Classical limits of associative algebras

Goal – to reproduce classical physics.

1. Algebra of dif. operators
2. High spin system as classical limit, the inverse spin $\frac{1}{j}$ would be a "Plank constant"

Specifically, we are aiming at

$$\frac{dF}{dt} = \{H, F\},$$ where F and H are functions on the phase space Φ,

$\{\ ,\ \}$ – Poisson brackets on the space Φ

$$\{F, G\} = \pi^{ij} \frac{\partial F}{\partial \varphi^i} \frac{\partial G}{\partial \varphi^j},$$ where π^{ij} – Poissonian bivector

$$\pi^{ij} = (\omega^{-1})^{ij},$$ where ω is a simplectic form.

classical mechanics.

Example 1.

Algebra of diff. operators (let us start with 1 var.)

Elements are $\sum_{k=0}^{N} a_k(x) \left(\frac{\partial}{\partial x}\right)^k$

It is clear that products of D.O. is a dif. operator →

$a_k(x)\left(\frac{\partial}{\partial x}\right)^k b_\ell(x)\left(\frac{\partial}{\partial x}\right)^\ell$ → we just

$$\to\ a_k(x)\, b_\ell(x) \left(\frac{\partial}{\partial x}\right)^{k+\ell} + \underline{\quad\quad} \left(\frac{\partial}{\partial x}\right)^{k+\ell-1} + \dots$$

However, it is just an algebra $\Rightarrow$ we need a family of algebras

modification $DA_\hbar$:

$$\sum_{k=0}^{N} \hbar^k a_k(x) \left(\frac{\partial}{\partial x}\right)^k$$

Commutator of two elements

$$\left[\hbar^k a_k(x)\left(\frac{\partial}{\partial x}\right)^k, \hbar^\ell b_\ell(x)\left(\frac{\partial}{\partial x}\right)^\ell\right]:$$

$$\hbar^k a_k(x)\left(\frac{\partial}{\partial x}\right)^k \hbar^\ell b_\ell(x)\left(\frac{\partial}{\partial x}\right)^\ell =$$

$$= \hbar^{k+\ell} \underset{(1)}{a_k(x)} b_\ell(x) \left(\frac{\partial}{\partial x}\right)^{k+\ell} +$$

$$+ \hbar^k \hbar^\ell \underset{(2)}{a_k(x)} \cdot k \frac{\partial b_\ell(x)}{\partial x} \cdot \left(\frac{\partial}{\partial x}\right)^{k+\ell-1} + \dots (\dots)$$

Term (1) is just like multiplication of pol. $a_k(x) p^k \cdot b_\ell(x) p^\ell =$

$= a_k(x) b_\ell(x) p^{k+\ell}$ – it is commutative.

Term (2): put it in a standard form in $DA_\hbar$:

$$(2) = \hbar\left(\hbar^{k+\ell-1} a_k(x) \cdot k \frac{\partial b_\ell(x)}{\partial x} \left(\frac{\partial}{\partial x}\right)^{k+\ell-1}\right.$$

$$(\dots) = \hbar^2(\;\dots\;) + \hbar^3(\;\dots\;) \text{ etc.}$$

Result of the ex: commutator of elements from $DA_\hbar$ is the following:

$\alpha \in DA_\hbar \Rightarrow$ its symbol $s\alpha$ – pol. in p

$$\sum_k \hbar^k a_k(x)\left(\frac{\partial}{\partial x}\right)^k \longrightarrow \sum_k a_k(x) p^k \in \mathrm{Fun}(\Phi)$$ with coord x and p

$$[\alpha, \beta] = \hbar\, C(\alpha,\beta) + \hbar^2(\ldots)$$

$$SC(\alpha,\beta) = \frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial p}$$ it is a Poissonian bracket on the space of functions of x and p for $\omega = dp \wedge dx$

This is a classical (actually, quasi-classical limit of a commutator)

Another way to look at $DA_\hbar$ is to say that it is generated by $f(x)$ and operators $\hbar\frac{\partial}{\partial x}$.

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Now, we will consider another example.

(example above ~~was~~ is used in 95% of books, now we consider the equally important example, studied in 5% of books).

Consider the vector space of pol. of degree N in two variables z_0 and z_1.

(sections of the bundle $\mathcal{O}(N)$ on $\mathbb{C}P^1$).

It has dimension $N+1$.

$V_N = \mathrm{Span}\{z_0^N, z_0^{N-1}z_1, \ldots, z_1^N\}$

Consider homog. dif. operators of the form: (4 of them)

$$z_i \frac{\partial}{\partial z_j} \qquad i,j = 0,1$$

1 of them $\to$ Euler operator acts in a noninteresting way

$$\mathcal{E} = z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1}$$

$$\mathcal{E}\, V_N = N \cdot V_N$$

Other are intersting – form an $sl(2)$ algebra

$$T_3 = z_1 \frac{\partial}{\partial z_1} - z_0 \frac{\partial}{\partial z_0}$$

$$T_+ = z_1 \frac{\partial}{\partial z_0} \qquad T_- = z_0 \frac{\partial}{\partial z_1}$$

T_+ – rizes the degree of z_1 by 1.

T_- – lowers . . . . 1.

T_3 - computes the difference.

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we would like to make V an Herm. vector space $\to$ we need a metric on the space of P.

$$\langle P_1, P_2 \rangle = \int\limits_{\mathbb{C}^2\times\mathbb{C}^2} \underbrace{d^2 z_0\, d^2 z_1}\, \overline{P_1(z)}\, P_2(z)\, e^{-|z_0|^2 - |z_1|^2}$$

– H. metric.

$$\langle P_1, \frac{\partial}{\partial z_i} P_2 \rangle \to \langle z_i P_1, P_2 \rangle$$

$$\int \overline{P_1}\, \frac{\partial P_2}{\partial z_0}\, e^{-|z_0|^2} = \int \overline{P_1}\, \frac{\partial}{\partial z_0}\left(P_2\, e^{-|z_0|^2}\right) + \int \overline{P_1}\, \overline{z_0}\, P_2\, e^{-|z_0|^2} = \int \overline{(z_0 P_1)}\, P_2\, e^{-|z_0|^2}$$

0

Thus, $T_3 = z_1 \frac{\partial}{\partial z_1} - z_0 \frac{\partial}{\partial z_0}$ is Hermitean

while $T_+^+ = T_-$

Herm. operators are T_3,

$$T_1 = T_+ + T_-$$

$$T_2 = -i(T_+ - T_-)$$

For $N=1$ $\quad V = \text{Span}(z_0, z_1)$

$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

This set of H. O. is called Pauli matrices in the literature.

V_2 – 3×3 matrices

$$T_+ \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad T_- \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

However, there are other operators

$$T_+^2 \sim \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_-^2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

and even H. operators, like $T_+^2 + T_-^2$

It is instructive to compute commutats of T matrixes:

$$[T_3, T_+] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2T_+$$

$$[T_3, T_\pm] = \pm 2 T_\pm$$

$$[T_+, T_-] = T_3.$$

$\to$ 2·algebra of $SO(3)$

So, it terms of T_i:

$$[T_a, T_b] = i\epsilon_{abc} T_c.$$

$SO(3)$ or $SU(2)$

T operators acting on V_N and V_N is a representation of $SU(2)$

Computation:

$$z_0^2, \quad z_0 z_1 \quad z_1^2$$

$$T_+ = z_1 \frac{\partial}{\partial z_0} \qquad T_+(z_0^2) = 2 z_0 z_1$$

$$T_+(z_0 z_1) = z_1^2$$

V_N is a representation of $SU(2)$ the lowest eigenvalue of T_3 is $-N$ (by convention, it is called $2j$)

j is called a spin of the represent.

Example: $V_1 = \mathrm{Span}(z_0, z_1)$ $\quad N=1$, $j=\frac{1}{2}$

$V_2 = \mathrm{Span}(z_0^2, z_1 z_0, z_1^2)$ $\quad N=2$, $j=1$

The dim $V_N = N+1$

$\dim V_N = 2j+1$

Originally, people considered only representations of $SO(3)$ not $SU(2)$

And these were representations for even N

like V_2 was 3-dim — vector of $SO(3)$

V_4 was 5-dim representation

and so on: $S^2 V_2$ — sym. matrices

m_{ab} $\quad a=1,2,3$, $b=1,2,3$

$$3 \cdot 4/2 = 6$$

However, $\mathrm{Tr}\, m$ was $SO(3)$ invariant.

so repr. was 5-dimensional.

(in terms of $SU(2)$ representations — it corr. to even N)

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change the algebra, but in a simple fashion: $T_a \to T_a^c = \frac{T_a}{N}$

In particular, eigenvalues of T_3 in V_N were $(-N, \dots, N)$

eigenvalues of $T_3^c \Rightarrow \underbrace{(-1, \dots, 1)}_{N+1 \text{ values.}}$

and in this way T_3^c somehow tends to what we may call a coordinate on an interval, whose eigenvalues are a segment]

$[T_a^c, T_b^c] = i \frac{\epsilon_{abc}}{N} T_c^c$, so when $N \to \infty$

we were approaching commutative algebra

(1) The spectrum of corresponding commutative algebra

(2) The Poissonian structure on this spectrum.

$\boxed{\text{Issue (1)}}$ There is well-known formula for so-called <u>quadratic Casimir operator</u> on V_N

$C = (T_1^2 + T_2^2 + T_3^2)$. 1) It commutes with all T_a –

– you may expect that it is a scalar, i.e. just a multiplication by number.

At the same time, it is a dif. operator of order not greater than 2.

There is a technical exp to get C in terms of E^2 and E, and then substitute E^2 by N.

One may compute that this could be 2

$$C = N(N+1)$$

For $T_a^c = \frac{T_a}{N}$

$\sum_{a=1}^{3} (T_a^c)^2 = 1 + O(\frac{1}{N})$, so in the limit $N \to \infty$ T_a^c become commuting variables, X_a, with the relation $\sum_{a=1}^{3} X_a^2 = 1$. And this is exactly a sphere S^2 in $\mathbb{R}^3$.

Now, we may compute the first term in $\frac{1}{N}$ in the commutator, and see that $[T_a^c, T_b^c] = \frac{1}{N} \epsilon_{abc} T_c^c$

This implies the following — consider the algebra A_N^c generated by T_a^c (acting on the given space V_N).

$\sum d^{a_1 \ldots a_n} T_{a_1}^c \ldots T_{a_n}^c \subset A_N^c$ — universal covering algebra with c-modification

A_N acts on V_N
(I do not study all operators acting on V_N, I study only operators, generated by T_a^c)

The difference is N-dependence.

coefficient α should be either N independent or go as $\left(\frac{1}{N}\right)^K$.

Then Herm. α would correspond to functions on S^2 (actually $\dfrac{\mathbb{R}(x_1, x_2, x_3)}{I(x_1^2+x_2^2+x_3^2=1)}$)

and

$$[\alpha_1, \alpha_2] = \frac{1}{N} i \cdot \{\alpha_1, \alpha_2\}$$

where $\{\alpha, \beta\} = \underbrace{\frac{\partial \alpha}{\partial x_a} \frac{\partial \beta}{\partial x_b} \epsilon_{abc} X^c}_{SU(2)}$

Poissonian bivector of the form

$\{Y_A, Y_B\} = f_{ABC} Y^C$, for f_{ABC} being a structure constant of a simple Lie algebra is known as a Kirillov Poisson bracket.

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statement, that symplectic form, corr. to this P. B. is a round Fubini-Study 2-form

$$\omega_{FS} = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$, where $S^2 = \mathbb{C} \cup \{\infty\}$ and z is a standard coord. on $\mathbb{C}$.