\mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2} \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathrm{IC}_{\mu_r} = \mathrm{IC}_{\tilde{\times}_i \bar{\mathcal{O}}_{\mu_i}}.$$
We have from the definition of $\star$
$$Rm_{\mu_\bullet,*}(\mathrm{IC}_{\tilde{\times}_i \bar{\mathcal{O}}_{\mu_i}}) = \bigoplus_{\lambda \leq |\mu_\bullet|} V^\lambda_{\mu_\bullet} \otimes \mathrm{IC}_\lambda.$$
Let $d = -\dim \mathcal{O}_\lambda$. Now apply the cohomology-stalk functor $\mathcal{H}^d_y$ to the above formula, and use the local-global spectral sequence, vanishing properties of cohomology of IC-complexes, and the semi-smallness (in the stratified sense) of $m_{\mu_\bullet}$ [GIVE CROSS-REF]. □

The fiber over $y \in \mathcal{O}_\lambda(k)$ of the LHS is the group $H^d(m^{-1}_{\mu_\bullet}(y), \mathrm{IC}_{\tilde{\times}_i \bar{\mathcal{O}}_{\mu_i}})$. The Lemma above results from a special case of the following general lemma, whose proof is the same as that sketched above.

Lemma 6.4.2. ([Hai03, Lem. 3.2]) *Let $f : X = \cup_\alpha X_\alpha \to Y = \cup_\beta Y_\beta$ be a semi-small morphism between proper stratified schemes such that for each α, $f(X_\alpha)$ is a union of certrain strata in Y. Suppose $y \in Y_\beta$. Let $d = -\dim Y_\beta$. There there is a canonical isomorphism*
$$H^d(f^{-1}(y), \mathrm{IC}_X) = \bar{\mathbb{Q}}_\ell^{C_{max}(y)}$$
where $C_{max}(y)$ is the set of irreducible components of $f^{-1}(y)$ having the maximal possible dimension $\frac{1}{2}(\dim X - \dim Y_\beta)$.

6.4.3. *PRV Conjecture.* The following is the PRV Conjecture, now a theorem proved by Shrawan Kumar [Kum90] and independently by Oliver Mathieu [Mat89].

Theorem 6.4.3. *Suppose μ_i, λ are dominant weights for any complex connected reductive group H and let V_{μ_i} and V_λ denote the corresponding irreducible highest weight representations. If $\lambda = \nu_1 + \cdots + \nu_r$ with $\nu_i \in W\mu_i$ for all i, then V_λ appears as a direct summand of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_r}$.*

We also need the **geometric PRV Conjecture**, which was discovered and proved by Richarz:

Theorem 6.4.4. ([Ri14, Lem. 5.5]) *If $\lambda \in X_+^\vee$ with $\lambda \leq |\mu_\bullet|$ is of the form $\lambda = \nu_1 + \cdots + \nu_r$ for $\nu_i \in W\mu_i$ for all i, then IC_λ appears as a direct summand of $\mathrm{IC}_{\mu_1} \star \cdots \star \mathrm{IC}_{\mu_r}$.*

Remark 6.4.5. The proof of geometric PRV is elementary, so you might think we can use the geometric Satake correspondence (after it is proved) to give an alternative proof of [Kum]. However, this would be circular, since in our current strategy, Theorem 6.4.3 is used to prove geometric Satake.

6.4.4. *Proof of the geometric PRV.*

- By induction, we easily reduce to case $r = 2$. By Lemma 6.4.1, we need to find $x \in \mathcal{O}_\lambda$ such that the dimension of $m_{\mu_\bullet}^{-1}(x)$ is at least $\langle \rho, |\mu_\bullet| - \lambda \rangle$.
- We have $\lambda = \nu_1 + \nu_2$. Choose $w \in W$ such that $w\nu_1$ is dominant, that is, $w\nu_1 = \mu_1$. Write $w\lambda = w\nu_1 + w\nu_2$.
- Let $S_{w\nu_\bullet} \cap \bar{\mathcal{O}}_{\mu_\bullet}$ be the intersection with $\bar{\mathcal{O}}_{\mu_\bullet} \subset \bar{\mathcal{O}}_{\mu_1} \times \bar{\mathcal{O}}_{\mu_1+\mu_2}$
$$S_{w\nu_\bullet} \cap \bar{\mathcal{O}}_{\mu_\bullet} := (S_{w\nu_1} \times S_{w\nu_1+w\nu_2}) \cap \overline{\mathcal{O}}_{\mu_\bullet}.$$
- By [NP01, Lem. 5.2], we have an isomorphism
$$S_{w\lambda} \cap \bar{\mathcal{O}}_\lambda = \prod_{\substack{\alpha>0 \\ w\alpha>0}} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{w\alpha, i},$$
and similarly for $(w\nu_i, \mu_i)$ replacing $(w\lambda, \lambda)$.
- Using this we get that $S_{w\nu_\bullet} \cap \bar{\mathcal{O}}_{\mu_\bullet} \cong (S_{w\nu_1} \cap \bar{\mathcal{O}}_{\mu_1}) \times (S_{w\nu_2} \cap \bar{\mathcal{O}}_{\mu_2})$ in the following way:
- $y = (y_1, y_2) \in (S_{w\nu_1} \times S_{w\nu_1+w\nu_2}) \cap \overline{\mathcal{O}}_{\mu_\bullet}$ can be written uniquely for certain $u_1, u_2 \in L^+U(k)$ as
$$y_1 = u_1 t^{w\nu_1} * \qquad , \qquad y_2 = u_1 t^{w\nu_1} u_2 t^{w\nu_2} * .$$
- Here $u_i t^{w\nu_i} * \in (S_{w\nu_i} \cap \bar{\mathcal{O}}_{\mu_i})$ for $i = 1, 2$.
- Since $w\nu_1$ is dominant we have $t^{w\nu_1} u_2 t^{-w\nu_1} \in L^+U(k)$, and so $y_2 = m_{\mu_\bullet}(y) \in \mathcal{O}_{w\nu_1+w\nu_2}$, so in fact $m_{\mu_\bullet}$ sends $S_{w\nu_\bullet} \cap \mathcal{O}_{\mu_\bullet}$ into $S_{w\lambda} \cap \mathcal{O}_\lambda$.
- In fact these two spaces are irreducible, and there is an open dense $Y \subset S_{w\nu_\bullet} \cap \mathcal{O}_{\mu_\bullet}$ mapped by $m_{\mu_\bullet}$ onto an open dense $Y' \subset S_{w\lambda} \cap \mathcal{O}_\lambda$.
- This is a little bit tricky, and is not explained in [Ri14]. One uses the fact that $m_{\mu_\bullet}$ is locally trivial in the stratified sense [GIVE CROSS-REF]. This means that
$$m_{\mu_\bullet} : m_{\mu_\bullet}^{-1}(V) \cap \mathcal{O}_{\mu_\bullet} \to V$$
is an *open* morphism for some small open $V \subset \mathcal{O}_\lambda$ containing a $t^{w\lambda}*$. In particular, its restriction over $S_{w\lambda} \cap \bar{\mathcal{O}}_\lambda$ is still open.
- The restriction of this last over the 1st factor preimage of the open dense $S_{w\nu_1} \cap \mathcal{O}_{\mu_1}$ is still an open map. The restriction to the intersection of this with some subset of the form $S_{\nu_1'} \times S_{\nu_1'+\nu_2'}$ with $\nu_1' + \nu_2' = w\lambda$ (these cover the preimage of $S_{w\lambda}$) must also be open.
- But by choice of our open subsets, we must have $\nu_1' = w\nu_1$ and hence $\nu_2' = w\nu_2$. This gives the open Y and shows that its image Y' is also *open* in $S_{w\lambda} \cap \mathcal{O}_\lambda$.
- Let $h = m_{\mu_\bullet}|Y$. Then by generic flatness of $h : Y \twoheadrightarrow Y'$, there exists $x \in Y'$ such that
$$\dim h^{-1}(x) = \dim Y - \dim Y' = \langle \rho, |\mu_\bullet| + w\lambda \rangle - \langle \rho, \lambda + w\lambda \rangle = \langle \rho, |\mu_\bullet| - \lambda \rangle.$$
- In particular $\dim m_{\mu_\bullet}^{-1}(x) \geq \langle \rho, |\mu_\bullet| - \lambda \rangle$, and hence equality by semi-smallness. We conclude that $V_{\mu_\bullet}^\lambda \neq 0$ using Lemma 6.4.1. $\square$

Remark 6.4.6. The above uses crucially that $\dim S_{w\lambda} \cap \overline{\mathcal{O}}_\lambda = \langle \rho, \lambda + w\lambda \rangle$. This is a consequence of the description in [NP01, Lem. 5.2] we already used. We do not need the finer result (which is also true) that for any $\nu \in \Omega(\lambda)$, $S_\nu \cap \overline{\mathcal{O}}_\lambda$ is equidimensional of dimension $\langle \rho, \lambda + \nu \rangle$.

6.5. **Preliminaries related to the KLV theorem.** Let H be any connected reductive group over $\bar{\mathbb{Q}}_\ell$ and let $K_0^+(H)$ denote the Grothendieck semiring attached to the category $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(H)$. The theorem of Kazhdan-Larsen-Varshavsky [KLV] is that H can be determined from $K_0^+(H)$.

In what follows X, X_+ will be *weights*. We will also now change our previous notation $\preceq$ to $\leq$ and replace the former $\preceq$ with a variant. More precisely, for $\lambda, \mu \in X_+$, define $\lambda \preceq \mu$ if $\mu - \lambda$ is an $\mathbb{R}_{\geq 0}$-linear combination of positive roots. Note $\lambda \leq \mu$ iff $\lambda \preceq \mu$ and the images of μ and λ in X/Q agree.

Define $\mathrm{Dom}_{\preceq \mu} = \{\nu \in X_+ \,|\, \nu \preceq \mu\}$.

Lemma 6.5.1 (KLV Lemma)**.** *For $\lambda, \mu \in X_+$, the following are equivalent:*

(i) $\lambda \preceq \mu$
(ii) $\mathrm{Conv}(W\lambda) \subseteq \mathrm{Conv}(W\mu)$
(iii) *There is a finite subset $F \subset X_+$ such that for all $k \in \mathbb{N}$:*

$$\mathrm{Dom}_{\preceq k\lambda} \subseteq WF + \sum_{i=1}^{k} W\mu$$

(iv) *There is a representation U such that for all $k \in \mathbb{N}$, every irreducible subquotient of $V_\lambda^{\otimes k}$ is a subquotient of $V_\mu^{\otimes k} \otimes U$.*

We omit the proof, given in [Ri14, Prop. B.3], but note that it is elementary except for the fact that it relies on the PRV Conjecture (proved in [Kum90, Mat89]). The most important equivalence is $(i) \Leftrightarrow (iv)$: this expresses $\lambda \preceq \mu$ purely in terms of the Grothendieck semiring.

7. Lecture 7

7.1. The results of Kazhdan-Larsen-Varshavsky.

7.1.1. *Tannakian reduction theorem.*

Theorem 7.1.1 (KLV Theorem [KLV])**.** *The based root system $\mathcal{B} = (X, R, X^\vee, R^\vee, \Delta)$ of H can be reconstructed from the semiring $K_0^+[H]$.*

Let v_μ be the image in $K_0^+[H]$ of the highest weight representation V_μ. Thus, the v_μ with $\mu \in X_+$ form a basis for $K_0^+[H]$. The first key lemma is the following.

Lemma 7.1.2 ($\preceq$ Lemma)**.** *The partial ordering $\preceq$ can be reconstructed from the semiring $K_0^+[H]$.*

Proof. By the KLV Lemma (Lemma 6.5.1), for weights $\lambda, \mu \in X$ we have $\lambda \preceq \mu$ if and only if there exists $u \in K_0^+[H]$ such that, for all $k \in \mathbb{N}$ and irreducible $v_\nu \in K_0^+[H]$, we have

$$v_\lambda^k - v_\nu \in K_0^+[H] \implies v_\mu^k u - v_\nu \in K_0^+[H].$$

N.B. We can formulate this without using "-", that is, without going outside of the semiring structure. □

The second key lemma is the following.

Lemma 7.1.3 (Δ Lemma)**.** *A weight $\alpha \in X$ belongs to Δ if and only if it is a $\preceq$-minimal nonzero element in X with the following property: there exists $\mu \in X_+$ such that $2\mu - \alpha \in X_+$ and $V_{2\mu-\alpha}$ appears in $V_\mu^{\otimes 2}$, i.e., $v_\mu^2 - v_{2\mu-\alpha} \in K_0^+[H]$.*

Proof. First suppose $\alpha \in \Delta$. Then take any $\mu \in X_+$ such that $\langle \mu, \alpha^\vee \rangle \geq 1$ (we can even take μ to be a suitable dominant weight plus α and thus arrange $\langle \mu, \alpha^\vee \rangle \geq 2$). Then $2\mu - \alpha \in X_+$ as $\langle \alpha, \beta^\vee \rangle < 0$ for $\beta \in \Delta \backslash \{\alpha\}$.

Now note that the weight $2\mu - \alpha$ appears with multiplicity at least 2 in $V_\mu^{\otimes 2}$, but with multiplicity exactly 1 in $V_{2\mu}$ (use Kostant's weight multiplicity via partitions theorem for the latter statement).

Remark 7.1.4. The Kostant weight multiplicity via partitions theorem gives the equality

$$\dim V_{2\mu}(2\mu-\alpha) = \sum_{w\in W_0} \operatorname{sgn}(w)\mathscr{P}\big(w(2\mu+\rho)-(2\mu-\alpha+\rho)\big).$$

Here $\mathscr{P}$ is defined as follows: if $\alpha_1,\dots,\alpha_N$ are the elements in Δ and $\nu\in X$, $\mathscr{P}(\nu)$ is the number of tuples $(n_1,\dots,n_N)\in\mathbb{Z}_{\geq 0}^N$ such that $n_1\alpha_1+\cdots+n_N\alpha_N=\nu$. Thus the key lemma, taking $\lambda=2\mu+\rho$, is the following:

Lemma 7.1.5. *If λ is regular dominant and $\langle\lambda,\alpha^\vee\rangle\geq 2$, then for $w\in W_0$, the condition $w\lambda-\lambda+\alpha\in\mathbb{Z}_{\geq 0}[\Delta]$ implies that $w=1$.*

Proof. Note first that $\lambda-\alpha$ is dominant. Suppose $w\neq 1$. Since λ is regular dominant, $w\lambda-\lambda$ is a non-empty sum of negative roots. Adding α cannot give a sum of positive roots, so by our hypothesis we must have $w\lambda-\lambda+\alpha=0$. Then $w\lambda=\lambda-\alpha$ is dominant, which implies since λ is regular that $w=1$, a contradiction. □

This means that $V_{2\mu-\alpha}$ must appear in $V_\mu^{\otimes 2}$. Next we prove that $\alpha\in\Delta$ implies α is **$\preceq$-minimal nonzero** with respect to the given properties.

Suppose $\beta\in X$ with $0\neq\beta\preceq\alpha$ has a dominant μ' for which $2\mu'-\beta\in X_+$ and $V_{2\mu'-\beta}$ appears in $V_{\mu'}^{\otimes 2}$. Then $\beta\in Q_+$, and it is then elementary to check that $\beta=\alpha$ (using $\alpha\in\Delta$ and $0\neq\beta\preceq\alpha$). This proves the "only if" direction.

Now suppose $\alpha\in X$ is minimal nonzero such that μ exists with the required properties. We must have $\alpha\in Q_+$. By Stembridge's Lemma ([NP01, 10.1]), there exists a positive root β such that

$$2\mu-\alpha\leq 2\mu-\beta<2\mu$$

and all three lie in X_+.

Write $\beta=\sum_i a_i\alpha_i$ for certain $\alpha_i\in\Delta$ and all $a_i\in\mathbb{N}$. There exists i such that $\langle\mu,\alpha_i^\vee\rangle>0$ (otherwise $\langle\mu,\beta^\vee\rangle=0$ and thus $\langle 2\mu-\beta,\beta^\vee\rangle=-2<0$, a contradiction).

Then $2\mu-\alpha_i\in X_+$ and by the above argument, $V_{2\mu-\alpha_i}$ appears in $V_\mu^{\otimes 2}$. By minimality $\alpha=\alpha_i$, that is, $\alpha\in\Delta$. This proves the "if" direction. □

7.1.2. *Sufficiency of reconstructing $(X_+,\leq)$.*

Lemma 7.1.6. *To reconstruct $\mathcal{B}$ from $K_0^+[H]$, it is enough to reconstruct the semigroup $(X_+,\leq)$ of dominant weights together with its partial order structure.*

Proof. The weight lattice X is the group completion of X_+, and is a finite free $\mathbb{Z}$-module. The dominance order $\leq$ extends uniquely to X, also denoted $\leq$. Then $X^\vee=\operatorname{Hom}(X,\mathbb{Z})$ is also determined, along with the canonical pairing

$$\langle -\,,-\rangle : X\times X^\vee\to\mathbb{Z}.$$

A weight $\alpha\in X\backslash\{0\}$ is in Δ if and only if $0\leq\alpha$ and α is minimal with this property. This reconstructs Δ.

Reconstructing $\alpha^\vee\in\Delta^\vee$: it is determined by its values on all $\mu\in X_+$. But $\langle\mu,\alpha^\vee\rangle$ is the unique $m\in\mathbb{Z}_{\geq 0}$ such that $2\mu-m\alpha\in X_+$ but $2\mu-(m+1)\alpha\notin X_+$.

The finite Weyl group $W_0\subset\operatorname{Aut}_{\mathbb{Z}}(X)$ is the finite subgroup generated by the reflections $s_{\alpha,\alpha^\vee}$ for $\alpha\in\Delta$. Then $R=W_0\cdot\Delta$ and $R^\vee=W_0\cdot\Delta^\vee$. Thus $\mathcal{B}$ is determined from $(X_+,\leq)$. □

7.1.3. *Proof of the KLV Theorem.*

- By Lemma 7.1.6 it is enough to reconstruct $(X_+,\leq)$ from the semiring $K_0^+[H]$.
- The irreducible objects in $K_0^+[H]$ are given by elements of X_+, which reconstructs the latter.
- By the $\preceq$ Lemma (Lemma 7.1.2), the partial order $\preceq$ is reconstructed from the semiring $K_0^+[H]$.
- The semigroup structure on X_+ is given by: $\nu=\lambda+\mu$ iff ν is the unique dominant weight which is $\preceq$-maximal with the property that $v_\lambda\cdot v_\mu-v_\nu\in K_0^+[H]$.
- Recall X is the group completion of X_+, so is determined. Also Δ is determined by the Δ Lemma (Lemma 7.1.3). Then also Q_+ is determined, and thus its group completion Q is determined.
- Finally, $\leq$ is determined from $\preceq$ and X/Q, so is determined. □

7.2. **Reconstruction Theorem on the geometric side.**

7.2.1. *First steps.* Write $(P_{L^+G}(\mathrm{Gr}_G), \star) = (\mathrm{Rep}(H), \otimes)$, for the connected reductive $\bar{\mathbb{Q}}_\ell$-group $H := G^\vee$. Let $(X', R', X'^\vee, R'^\vee, \Delta')$ be the based root system for H.

We know that X' is the group completion of X'_+, which can be identified with the irreducible elements of $P_{L^+G}(\mathrm{Gr}_G)$, that is, with the classes $[\mathrm{IC}_\mu] \in K_0^+[H]$ where $\mu \in X_*(T)^+$. Thus we can already identify $X'_+ = X_*(T)^+$, where the notion of dominant cocharacter comes from our choice of Borel pair $T \subset B \subset G$.

We therefore have a bijection $X_+^\vee \to X'_+$ given by $\mu \mapsto [\mathrm{IC}_\mu]$. It is enough (by the proof of the KLV Theorem) to show that this extends to a bijection of partially ordered semigroups

$$(X_+^\vee, \leq) \ \tilde{\to}\ (X'_+, \leq'),$$

where $(X, R, X^\vee, R^\vee, \Delta) = (X^*(T), R, X_*(T), R^\vee, \Delta)$ is the based root system for (G, B, T).

7.2.2. *Game plan.*

- Use the PRV Conjecture and the geometric PRV result to prove that $\lambda \preceq \mu$ is equivalent to $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$.
- Then one puts together the two lemmas below (the Geometric Δ Lemma and the Geometric Multiplicity Lemma) to prove that $Q_+^\vee$ corresponds to Q'_+ (equivalently, $\Delta^\vee = \Delta'$) and thus that $\leq$ corresponds to $\leq'$.
- We then conclude as desired

  $$(X_+^\vee, \leq) \ \tilde{\to}\ (X'_+, \leq'),$$

  which finishes the proof of the Main Theorem, Theorem 1.5.1, modulo ULA and fusion ingredients.

7.2.3. *The proof, continued.*

- We claim for $\lambda, \mu \in X_+^\vee$ that $\lambda \preceq \mu$ iff $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$.
- Assume $\lambda \preceq \mu$ and choose the finite subset $F \subset X_+^\vee$ as in the KLV Lemma. Let $\mathcal{A} = \oplus_{\nu \in F} \mathrm{IC}_\nu$ and suppose IC_χ appears in $\mathrm{IC}_\lambda^{\star k}$ for some $k \in \mathbb{N}$.
- Thus $\chi \leq k\lambda$ and by the KLV Lemma, $\chi \in WF + \sum_{i=1}^k W\mu$.
- By the **geometric PRV**, IC_χ appears as a direct summand of $\mathrm{IC}_\mu^{\star k} \star \mathcal{A}$.
- We have proved: there exists $\mathcal{A}$ such that for all $k \in \mathbb{N}$, if $[\mathrm{IC}_\chi]$ appears in $[\mathrm{IC}_\lambda]^{\star k}$, then it appears in $[\mathrm{IC}_\mu]^{\star k} \star \mathcal{A}$. Thus by the KLV Lemma, we have proved $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$.
- Conversely, assume $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$. By (iv) of the KLV Lemma, by looking at supports we see that $\exists \nu \in X_+^\vee$ that that

  $$\overline{\mathcal{O}}_{k\lambda} \subset \overline{\mathcal{O}}_{k\mu+\nu}$$

  for infinitely many $k \in \mathbb{N}$. This implies $\lambda \preceq \mu$.

7.2.4. *Additive and $\preceq$ structures match.*

- We have proved $\preceq$ and $\preceq'$ correspond.
- For $\lambda, \mu \in X_+^\vee$, we claim that $[\mathrm{IC}_\lambda] + [\mathrm{IC}_\mu] = [\mathrm{IC}_{\lambda+\mu}]$ in X'_+. Indeed, $[\mathrm{IC}_\lambda] + [\mathrm{IC}_\mu]$ is the class of the maximal element appearing in $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$, which is $[\mathrm{IC}_{\lambda+\mu}]$, as one sees from the definition of $\star$.
- It remains to show that $\leq$ and $\leq'$ correspond. It is enough to show that under $X_+^\vee \tilde{\to} X'_+$, we have $Q_+^\vee \tilde{\to} Q'_+$.
- For this the key is a **geometric Δ Lemma**.

Lemma 7.2.1 (Geometric Δ Lemma)**.** *A coweight $\lambda \in X^\vee$ has $\lambda \in \Delta^\vee$ if and only if it is $\preceq$-minimal among nonzero coweights for which there exists a $\mu \in X_+^\vee$ with $2\mu - \lambda \in X_+^\vee$ and $\mathrm{IC}_{2\mu-\lambda}$ appears in $\mathrm{IC}_\mu^{\star 2}$.*

7.2.5. *Proof of Lemma 7.2.1.*

- Proof of "only if": Assume $\lambda = \alpha^\vee \in \Delta^\vee$. Let $\mu \in X_+^\vee$ be such that $\langle \alpha, \mu \rangle \geq 1$
- We clearly have $2\mu - \alpha^\vee \in X_+^\vee$, since $\langle \beta, \alpha^\vee \rangle < 0$ for any $\beta \in \Delta \backslash \{\alpha\}$. Next, we need to show that $\mathrm{IC}_{2\mu-\alpha^\vee}$ appears in $\mathrm{IC}_\mu^{\star 2}$.

Lemma 7.2.2 (Geometric multiplicity Lemma)**.** *In the above situation,* $\mathrm{IC}_{2\mu-\alpha^\vee}$ *appears in* $\mathrm{IC}_\mu^{\star 2}$.

Proof. First consider the case $\langle \alpha, \mu \rangle = 1$. Then $s_\alpha(\mu) = \mu - \alpha^\vee$, and $2\mu - \alpha^\vee = \mu + s_\alpha(\mu)$. By the **Geometric PRV**, we see that $\mathrm{IC}_{2\mu-\alpha^\vee}$ appears in $\mathrm{IC}_\mu^{\star 2}$.

Now consider the case where $\langle \alpha, \mu \rangle \geq 2$.

Let $S_{2\mu-\alpha^\vee}$ be the LU-orbit of $t^{2\mu-\alpha^\vee} * \in \mathrm{Gr}_G$. Consider the $\mu_\bullet = (\mu, \mu)$ and the convolution morphism $m_{\mu_\bullet} : \bar{\mathcal{O}}_\mu \tilde{\times} \bar{\mathcal{O}}_\mu \to \bar{\mathcal{O}}_{2\mu}$.

We can write $\mathrm{IC}_\mu \star \mathrm{IC}_\mu = \oplus_{\lambda \leq 2\mu} V_{\mu_\bullet}^\lambda \otimes \mathrm{IC}_\lambda$. Take $\lambda = 2\mu - \alpha^\vee$.

By Lemma 6.4.1, it is enough to find at least one irreducible component of $m_{\mu_\bullet}^{-1}(S_{2\mu-\alpha^\vee} \cap \mathcal{O}_{2\mu-\alpha^\vee})$ having the maximal possible dimension

$$\langle \rho, 4\mu - \alpha^\vee \rangle.$$

We proceed with the following steps:

- We can write $m_{\mu_\bullet}^{-1}(S_{2\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_{2\mu})$ as the disjoint union
$$m_{\mu_\bullet}^{-1}(S_{2\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_{2\mu}) = (S_\mu \cap \bar{\mathcal{O}}_\mu) \tilde{\times} (S_{\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_\mu) \coprod (S_{\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_\mu) \tilde{\times} (S_\mu \cap \bar{\mathcal{O}}_\mu).$$
- The factor $S_{\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_\mu$ contains (details below) the irreducible subset $\mathcal{S}$
$$\Big(\prod_{\substack{\gamma>0 \\ \langle \gamma, \mu-\alpha^\vee \rangle > 0}} (\prod_{i=0}^{\langle \gamma, \mu-\alpha^\vee \rangle - 1} U_{\gamma+i}) \cdot U_{\alpha-1} \Big) t^{\mu-\alpha^\vee} * .$$
- The $\Big(\cdots \Big)$ term lies in $L^+U \cdot U_{\alpha-1}$ and its intersection with $\mathrm{Stab}_{LU}(t^{\mu-\alpha^\vee} *) = t^{\mu-\alpha^\vee} \, L^+U \, t^{-(\mu-\alpha^\vee)}$ is trivial. **To handle $U_{\alpha-1}$, this is where we use the assumption** $\langle \alpha, \mu \rangle \geq 2$.
- Thus $\mathcal{S}$ is irreducible of dimension $\langle 2\rho, \mu - \alpha^\vee \rangle + 1 = \langle \rho, 2\mu - \alpha^\vee \rangle$. Also, $S_\mu \cap \bar{\mathcal{O}}_\mu$ is irreducible of dimension $\langle \rho, 2\mu \rangle$, and has a similar description, like $\mathcal{S}$.
- Since $t^\mu U_{\alpha-1} t^{-\mu} \subset L^+U$, the factor contains an irreducible component of dimension $\langle \rho, 2\mu + 2\mu - \alpha^\vee \rangle$ which is entirely contained in $m_{\mu_\bullet}^{-1}(S_{2\mu-\alpha^\vee} \cap \mathcal{O}_{2\mu-\alpha^\vee})$. This proves the Geometric Multiplicity Lemma (Lemma 7.2.2).

□

Remark 7.2.3. Here we give more details about the inclusion $U_{\alpha-1} t^{\mu-\alpha^\vee} \subset S_{\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_\mu$, under the assumption $\langle \alpha, \mu \rangle \geq 1$ (we don't need ≥ 2 here). This boils down to a computation of 2×2 matrices over $k(\!(t)\!)$. Fix integers a, b with $a - b \geq 1$, and let $x \in k$. Then we have

$$\begin{bmatrix} 1 & xt^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^{a-1} & 0 \\ 0 & t^{b+1} \end{bmatrix} = \begin{bmatrix} t^{a-1} & xt^b \\ 0 & t^{b+1} \end{bmatrix}.$$

By looking at valuations of minors, we see that this belongs to $K \begin{bmatrix} t^a & 0 \\ 0 & t^b \end{bmatrix} K$ where $K = \mathrm{GL}_2(k[\![t]\!])$, whenever $x \neq 0$.

This implies that $U_{\alpha-1} t^{\mu-\alpha^\vee} \subset S_{\mu-\alpha^\vee} \cap \bar{\mathcal{O}}_\mu$.

7.2.6. *End of proof of Geometric* Δ *Lemma* (*Lemma 7.2.1*). We proceed as follows:

- Now we return to the proof of the Geometric Δ Lemma. We will argue in a way that is very parallel to the proof of the Δ Lemma, with the Geometric Multiplicity Lemma playing the role that Kostant's weight multiplicity theorem played before.
- Although the argument is parallel, let us spell out again how it works. Recall we start with $\lambda = \alpha^\vee \in \Delta^\vee$, and we may choose any $\mu \in X_+^\vee$ such that $\langle \alpha, \mu \rangle \geq 2$. We clearly have $2\mu - \alpha^\vee \in X_+^\vee$, since $\langle \beta, \alpha^\vee \rangle < 0$ for any $\beta \in \Delta \backslash \{\alpha\}$.

- Now **the geometric multiplicity lemma shows that** $\mathrm{IC}_{2\mu-\alpha^\vee}$ **must appear in** $\mathrm{IC}_\mu^{\star 2}$. Similarly to before, we see that $\alpha^\vee$ is **minimal nonzero** such that a μ with the required properties exists. This proves the "only if" direction of the Geometric Δ Lemma.
- Conversely, assume $\lambda \in X^\vee$ is minimal nonzero with the property that a μ with the required properties exists. So we have $2\mu-\lambda \in X^\vee_+$ and $\mathrm{IC}_{2\mu-\lambda}$ appears in $\mathrm{IC}_\mu^{\star 2}$. This implies $\lambda \in Q^\vee_+$.
- By Stembridge, there exists a positive coroot $\beta^\vee$ with
$$2\mu - \lambda \leq 2\mu - \beta^\vee < 2\mu$$
and all three are dominant.
- Write $\beta^\vee = \sum_i a_i\alpha_i^\vee$ where $a_i \in \mathbb{N}$ and $\alpha_i \in \Delta$ for all i.
- As before, for some i with $a_i \geq 1$, we have $\langle \alpha_i, \mu\rangle \geq 1$. This means $2\mu - \alpha_i^\vee \in X^\vee_+$ and (as in "only if", relying on the full strength of the Geometric Multiplicity Lemma), $\mathrm{IC}_{2\mu-\alpha_i^\vee}$ appears in $\mathrm{IC}_\mu^{\star 2}$.
- By minimality, we must have $\alpha_i^\vee = \lambda$, that is, $\lambda \in \Delta^\vee$. This proves the Geometric Δ Lemma (Lemma 7.2.1). □

7.2.7. *End of the proof of the geometric reconstruction theorem.*

- Finally, we can complete the proof of the Reconstruction Theorem, which identifies H with $\widehat{G}$.
- Recall, we were almost done, and only had to show that $Q'_+ = Q^\vee_+$. This follows from $\Delta' = \Delta^\vee$.
- This last identification follows immediately if we put together the Δ Lemma and the Geometric Δ Lemma. □

Remark 7.2.4. If you look at [Ri14], you will notice that the Geometric Δ Lemma does not appear there, and instead the argument looks simpler. However, there is a mistake in one small computation in [Ri14]: for $\alpha \in \Delta$, and $\mu \in X_+$ with $\langle \mu, \alpha^\vee\rangle = 2$, it is asserted that $s_\alpha(\mu) = \mu - \alpha$; however this is false and in fact $s_\alpha(\mu) = \mu - 2\alpha$. This mistake has serious consequences, as it is no longer possible to assert that the PRV Conjecture shows that $V_{2\mu-\alpha}$ appears in $V_\mu \otimes V_\mu$ (since $2\mu - \alpha$ is not equal to $\mu + s_\alpha(\mu)$, and in fact only $2\mu - 2\alpha = \mu + s_\alpha(\mu)$). A similar state of affairs occurs in using the geometric PRV Conjecture. Therefore, a different argument is needed and this is the origin of our Δ-Lemma (resp. our geometric Δ Lemma).

7.3. **Beilinson-Drinfeld Grassmannians.** We make some closing remarks about how the Beilinson-Drinfeld Grassmannians allow for the construction of a fusion product on ULA perverse sheaves, which yields the commutativity constraints and the $\otimes$-preservation of the fiber functor $R^*\Gamma(-)$.

7.3.1. *Definition.* The main remaining difficulty is to prove the commutativity constraint, and for this the standard method is to express the convolution of perverse sheaves on $\mathrm{Gr}_{\mathcal{G}}$ in terms of the fusion product of certain sheaves on the Beilinson-Drinfeld Grassmannian.

For the moment we let k denote any finite or separably closed field. Let X be a smooth geometrically connected curve over k (for simplicity, we assume it is quasi-projective). Given $R \in \mathrm{Aff}_k$ set $X_R := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. Let Σ be the presheaf on Aff_k defined by

$$\Sigma(R) = \{D \subset X_R \text{ relative effective Cartier divisors (in particular } D \to \mathrm{Spec}(R) \text{ is finite and flat)}\}.$$

Note that Σ is represented by the ind-scheme $\coprod_n X^n/S_n$.

For the next material, we assume $G = \mathcal{G}$ is split, so we can consider it as a constant group scheme over the curve X. In the final version, this assumption will be removed but the definitions look slightly different, see [HR20b].

Definition 7.3.1. Let $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}$ be the presheaf on Aff_k defined by sending a k-algebra R to the set $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}}(R)$ of isomorphism classes of triples $(D, \mathcal{F}, \beta)$ where

- $D \in \Sigma(R)$
- $\mathcal{F}$ is a $\mathcal{G}$-torsor on X_R
- $\beta : \mathcal{F}|_{X_R \setminus D} \xrightarrow{\sim} \mathcal{F}_0|_{X_R\setminus D}$, a trivialization, i.e., $\mathcal{F}_0$ is the trivial $\mathcal{G}$-torsor.

Proposition 7.3.2. *The forgetful functor* $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}} \to \Sigma$ *is representable by an ind-projective ind-scheme over* Σ.

Proof. A detailed proof appears (also in the non-split context) in [HR20b, §3.8 – 3.10]. Here is a rough outline:

- First show $\mathrm{Gr}^{\mathrm{BD}}_{\mathrm{GL}_n} =$ "$R[\![D]\!]$-lattices in $R(\!(D)\!)^n$" is representable
- Show that $\mathcal{G} \hookrightarrow \mathcal{H}$ being a closed immersion of groups such that $\mathcal{H}/\mathcal{G}$ is quasi-affine (resp. affine) implies that $\mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G}} \to \mathrm{Gr}^{\mathrm{BD}}_{\mathcal{H}}$ is represented by a quasi-compact immersion (resp, closed immersion).

□

7.3.2. *Global loop groups and relation to usual affine Grassmannians.* We need global versions $\mathcal{L}G$ and $\mathcal{L}^+\mathcal{G}$ of LG and $L^+\mathcal{G}$, respectively.

For $D \in \Sigma(R)$, let $\mathrm{Spf}(\hat{\mathcal{O}}_{X,D})$ denote the formal completion of X_R along D. (This is the **definition** of the R-algebra $\hat{\mathcal{O}}_{X,D}$.) Let $\hat{D} := \mathrm{Spec}(\hat{\mathcal{O}}_{X,D})$ and $\hat{D}^\circ = \hat{D} \backslash D$.

Definition 7.3.3. Define the fpqc sheaves $\mathcal{L}G$ and $\mathcal{L}^+\mathcal{G}$:

$$\mathcal{L}G : R \mapsto \{(s, D) \mid D \in \Sigma(R),\ s \in G(\hat{D}^\circ)\}$$
$$\mathcal{L}^+\mathcal{G} : R \mapsto \{(s, D) \mid D \in \Sigma(R),\ s \in \mathcal{G}(\hat{D})\}.$$

The following is the Beauville-Laszlo theoorem in this context:

Lemma 7.3.4. *We have the following statements:*

(i) $\mathcal{L}G$ *is an ind-group scheme over* Σ*, and represents the functor on* k*-algebras parametrizing isomorphisms classes of quadruples* $(D, \mathcal{F}, \beta, \sigma)$ *where* $(D, \mathcal{F}, \beta)$ *are as before, and* $\sigma : \mathcal{F}_0|_{\hat{D}} \xrightarrow{\sim} \mathcal{F}|_{\hat{D}}$ *is an isomorphism.*
(ii) $\mathcal{L}^+\mathcal{G}$ *is an affine group scheme over* Σ *with geometrically connected fibers.*
(iii) $\mathcal{L}G \to \mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G},X}$, $(D, \mathcal{F}, \beta, \sigma) \mapsto (D, \mathcal{F}, \beta)$ *is a right* $\mathcal{L}^+\mathcal{G}$*-torsor and induces an isomorphism of fpqc-sheaves over* Σ
$$\mathcal{L}G/\mathcal{L}^+\mathcal{G} \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G},X}.$$

Proof. WLOG X is affine. Then fppf-locally on R any $D \in \Sigma(R)$ is of form $V(f)$. Then the moduli part of (i) follows from the Beauville-Laszlo descent lemma. Ind-representability follows from (ii, iii). This proves (i).

Part (ii): Let $D^{(i)}$ the the i-th infinitesimal neighborhood of D . Note that

$$\mathcal{L}^+\mathcal{G} \times_{\Sigma,D} \mathrm{Spec}(R) \xrightarrow{\sim} \varprojlim_i \mathrm{Res}_{D^{(i)}/R}(\mathcal{G}).$$

This proves (ii).

Part (iii): The main point is that after an fpqc cover $R \to R'$, any $\mathcal{G}$-torsor on $\hat{D}_R$ aquires a section over $\hat{D}_{R'}$. This follows because $\mathcal{F}$ admits an R'-section, which extends over $\hat{D}_{R'}$ by smoothness (and affineness of $\mathcal{F}$) – this is a form of Grothendieck's Algebraization (Existence) Theorem. □

Abbreviate $\mathcal{G}r := \mathrm{Gr}^{\mathrm{BD}}_{\mathcal{G},X}$ from now on.

Remark 7.3.5. The group $\mathcal{L}G$ acts on $\mathcal{G}r$ over $D \in \Sigma(R)$ as follows: for $g \in G(\hat{D})$,

$$(g, D) \cdot (\mathcal{F}, \beta, D) = (g\mathcal{F}, g\beta, D)$$

where the right hand side is defined using Beauville-Laszlo gluing. We also have a union operation

$$\cup\ :\ \Sigma \times \Sigma \to \Sigma$$
$$(D_1, D_2) \mapsto D_1 \cup D_2.$$

Remark 7.3.6. Here is the connection with the usual affine Grassmannian. Note that X identifies with a connected component of Σ. Any point $x \in X(k)$ is identified with $D_x \in \Sigma(k)$, and $\hat{D}_x \cong$

Spec($k[\![t]\!]$), where t be a local parameter of X in x. Then as fpqc-sheaves we identify the fibers over $x \in \Sigma$:

$$\mathcal{L}G_x \cong LG$$
$$\mathcal{L}^+\mathcal{G}_x \cong L^+\mathcal{G}$$
$$\mathcal{G}r_{,}x \cong \mathrm{Gr}_{\mathcal{G}}.$$

7.3.3. *Convolution diagram towards fusion.*

Definition 7.3.7. For $k \geq 1$ we define the convolution version $\widetilde{\mathcal{G}r}_k$ to be the ind-scheme over Σ^k sending a k-algebra R to the set of isomorphism classes of tuples $((D_i, \mathcal{F}_i, \beta_i)_{i=1,\dots,k})$ with

- $D_i \in \Sigma(R)$
- $\mathcal{F}_i$ is aG-torsor on X_R
- $\beta_i : \mathcal{F}_i|_{X_R \setminus D_i} \xrightarrow{\sim} \mathcal{F}_{i-1}|_{X_R \setminus D_i}$, $i = 1, \dots, k$

Proposition 7.3.8. *The morphism $\widetilde{\mathcal{G}r}_k \to \Sigma^k$ is represented by a strict ind-scheme which is ind-proper over Σ^k.*

Let $\mathcal{L}^+G_X := \mathcal{L}^+G \times_\Sigma X$, $\mathcal{L}G_X := \mathcal{L}G \times_\Sigma X$, and $\mathcal{G}r_X := \mathcal{G}r \times_\Sigma X$. Similarly base-changing along $X^I \to \Sigma$ we defined $\mathcal{L}^+G_I = \mathcal{L}^+G_{X^I}$, etc.

There is a class of **ULA perverse sheaves** $P^{\mathrm{ULA}}_{\mathcal{L}^+G_X}(\mathcal{G}r_X)$ on $\mathcal{G}r$ which have good properties as below. Namely, for each finite set I there is a convolution diagram

$$\textstyle\prod_i \mathcal{G}r_{X,i} \xleftarrow{p_I} \mathcal{L}G_I \xrightarrow{q_I} \widetilde{\mathcal{G}r}_I \xrightarrow{m_I} \mathcal{G}r_I.$$

with the usual good properties. The first key result which leads to fusion and the commutativity constraint is the following.

Let $j_I : U_I \hookrightarrow X^I$ be the open embedding of the locus of points with pairwise distinct coordinates.

Proposition 7.3.9. *Suppose $\mathcal{A}_i \in P^{\mathrm{ULA}}_{\mathcal{L}^+G_X}(\mathcal{G}r_X)$, for $i \in I$. There exists a unique ULA perverse sheaf $\widetilde{\boxtimes}_i\mathcal{A}_i$ on $\widetilde{\mathcal{G}r}_I$ such that*

$$p_I^*(\boxtimes_i \mathcal{A}_i) \cong q_I^*(\widetilde{\boxtimes}_i \mathcal{A}_i)$$

and moreover

$$m_{I,*}\big(\widetilde{\boxtimes}\mathcal{A}_i\big) \cong j_{I,!*}\big(\boxtimes_i \mathcal{A}_i \,|\, U_I\big).$$

But this still lives on X^I, not X.

7.3.4. *Fusion product.* We define the k-fold fusion product on $P^{\mathrm{ULA}}_{\mathcal{L}^+G_X}(\mathcal{G}r_X)$, by considering a diagram with $|I| = k$

$$\begin{tikzcd}
\mathrm{Gr}_X \arrow[r, "i_I"] \arrow[d] & \mathcal{G}r_I \arrow[d] & \mathrm{Gr}^I_X|U_I \arrow[l, "j_I"'] \arrow[d] \\
X \arrow[r, "\Delta"] & X^I & U_I. \arrow[l, "j_I"']
\end{tikzcd}$$

Definition 7.3.10. If $|I| = k$ and we fix a total order on I, set

$$*_i\mathcal{A}_i = i_I^*[-k+1]j_{I,!*}\big(\boxtimes_i \mathcal{A}_i \,|\, U_I\big).$$

This manifestly obeys a commutativity constraint. This ends up forcing the commutativity constraint on $(P_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}), \star)$, and it also plays a role in the proof that the fiber functor $\omega = R^*\Gamma(-)$ is a $\otimes$-functor.

In the final version of these notes, this will be explained. Moreover, the definition of the fusion product will be given and it will be related to the above. However, there will be nothing new in this part, because this topic is already treated well in the articles of Baumann-Riche, Richarz, Riche, and Xinwen Zhu.

8. Appendices

8.0.1. *Proof of Cartan, Iwasawa, and refined Iwasawa decomposition from the Bruhat-Tits decomposition.*

8.0.2. *Remarks about the thick subcategory consisting of the essential image of $p^*[d]$.* Ths section will describe a general criterion for descent of a simple perverse sheaf along $p^*[d]$.

8.0.3. *Proof of Lemma 3.3.11.* This section describes a general criterion of descent of suitably equivariant perverse sheaves for a functor of the form $p^*[d]$.

Write $\pi : G_Y \to Y$ for the structure morphism. Assume temporarily that f has a section $s : Y \to X$. This gives an morphism $a : G_Y \to X$ by sending $g \in G_Y \times_Y X$ to $g \cdot s(\pi(g)) \in X$.

Lemma 8.0.1. *The morphism a is smooth.*

Proof. Since a is locally of finite presentation it is enough to show it is flat and all geometric fiber are smooth over the corresponding residue field (see [BLR90, 2.4, Prop. 8]). The geometric fibers of a are isomorphic to the stabilizer subgroups $G_{Y,x}$ (which are smooth by hypothesis). So it is enough to show that a is flat.

First, a is surjective (this is clear on geometric points, which suffices). We have for any point $y \in Y$ a surjective morphism $G_Y \times_Y \{y\} \to X_y$. Since the left hand side is a smooth and connected group scheme, the right hand side is irreducible.

Now $G_Y \to Y$ is flat, so by the fiberwise flatness criterion (see [StaPro, Lem. 37.16.4]) it is enough to show that $G_y \times_Y \{y\} \xrightarrow{a_y} X_y$ is flat for all $y \in Y$. But a_y is generically flat (since X_y is reduced hence integral, being smooth over $\mathrm{Spec}(k(y))$ hence flat everywhere on X_y by the transitivity of the group action on the fibers. □

Let $A : G_Y \times_Y X \to X$ denote the action morphism. We have a commutative diagram

$$\begin{tikzcd}
G_Y \times_Y X \arrow[r, shift left, "A"] \arrow[r, shift right, "p_2"'] & X. \\
G_Y \times_Y X \arrow[u, "\mathrm{id}\times s"] \arrow[ur, shift left, "a"] \arrow[ur, shift right, "sfa"'] &
\end{tikzcd}$$

So for any equivariant sheaf $\mathcal{F}$, we have an isomorphism $A^*\mathcal{F} \cong p_2^*\mathcal{F}$, which implies

$$a^*\mathcal{F} \cong a^* f^* s^* \mathcal{F}.$$

Since a and $f \circ a$ are both smooth with geometrically connected fibers, we deduce that $s^*\mathcal{F}$ is perverse (up-to-shift). Indeed, $(fa)^*$ kills all **other** ${}^p\mathrm{H}^i(s^*\mathcal{F})$, and $(fa)^*$ (suitably shifted) is a fully faithful conservative functor. Then $f^*s^*\mathcal{G}$ is also perverse (up-to-shift). Now applying the full-faithfulness of a^* (suitably shifted), we deduce that

$$\mathcal{F} \cong f^* s^* \mathcal{F},$$

meaning that $\mathcal{F}$ descends to Y.

In general, there exists an étale cover $U \to Y$ such that $f_U : X \times_Y U \to U$ has a section. Using the $G_Y \times_Y U$-group action, the above shows that $\mathcal{F}_U$ descends to a unique perverse (up-to-shift) sheaf on U. Now using ([BBD82, 3.2.4]) to glue the perverse sheaves on U, we get the desired descent of $\mathcal{F}$. □

8.0.4. *Proof of α_2-equivariance and descent.* **Here we prove that $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is α_2 equivariant, and the α_2-action of $L^+\mathcal{G} \times L^+\mathcal{G}$ on $LG \times LG$ satisfies the axioms of Lemma 3.3.11, so that there exists a unique perverse sheaf $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2$ with $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong q^*(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)$.**

We abbreviate $K = L^+\mathcal{G}$, $L = LG$, and write the constant sheaf as $\mathbf{1} = \bar{\mathbb{Q}}_\ell$ (on any space, which is understood from context). We also switch to left actions instead of right actions. Recall the α_1 action of $K \times K$ on $L \times L$ is given by $(k_1, k_2) \cdot (x, y) = (xk_1^{-1}, yk_2^{-1})$. The action α_2 is given by $(k_1, k_2) \cdot (x, y) = (xk_1^{-1}, k_1 y k_2^{-1})$.

Definition 8.0.2. We define the morphism
$$a : K \times K \times K \times L \times L \to L \times L$$
$$(k, k'', k', x, y) \mapsto (xk^{-1}, k''yk'^{-1}).$$

Note that $a = a_2 \circ (p_2 \times \alpha_1)$, where

$$p_2 \times \alpha_1 : (k, k'', k', x, y) \mapsto (k'', \alpha_1(k, k', x, y))$$
$$a_2 : (k'', x, y) \mapsto (x, k''y).$$

We claim that the left K-equivariance of $\mathcal{F}_2$ implies that $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is α_2-equivariant. By abuse of notation, write p as
$$p \times p' : L \times L \to \mathrm{Gr}_{\mathcal{G}} \times \mathrm{Gr}_{\mathcal{G}}.$$
Then we see that $p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2$ is a_2-equivariant. Therefore,
$$a_2^*(p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2) \cong 1 \boxtimes p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2.$$
By α_1-equivariance,
$$\alpha_1^*(p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2) \cong 1 \boxtimes 1 \boxtimes p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2,$$
hence
$$(p_2 \times \alpha_1)^*(1 \boxtimes p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2) \cong 1 \boxtimes 1 \boxtimes 1 \boxtimes p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2,$$
Therefore
$$a^*(p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)) \cong 1 \boxtimes 1 \boxtimes 1 \boxtimes p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2. \tag{8.0.1}$$
But $\alpha_2 = a \circ \Delta_1$, where
$$\Delta_1 : (k, k', x, y) \mapsto (k, k, k', x, y).$$
Pulling back (8.0.1) with Δ_1^*, we get
$$\alpha_2^*(p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)) \cong 1 \boxtimes 1 \boxtimes p^*\mathcal{F}_1 \boxtimes p'^*\mathcal{F}_2,$$
as desired. □

(It remains to check the action α_2 satisfies the axioms of Lemma 3.3.11 – see final version of these notes...)

8.0.5. *Proof that convolution morphisms are locally trivial in the stratified sense.* **The point is this is for the non-split case, so without big cells....**

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