

Recent developments in Seiberg-Witten theory

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(~~Lecture 1: Basics of SW theory~~) DONE!
Lecture 2 : ① Finite-lim approx. ② Diff vs. Homeo in 4D
Lecture 3 : SW Floer homotopy theory

Today. { § 1. Finite-lim approximation
§ 2. Diff vs. Homeo in 4D

§ 1. Finite-dim approximation (Furuta, Bauer-Furuta)

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Overview.

An approximation of the SW equations

($\longleftrightarrow$ a map between
Hilbert spaces)

$\rightsquigarrow$
yields

(i) An invariant of (spin^c) 4-manifolds
(called the Bauer-Furuta invariant)

$\rightsquigarrow$ the SW invariant.
relates

(ii) Constraints on the intersection form
(including Donaldson's diagonalization).

Review

X^4 : oriented closed smooth 4-manifold
equipped with a Riemannian metric.

$\mathcal{G}$: a spin structure on X (classified by $H^2(X; \mathbb{Z})$)

$$\left(\begin{array}{c} S^{\pm} \\ \downarrow \mathcal{D}^2 \\ X \end{array}, \begin{array}{c} L \\ \downarrow \mathcal{D} \\ X \end{array} \right) \quad \mathcal{Q}^2(X)$$

$$\rightsquigarrow SW : \underbrace{A(L) \oplus P(S^+)}_{\text{v(1)-conn.}} \rightarrow i \underbrace{\mathcal{Q}^+(X)}_{\parallel} \oplus P(S^-)$$

$\downarrow$
sw eq

$$\hookrightarrow \{ * \omega = \omega \}$$

$$\mathcal{G}(X) = \text{Map}(X, \mathcal{V}(1)).$$

Assume $b_1(x) = 0$ (for simplicity)

Output:

As a finite-dim. approx, we'll get:



$$f: V^+ \rightarrow W^+$$

$\swarrow \quad \searrow$
 $S' \quad S'$

1 point compactification

where $\begin{cases} V = V(\mathbb{R}) \oplus V(\mathbb{C}) \\ W = W(\mathbb{R}) \oplus W(\mathbb{C}) \end{cases}$ finite dim real and complex vector spaces

$S' \hookrightarrow V(\mathbb{R}), W(\mathbb{R})$ trivial

$\hookrightarrow V(\mathbb{C}), W(\mathbb{C})$ multiplication

$$\begin{cases} \dim_{\mathbb{R}} V(\mathbb{R}) - \dim_{\mathbb{R}} W(\mathbb{R}) = \operatorname{ind}_{\mathbb{R}} (d^{\dagger} \oplus d^{\dagger}: \mathcal{Q}' \rightarrow \mathcal{Q}' \oplus \mathcal{Q}'') - 1 = b^{+} \\ \dim_{\mathbb{C}} V(\mathbb{C}) - \dim_{\mathbb{C}} W(\mathbb{C}) = \operatorname{ind}_{\mathbb{C}} D_A = \frac{c(L)^2 - \sigma(X)}{8} \end{cases}$$

How to get f :  picking a reference connection

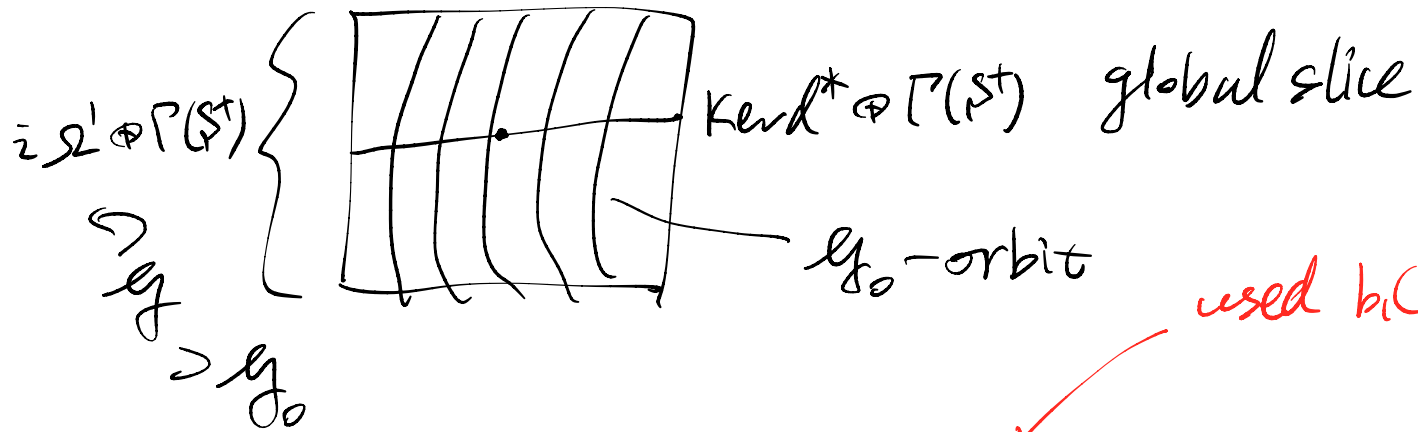
$$\mathcal{A}(L) \oplus \mathcal{P}(S^{\dagger}) \cong i \mathcal{Q}'(X) \oplus \mathcal{P}(S^{\dagger})$$

$$g_0 := \{ u \in \mathcal{G} \mid u(x_0) = 1 \}$$

$\uparrow$
 $x_0 \in X$: a point.

$$1 \rightarrow g_0 \hookrightarrow \mathcal{G} \xrightarrow{ev_{x_0}} S' \rightarrow 1.$$

$\uparrow$
based gauge group.



used $b_i(X) = 0$.

$$\begin{array}{ccc}
 (\text{ker} d^* \oplus P(S^+)) \times g_0 & \xrightarrow{\cong} & i\Omega' \oplus P(S^+) \\
 \cap & & \uparrow \\
 \Omega' & & g_0\text{-equiv.}
 \end{array}$$

$$\text{coker}(g_0 \hookrightarrow g)$$

$$\parallel$$

$$S' \hookrightarrow \Omega^* \text{ trivially}$$

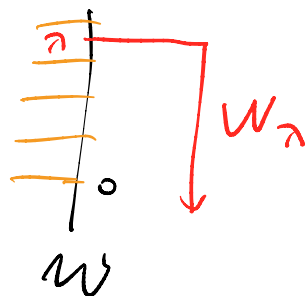
$\hookrightarrow P(S^\pm)$ scalar multiplication.

$$\begin{array}{ccc} \textcircled{v} & & \textcircled{w} \\ \text{ii} & & \text{ii} \\ SW : \text{Kend}^* \oplus P(S^+) \longrightarrow i \mathcal{R}^+ \oplus P(S^-) : S'\text{-equiv.} \\ \textcircled{\cup} & & \textcircled{\cup} \\ L^*L & & LL^* \end{array}$$

$$L := "dSW_{(0,0)}", \quad L^* : \text{formal adjoint}$$

$\{ \text{eigenvalues of } L^*L (= LL^*) \} \in \mathbb{R}_{\geq 0}$

Take $\lambda > 0$


$$V_\lambda, W_\lambda : \text{finite dim}$$

$$SW : V \rightarrow W$$

$L + \underbrace{Q}_{\text{the non-linear part of the SW eq. (quadratic)}}$

$P_\lambda : W \rightarrow W_\lambda : L^2\text{-orthogonal proj.}$

$$\boxed{f_\lambda := P_\lambda \circ SW = \underbrace{L + P_\lambda Q}_{L(V_\lambda) \subset W_\lambda} : \underbrace{V_\lambda}_{\substack{V \\ \parallel \\ S^1}} \rightarrow \underbrace{W_\lambda}_{\substack{W \\ \parallel \\ S^1 \oplus P(S^{\pm})}}}$$

$$\begin{cases} V = V(\mathbb{R}) \oplus V(\mathbb{C}) \\ W = W(\mathbb{R}) \oplus W(\mathbb{C}) \end{cases}$$

Compare f with SW

$$\left\{ L + (t \text{ id} + (1-t))\alpha : V \rightarrow W \right\}_{t \in [0,1]}$$

$(t \in [0,1])$

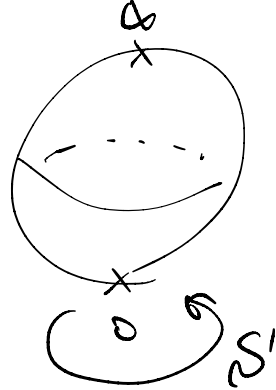
$\left[\begin{array}{l} \mathcal{M} = SW^{-1}(0)/\mathcal{G} : \text{compact} \\ \downarrow \text{a bit more} \\ f \text{ is proper } (\lambda \gg 0) \end{array} \right.$

by perturbing
this 1-parameter
family

$f^{-1}(0)/S \sim \mathcal{M}$
cobordant

$$\leadsto f : V^+ \rightarrow W^+ : \underline{\underline{\text{continuous}}}$$

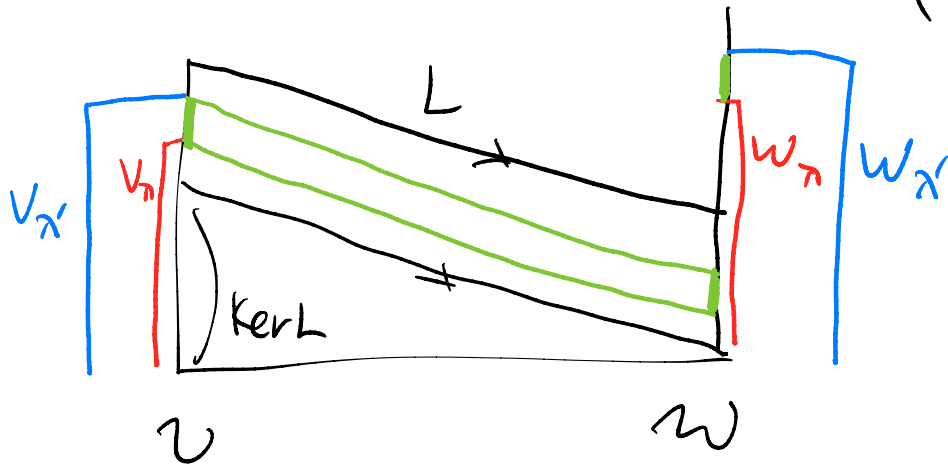
$\swarrow \searrow$
 S'



① Bauer-Furuta invariant.

Ambiguity of ... choices of

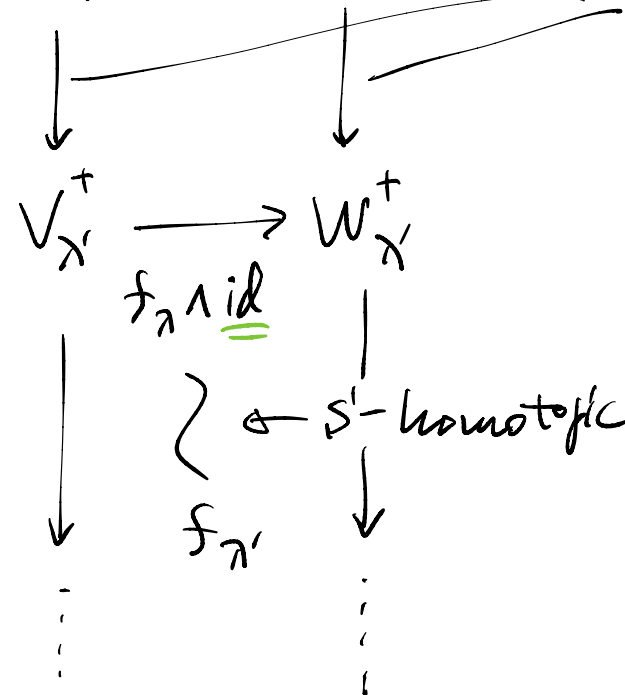
- $$\left\{ \begin{array}{l} \text{① fin. dim. approx.} \\ \text{② metric} \end{array} \right.$$



$$L|_{\ker L^\perp} : \ker L^\perp \hookrightarrow \text{Im } L$$

$$V_\lambda^+ \xrightarrow{f_\lambda} W_\lambda^+$$

$$\llcorner \underline{\underline{\lambda(V_{\lambda'} / V_\lambda)^+}} \rceil$$



S' -stable homotopy class

$$\leadsto \lim_{\lambda \rightarrow \infty} [f_\lambda : V_\lambda^+ \rightarrow W_\lambda^+]_{S'}$$

!!

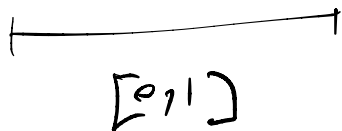
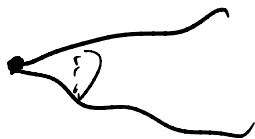
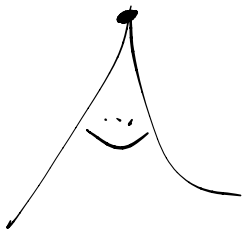
$$BF(X, S)$$

Bauer-Furuta invariant.

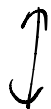
inductive system

$$\left[f_\lambda^{-1}(0) / S' \right]_{S'} \xrightarrow{\text{cobordant}} \mathcal{M}(g) (\lambda \gg 0)$$

$$b^+ = 0 \quad b^+ = 1$$



$$SW(X, S) \quad b^+ > 1$$



ordinary cohomology

Rem. You may apply any kind of
generalized cohomology theory to f

(such as K -theory) $\longrightarrow$ Furuta's $10/8$ -inequality.

$$V^+ \xrightarrow{f} W^+$$

(G-K-theoretic
Thom class)

$$\begin{array}{ccc} \exists \alpha \cup T_G^K(V) & \longleftarrow & T_G^K(W) \\ \cap & & \cap \\ K_G(V^+) & \xleftarrow{f^*} & K_G(W^+) \end{array}$$

$$\alpha \in K_G^*(pt) = R(G)$$

" K_G -degree of f "

$$G = S^1$$

$$\left(\begin{array}{l} = \text{Pin}(2) \quad 5 = \text{spin} \\ \downarrow \mathbb{Z}/2 \\ O(2) \end{array} \right)$$

$$\begin{array}{c} W \\ \downarrow \\ ft \end{array}$$

"Hurevitz map"

homotopy $\leftrightarrow$ homology.

$$\underline{b^+ > 1}$$

• How to derive Donaldson's diagonalization

$$b^+(X) = 0 \quad (\text{want: } Q_X : \text{diagonal})$$

$\mathbb{Z}/2$ -coefficient

$$\begin{array}{ccc}
 V^+ & \xrightarrow{f} & W^+ \\
 \uparrow & & \uparrow \\
 V^+(\mathbb{R}) & \xrightarrow[\cong]{f_{S'}} & W(\mathbb{R})^+ \\
 \nearrow & & \\
 b^+ = 0 & &
 \end{array}
 \quad \leadsto \quad
 \begin{array}{ccc}
 H_{S'}^*(V^+) & \xleftarrow{f^*} & H_{S'}^*(W^+) \supseteq \tau_{S'}(W^+) \\
 \downarrow & & \downarrow \\
 H_{S'}^*(V(\mathbb{R})^+) & \xleftarrow[\cong]{(f_{S'})^*} & H_{S'}^*(W(\mathbb{R})^+)
 \end{array}$$

S' -Thom class

$$\begin{array}{ccc}
 \leadsto \exists \underbrace{\alpha \cdot e_{S'}(V(\mathbb{C}))}_{\uparrow} = e_{S'}(W(\mathbb{C})) & \text{in } H_{S'}^*(pt) \\
 \uparrow & \uparrow S'\text{-Euler class} & \parallel \\
 H_{S'}^*(pt) & & \mathbb{Z}/2[\sigma]
 \end{array}$$

$$\leadsto \dim_{\mathbb{C}} V(\mathbb{C}) - \dim_{\mathbb{C}} W(\mathbb{C}) \leq 0 \quad (45)$$

$$\frac{\frac{c_1(L)^2 - \overline{\sigma(X)}}{8}}{\quad} = \text{ind}_{\mathbb{C}} \mathcal{P}_A$$

$\leadsto Q_X$: diagonal. //

Elkie's then

purely alg.
 (θ -functions)

$Q : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z} : \text{neg-def form}$

$\left[c \in \mathbb{Z}^n, Q(v,v) \equiv Q(c,v) \pmod{2} \Rightarrow Q(c,c) + n \leq 0 \right]$

$\Leftarrow \text{easy}$

$\Leftrightarrow Q \cong_{\mathbb{Z}} \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$

eg. $-E_8$ spin, neg-def, rank 8

5 : spin str on $-E_8$

$$\underbrace{c_1(5)^2 - \sigma(X)}_0 = 8 > 0.$$

cf. For smooth X ,

$$c_1(5)^2 - \sigma(X) \leq 0.$$

$$X = \#(-\mathbb{CP}^2) \quad \begin{cases} c_1 = E_1 \# \dots \# E_n \\ \sigma = -n \end{cases} \Rightarrow c_1^2 - \sigma = 0.$$

cf. moduli space.

$$b^+(X) = 0, \quad b_1(X) = 0$$

Suppose that Q_X is not diagonal.

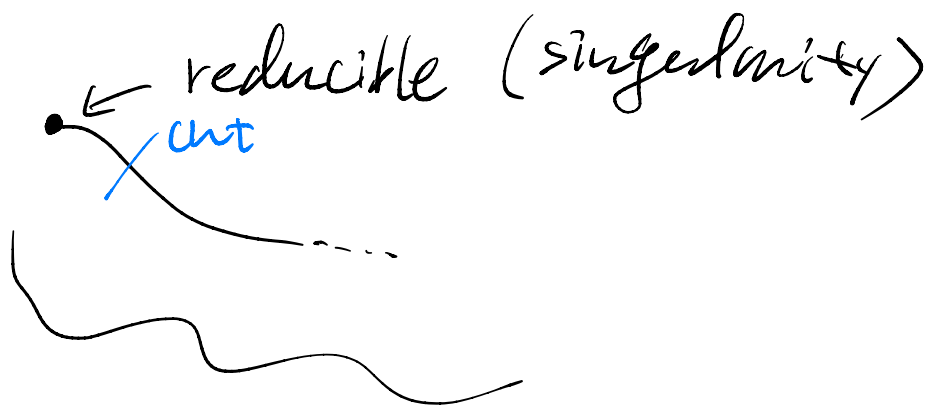
$$\Rightarrow \exists S, \quad \underbrace{c_1(S)^2 - \sigma(X)}_{\text{Elkies}} > 0.$$

$$\parallel$$
$$8 \cdot \text{ind}_{\mathbb{C}}^{\mathbb{D}}$$

$\Rightarrow \dim M$: odd positive integer

$$b_0(X) = 1$$

$$b^+ = 0$$



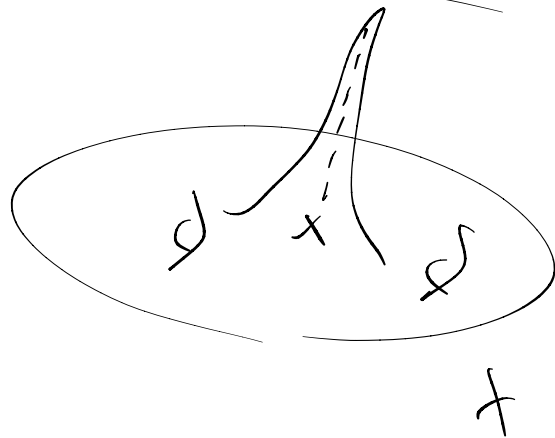
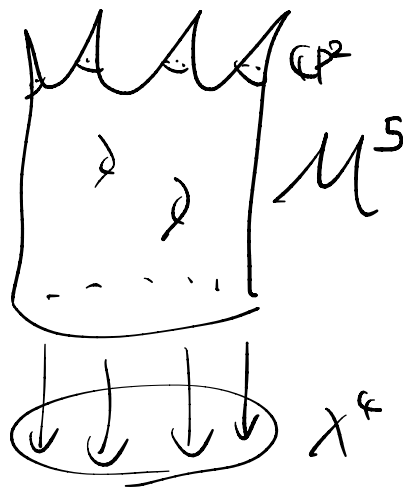
$$\dim M = 1$$

compact

contradiction

cf. Donaldson

gluing



§2. Diff vs. Homeo in 4D.

Slogan: C^0 -category & C^∞ -category
are essentially same in $\dim \leq 3$.

} automorphism version

Fact

X : orientable closed smooth manifold of $\dim \leq 3$
 $\Rightarrow \text{Diff}(X) \hookrightarrow \text{Homeo}(X)$ is a weak homotopy eqn

$\perp X^4$: 4-mfld, $\text{Diff}(X) \xrightarrow[\neq]{\hookrightarrow} \text{Homeo}(X)$.

For some

Then (Friedman-Morgan (1988), Donaldson (1990))

$\pi_0 \text{Diff}(X) \rightarrow \pi_0 \text{Homeo}(X)$ is not surjective

for $X = \begin{cases} \mathbb{CP}^2 \# n(-\mathbb{CP}^2) & \text{with } n > 9 \dots (\text{FM}) \\ \text{or} \\ K3 & \dots (\text{D.}) \end{cases}$

Then (Ruberman (1998), Baraglia-K. (2020),

Kronheimer-Mrowka (2020), J. Lin (2020))

$\pi_0 \text{Diff}(X) \rightarrow \pi_0 \text{Homeo}(X)$ is not injective

$$\text{for } X = \begin{cases} (2n) (\mathbb{C}P^2 \# m(-\mathbb{C}P^2)) & (n \geq 2, m \gg 0) & \dots R \\ n(K3 \# S^2 \times S^2) & (n \geq 2) & \dots B-K \\ K3 \# K3 & & \dots K-M \\ K3 \# K3 \# S^2 \times S^2 & & \dots L \end{cases}$$

Thm (B-k. 2019)

$\left[\pi_1 \text{Diff}(K3) \rightarrow \pi_1 \text{Homeo}(K3) \right]$ is not surjective.

cf. (Watanabe, 2018)

$\left[\pi_1 \text{Diff}(S^4) \rightarrow \pi_1 \text{Homeo}(S^4) \right]$ is not injective.

How to prove?

$\pi_n \text{Diff} \not\rightarrow \pi_n \text{Homeo}$:

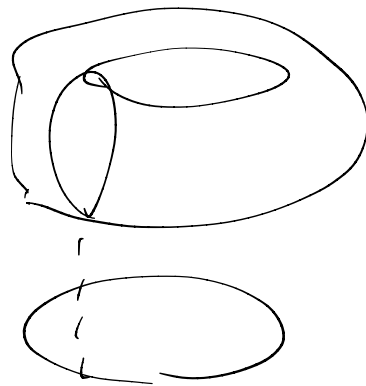
--- define invariants of $\begin{array}{ccc} X^q & \rightarrow & E \\ & & \downarrow \\ & & B \end{array} : \text{Diff}(X)\text{-bundle}$

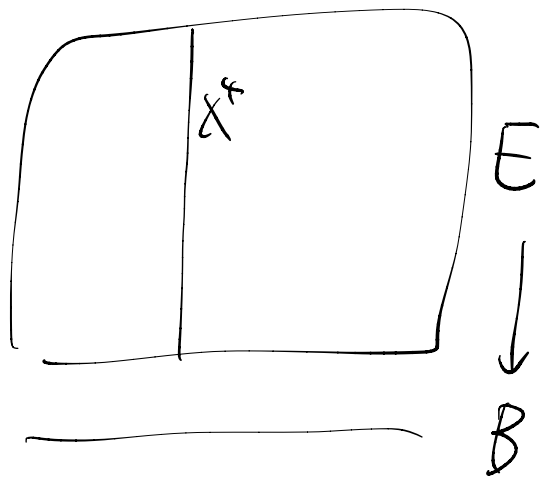
eg. $\text{sw} \begin{pmatrix} E \\ \downarrow X^q \\ S^1 \end{pmatrix} \neq \text{sw} \begin{pmatrix} E' \\ \downarrow X \\ S^1 \end{pmatrix}$

$\Rightarrow E \not\stackrel{co}{\cong} E'$

if $E \stackrel{co}{\cong} E'$

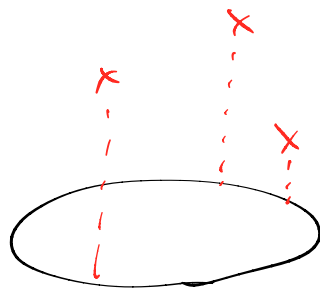
$\Rightarrow \pi_0 \text{Diff}(N) \rightarrow \pi_0 \text{Homeo}(N)$
is not injective.





$$\dim \mathcal{M} = -\dim B$$

"parametrized
moduli space over B "
is 0-dim.



$$\dim \mathcal{M} = -1$$

$$B = S^1$$

$\pi_n \text{Diff} \not\rightarrow \pi_n \text{Homeo}$:

... Give a constraint on

$$\begin{array}{ccc} X^q & \rightarrow & E \\ & & \downarrow \\ & & B \end{array} \quad \text{Diff}(X)\text{-bundle}$$

eg. $E \hookrightarrow H^+(X) \rightarrow E_{H^+}$ vect. bundle

$$\begin{array}{ccc} & & \downarrow \mathbb{R}^4 \\ & & B \end{array}$$

Get constraint on E_{H^+} .

$$f: V^+ \rightarrow W^+ \rightsquigarrow f: V^+ \rightarrow W^+ \quad \begin{array}{c} \text{family} \\ \text{of } f \end{array} \quad \begin{array}{ccc} & & \downarrow \checkmark \\ & & B \end{array} \quad \begin{array}{c} \text{vector} \\ \text{bundles} \end{array}$$