Lecture No 15 April 26, 2022 (Tue)

Relation between PDE and Brown motion:

- As a simple application of Itô's formula and Doob's optional sampling theorem, we give a probabilistic representation of the solution of $\Delta u = 0$ on a bounded set $D$ with Dirichlet boundary condition.

[Theorem 13.2] (Kakutani's theorem) Let $D \subset \mathbb{R}^d$ be a bounded open set and let $u \in C^2(D) \cap C(\bar{D})$ satisfy

$$\begin{cases} \Delta u(x) \equiv \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0, & x \in D \\ u(x) = f(x), & x \in \partial D, \end{cases}$$

with $f \in C(\partial D)$. For simplicity, we assume that $u$ can be extended to $D^c$ such that $u \in C_b^2(\mathbb{R}^d)$.

$\Longrightarrow$

$\Longrightarrow$ We have

$$u(x) = E_x[f(B_\sigma)], \quad x \in D,$$

where $E_x$ denotes the expectation for the $d$-dimensional Brown motion $(B_t)$ starting at $x$ and $\sigma$ is the first hitting time to the boundary $\partial D$:

$$\sigma := \inf\{t > 0; B_t \in \partial D\}$$

(As before, we set $\sigma = \infty$, if the set $\{\cdots\}$ inside of inf is $\emptyset$.) Actually, $\sigma < \infty$ a.s. holds. In particular, the solution of the above Dirichlet problem is unique. □

- Later, Theorem 13.2 will be extended from $\Delta$ to more general elliptic differential operators by means of stochastic differential equations.

[Proof of Theorem 13.2] • First, we show $\sigma < \infty$ a.s.

$\because$ Since $D$ is bounded, ${}^{\exists}R > 0$ s.t. $D \subset B(x, R)$, which is a closed ball with center $x$ and radius $R > 0$. Therefore,

$$P_x(\sigma = \infty) \leq P_x(\sigma_{\partial B(x,R)} = \infty) \leq P_x(\sigma_{\partial B(x,R)} > t) \quad ({}^{\forall}t > 0)$$
$$\leq P_0(B_t \in B(0, R)) = \int_{B(0,R)} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} dx \underset{t\to\infty}{\longrightarrow} 0 \quad \square$$

• Since we assume $u \in C_b^2(\mathbb{R}^d)$, one can apply Itô's formula to $u(B_t)$ and obtain, in the form of stochastic differential:

$$du(B_t) = \frac{\partial u}{\partial x_i}(B_t)dB_t^i + \frac{1}{2}\frac{\partial^2 u}{\partial x_i \partial x_j}(B_t)dB_t^i dB_t^j.$$

However, recalling $dB_t^i dB_t^j = \delta^{ij} dt$ and writing in integrated form, we obtain

$$u(B_t) = u(x) + \sum_{i=1}^{d} \int_0^t \frac{\partial u}{\partial x_i}(B_s)dB_s^i + \frac{1}{2}\int_0^t \Delta u(B_s)ds.$$

In particular, $u(B_{t\wedge\sigma})$ is a bounded martingale.
(filtration $(\mathcal{F}_t)$ is the same as that for $(\mathcal{F}_t)$-Brown motion $B$)

$\because$

- boundedness follows from $u \in C_b^2(\mathbb{R}^d)$ (or $u$ is bounded on $\bar{D}$)
- Denote the stochastic integral in the RHS of $u(B_t)$ by $M_t$. Then, $M$ is a martingale. However, since $\sigma$ is a Markov time, by Doob's optional sampling theorem, $M_{t\wedge\sigma}$ is a martingale.
- Denote the last term in the RHS of $u(B_t)$ by $A_t$. Then, since $\Delta u(B_s) = 0, s < \sigma$, we see $A_{t\wedge\sigma} = 0$. □

Therefore, we obtain

$$u(x) \underset{^\forall t>0}{=} E_x[u(B_{t\wedge\sigma})] \underset{t\to\infty}{\longrightarrow} E_x[u(B_\sigma)] = E_x[f(B_\sigma)].$$

Indeed,

- the 1st identity follows from the martingale property shown above,
- the limit as $t \to \infty$ is shown by Lebesgue's convergence theorem noting that $\sigma < \infty$ a.s. and $u$ is bounded.
- The last equality follows from $B_\sigma \in \partial D$ and $u = f$ on $\partial D$. In fact, "$B_\sigma \in \partial D$" is shown noting that, by the definition of $\sigma$, $^\exists t_n \downarrow \sigma$ s.t. $B_{t_n} \in \partial D$ and $\partial D$ is a closed set.

This concludes the proof of Theorem 13.2. □

## §14 Stochastic differential equations

- For a given time-dependent vector field $b = b(t, x)$ on $\mathbb{R}^d$, one can consider an ordinary differential equation (ODE) for $X_t \in \mathbb{R}^d$:

$$\dot{X}_t \left(= \frac{dX_t}{dt}\right) = b(t, X_t).$$

- The stochastic differential equation (SDE) is formally a randomization of ODE:

$$\dot{X}_t = b(t, X_t) + \alpha(t, X_t)\dot{B}_t.$$

- $\dot{B}_t$ actually does not exist, but formally it is given by $\dot{B}_t = \lim_{h\downarrow 0} \frac{1}{h}(B_{t+h} - B_t)$ so that, by the independence of increments, $\dot{B}_t$ is considered to be independent for different $t$.
- In other words, we consider the system with an external noise which is created independently for each $t$.
- $\alpha = \alpha(t, x)$ describes the strength of noise at $(t, x)$.

- More precisely, let continuous functions

$$\alpha : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times N} \text{ denoted by}$$
$$\alpha = \alpha(t, x) \equiv (\alpha^i_k(t, x))_{1\le i\le d, 1\le k\le N},$$
$$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \text{ denoted by}$$
$$b = b(t, x) \equiv (b^i(t, x))_{1\le i\le d}$$

  and

  $B_t = (B^k_t)_{1\le k\le N}$: $N$-dimensional $(\mathcal{F}_t)$-Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$

  be given.

- We denote $\mathbb{R}^{d\times N} = \{d \times N \text{ real matrices}\}$.

- Then, stochastic differential equation (SDE) is the following formal equation for $X_t = (X_t^i)_{1 \le i \le d} \in \mathbb{R}^d$, written in terms of stochastic differentials

$$dX_t = \alpha(t, X_t)\, dB_t + b(t, X_t)\, dt \tag{1}$$

or written componentwisely

$$dX_t^i = \sum_{k=1}^{N} {\alpha^i}_k(t, X_t)\, dB_t^k + b^i(t, X_t)\, dt, \quad 1 \le i \le d.$$

- We call $\alpha$ the diffusion coefficient and $b$ the drift coefficient.

Mathematically, the SDE (1) is defined in an integrated form:

[Definition 14.1] $X = (X_t)_{t\geq 0}$ is called a solution of the SDE (1) starting at $x \in \mathbb{R}^d$, if $X$ is an $(\mathcal{F}_t)$-adapted and measurable (as a map $X : [0,\infty) \times \Omega \to \mathbb{R}^d$) $\mathbb{R}^d$-valued continuous stochastic process defined on $(\Omega, \mathcal{F}, P)$ and satisfies

1. For ${}^\forall i, k$, $\alpha^i{}_k(t, X_t) \in \mathcal{L}^2(\mathcal{F}_t), b^i(t, X_t) \in L^1_{\text{loc}}([0,\infty))$ a.s. (i.e. $\int_0^T |b^i(t, X_t)|\, dt < \infty, \forall T \geq 0$ a.s.) and
2. the following stochastic integral equation is satisfied:

$$X_t = x + \int_0^t \alpha(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds \tag{2}$$

or componentwisely,

$$X_t^i = x^i + \sum_{k=1}^N \int_0^t \alpha^i{}_k(s, X_s)\, dB_s^k + \int_0^t b^i(s, X_s)\, ds,\ 1 \leq i \leq d.$$

Especially, $(\mathcal{F}_t^B)$-adapted solution is called a strong solution. □

[Remark] • By the condition 1, each term in the RHS of (2) has meaning. • Strong solution is a solution.

- In the theory of ODEs, the simplest condition to guarantee the existence and uniqueness of a solution is the Lipschitz condition for the coefficients.
- Similar theorem holds also for the SDEs, which is a stochastic version of Cauchy's theorem.
- Euclidean norms of $\alpha \in \mathbb{R}^{d\times N}$ and $b \in \mathbb{R}^d$ are defined by

$$||\alpha|| := \sqrt{\sum_{i=1}^{d}\sum_{k=1}^{N}({\alpha^i}_k)^2}, \qquad |b| = \sqrt{\sum_{i=1}^{d}(b^i)^2}.$$

[Theorem 14.1] Assume that the coefficients $\alpha, b$ are continuous in $(t, x)$ and (globally) Lipschitz continuous in $x$, i.e. for $^\forall T > 0$, there exists $K = K_T > 0$ such that

$$||\alpha(t,x) - \alpha(t,y)|| + |b(t,x) - b(t,y)| \leq K|x-y|$$

for $^\forall t \in [0, T]$, $^\forall x, y \in \mathbb{R}^d$. Then, for each starting point $x$, a strong solution $X = (X_t^i)$ of (1) exists and the solution is unique. i.e. If two solutions $X, X'(\in \mathcal{L}_T^2)$ with the same starting point exist, then $P(X_t = X_t', {}^\forall t \geq 0) = 1$ holds. We call such uniqueness the pathwise uniqueness. □

- The Lipschitz condition and the continuity of the coefficients imply the linear growth condition (which is non-explosion condition for the solution):

  $$||\alpha(t,x)|| + |b(t,x)| \leq K'(1+|x|), \quad t \in [0, T], x \in \mathbb{R}^d,$$

  with some $K' = K_T'$.

[Proof of Theorem 14.1]

[Step 1] Construction of strong solution:

• It is enough to consider for $t \in [0, T]$ for every fixed $T > 0$. We apply Picard's successive approximation. Namely, we set $X_t^{(1)} = x$ for $n = 1$ and then, if $X_t^{(n-1)}$ is determined, we define $X_t^{(n)}, n \geq 2$ by

$$X_t^{(n)} = x + \int_0^t \alpha(s, X_s^{(n-1)})\, dB_s + \int_0^t b(s, X_s^{(n-1)})\, ds. \quad (3)$$

- First we show by induction that $X^{(n)} \in \mathcal{L}_T^2$ and $X_t^{(n)}$ is $(\mathcal{F}_t^B)$-adapted. (Once this is shown, we see that the stochastic integral in (3) is well-defined.)
  - Indeed, this is obvious for $n = 1$.
  - If this is shown for $n-1$, since $\alpha$ is of linear growth, by $X^{(n-1)} \in \mathcal{L}_T^2$, we see $\alpha(s, X_s^{(n-1)}) \in \mathcal{L}_T^2$ so that the stochastic integral in (3) is well-defined and $\in \mathcal{L}_T^2$ is obtained by Itô isometry.
  - For the drift term, one can show it belongs to $\mathcal{L}_T^2$.
  - Thus, we observe $X^{(n)} \in \mathcal{L}_T^2$.
  - The $(\mathcal{F}_t^B)$-adaptedness of $X_t^{(n)}$ follows by considering the stochastic integral under the filtration $(\mathcal{F}_t^B)$.

- To show the convergence of $X^{(n)}$, for $^\forall n \geq 3$, $t \in [0, T]$,

$$
\begin{aligned}
&E\left[\sup_{0\le r\le t}|X_r^{(n)}-X_r^{(n-1)}|^2\right]\\
\le&2E\left[\sup_{0\le r\le t}\left|\int_0^r\{\alpha(s,X_s^{(n-1)})-\alpha(s,X_s^{(n-2)})\}\,dB_s\right|^2\right]\\
&\qquad+2E\left[\sup_{0\le r\le t}\left|\int_0^r\{b(s,X_s^{(n-1)})-b(s,X_s^{(n-2)})\}\,ds\right|^2\right]\\
\le&8E\left[\int_0^t||\alpha(s,X_s^{(n-1)})-\alpha(s,X_s^{(n-2)})||^2\,ds\right]\\
&\qquad+2tE\left[\int_0^t|b(s,X_s^{(n-1)})-b(s,X_s^{(n-2)})|^2\,ds\right]\\
\le&C_1\int_0^t E\left[|X_s^{(n-1)}-X_s^{(n-2)}|^2\right]ds\le C_1\int_0^t E\left[\sup_{0\le r\le s}|X_r^{(n-1)}-X_r^{(n-2)}|^2\right]ds,
\end{aligned}
$$

where $C_1 = (8 + 2T)K_T^2$. For the 1st inequality, we first compute $X_r^{(n)} - X_r^{(n-1)}$ by (3) and then use $(a + b)^2 \leq 2(a^2 + b^2)$, $a, b \in \mathbb{R}$. For the 2nd, we apply Doob's inequality, Itô isometry and Schwarz's inequality. For the 3rd, we use the Lipschitz continuity of $\alpha, b$.

• We have shown for $^\forall n \geq 3$, $t \in [0, T]$:

$$E\left[\sup_{0\leq r\leq t} |X_r^{(n)} - X_r^{(n-1)}|^2\right] \leq C_1 \int_0^t E\left[\sup_{0\leq r\leq s} |X_r^{(n-1)} - X_r^{(n-2)}|^2\right] ds$$

From this estimate and by induction, we obtain

$$E\left[\sup_{0\leq r\leq t} |X_r^{(n)} - X_r^{(n-1)}|^2\right] \leq C_2 \frac{(C_1 t)^{n-2}}{(n-2)!}, \quad ^\forall t \in [0, T], ^\forall n \geq 2 \qquad (4)$$

where $C_2 := E[\sup_{0\leq t\leq T} |X_t^{(2)} - X_t^{(1)}|^2] < \infty$. (This can be easily checked.) Thus, by using Schwarz's inequality, we have

$$E\left[\sum_{n=2}^{\infty} \sup_{0\leq t\leq T} |X_t^{(n)} - X_t^{(n-1)}|\right] = \sum_{n=2}^{\infty} E\left[\sup_{0\leq t\leq T} |X_t^{(n)} - X_t^{(n-1)}|\right]$$

$$\leq \sum_{n=2}^{\infty} E\left[\sup_{0\leq t\leq T} |X_t^{(n)} - X_t^{(n-1)}|^2\right]^{\frac{1}{2}} \leq \sum_{n=2}^{\infty} a_n < \infty,$$

where $a_n := \sqrt{C_2 \frac{(C_1 T)^{n-2}}{(n-2)!}}$. Note that the last series converges, since $\frac{a_{n+1}}{a_n} = \sqrt{\frac{CT}{n-1}} \to 0$.

In particular, we have shown that

$$\sum_{n=2}^{\infty} \sup_{0\le t\le T} |X_t^{(n)} - X_t^{(n-1)}| < \infty \quad \text{a.s.}$$

This implies that $X_t^{(n)}$ is a Cauchy sequence in the space $(C([0, T], \mathbb{R}^d), \|\cdot\|_\infty)$ a.s. so that it converges uniformly on $[0, T]$ a.s. Set the limit $X_t$. Then, it is a strong solution of the SDE(1). Indeed,

- The limit $X_t$ is $(\mathcal{F}_t^B)$-adapted, since $X_t^{(n)}$ are $(\mathcal{F}_t^B)$-adapted.

- By the estimate (4), $X_t^{(n)}$ is a Cauchy sequence also in the space $\mathbb{L}_T^2 = L^2([0, T] \times \Omega)$ so that $X^{(n)} \to X$ in $\mathcal{L}_T^2$. Thus, by the Lipschitz continuity of $\alpha$ and Itô isometry, we have

$$\int_0^t \alpha^i{}_k(s, X_s^{(n-1)})\, dB_s^k \longrightarrow \int_0^t \alpha^i{}_k(s, X_s)\, dB_s^k \quad \text{in } \mathcal{M}_{c,T}^2,$$

where $\mathcal{M}_{c,T}^2$ equips the inner product $E[\langle\cdot,\cdot\rangle_T]$. In particular, this convergence holds in $L^2(\Omega)$ for any fixed $t \in [0, T]$.

• On the other hand, for the drift term, since $X_s^{(n-1)} \to X_s$ uniformly in $s \in [0, t]$ (a.s.), we have

$$\int_0^t b^i(s, X_s^{(n-1)})\, ds \longrightarrow \int_0^t b^i(s, X_s)\, ds \quad \text{a.s.}$$

• Thus, letting $n \to \infty$ (along a subsequence) in (3), we see that (2) holds for ${}^\forall t \in [0, T]$ (a.s.) However, since the both sides are continuous in $t$, we obtain

$$P((2) \text{ holds for } {}^\forall t \in [0, T]) = 1,$$

which shows that $X_t$ is a desired strong solution.

[Step 2] Uniqueness of the solution: Assume that we have two solutions $X, X' \in \mathcal{L}_T^2$. Then, as above, we obtain

$$E[|X_t - X_t'|^2] \le C_1 \int_0^t E[|X_s - X_s'|^2]\, ds, \quad t \in [0, T].$$

Thus, by Gronwall's lemma stated below, we have $E[|X_t - X_t'|^2] = 0$. Recalling the continuity of $X, X'$ in $t$, we finally obtain $P(X_t = X_t' \text{ for } {}^\forall t \ge 0) = 1$. □

We recall Gronwall's lemma which is introduced and used in the usual course of ODEs.

[Lemma 14.2] Assume $u(t) \geq 0$ is a continuous function on $[0, T]$ and satisfies with ${}^{\exists}C_1, C_2 \geq 0$ that

$$u(t) \leq C_1 + C_2 \int_0^t u(s)\, ds, \quad t \in [0, T].$$

Then, we have $u(t) \leq C_1 e^{C_2 t}, t \in [0, T]$. □