The formal moment map geometry of the space of symplectic connections

Day 2:

I) Fedosov star product

(M,ω) sympl mfd, $\nabla \in \mathcal{E}(M,\omega)$, $\Omega \in \Omega^2(M)[[\nu]]$ closed $\longmapsto$ $*_{\nabla,\Omega}$ a star product

A) Weyl algebra bundle

Def 7: The formal Weyl algebra bundle is the bundle of algebra over M:

$$\mathcal{W} = \coprod_{x\in M} W_x \cong \prod_p S^p T^*M[[\nu]].$$

$\downarrow$

M

Weyl algebra associated to T_xM

$\{ a(y,\nu) = \sum_{\substack{2k+r\\ \geq 0}} \nu^k a_{k,i_1\dots i_r} y^{i_1}\dots y^{i_r} \}$.

with $\circ$-product.

Structural group:

For $A \in Sp(T_xM,\omega_x)$ i.e. $A: T_xM \to T_xM$ linear isom $\omega_x(Au, Av) = \omega_x(u,v)$ $u,v\in T_xM$

12h

Define $\rho(A)\left(\sum_{2k+r\geq 0} \nu^k a_{k,i_1\dots i_r} y^{i_1}\dots y^{i_r}\right)$

$$:= \sum_{2k+r\geq 0} \nu^k a_{k,i_1\dots i_r} (A^{-1})^{i_1}_{j_1}\dots (A^{-1})^{i_r}_{j_r} y^{j_1}\dots y^{j_r}$$

One checks (exercise):

$$\rho(A)(a\circ b) = \rho(A)a \circ \rho(A) b.$$

Working with Darboux charts: Structural gp is $Sp(2n)$ symplectic lin gp.

Denote by $(\Gamma W, \circ)$ the algebra of sections of W.

Sections of W are of the form: (locally)

$$a(x,y,\nu) = \sum_{2k+r\geq 0} \nu^k \underbrace{a_{k,i_1\dots i_r}}(x)\, y^{i_1}\cdots y^{i_r}$$

are symmetric in the i's defining a tensor field on M.

The $\circ$-product on sections is defined fiberwisely:

$$(a\circ b)(x,y,\nu) = \left[\exp\left(\frac{\nu}{2}\Lambda^{ij}\partial_{y^i}\partial_{z^j}\right)a(x,y,\nu)\,b(x,z,\nu)\right]\Big|_{y=z}$$

The center of this algebra is $\underline{C^\infty(M)[[\nu]]}$.

$\subset \Gamma W$

B) Connections on the Weyl algebra.

Consider $D \in \mathcal{E}(M,\omega)$, it induces a connection / covariant derivative of section of $\mathcal{W}$.

Def 8: The space $\Gamma \mathcal{W} \otimes \Lambda^* M$ of $\mathcal{W}$-valued forms is the sections of $\mathcal{W} \otimes \Lambda^* M$, locally they write:

a q-forme writes as

$$\sum_{2k+q \geq 0} \nu^k \, a_{k, \underbrace{i_1 \dots i_p}_{\text{symmetric}}, \underbrace{j_1 \dots j_q}_{\text{anti-symmetric}}}(x)\, y^{i_1} \dots y^{i_p} dx^{j_1} \wedge \dots \wedge dx^{j_q}$$

The $\circ$-product extends to $\Gamma \mathcal{W} \otimes \Lambda^* M$ by

$$(\underset{\Gamma\mathcal{W}}{a} \otimes \underset{\Lambda^* M}{\alpha}) \circ (b \otimes \beta) = (a \circ b) \otimes \alpha \wedge \beta$$

$\Gamma \mathcal{W} \otimes \Lambda^* M$ is made into a graded Lie algebra wrt the $\circ$-graded commutator:

$$[s, s']_\circ = s \circ s' - (-1)^{q_1 q_2} s' \circ s \quad \text{for } \begin{array}{l} s \in \Gamma \mathcal{W} \otimes \Lambda^{q_1} M \\ s' \in \Gamma \mathcal{W} \otimes \Lambda^{q_2} M \end{array}$$

i.e. For $s \in \Gamma \mathcal{W} \otimes \Lambda^q$, $s' \in \Gamma \mathcal{W} \otimes \Lambda^{q'}$, $s'' \in \Gamma \mathcal{W} \otimes \Lambda^{q''}$

- $[s, s']_\circ \in \Gamma \mathcal{W} \otimes \Lambda^{q+q'}$
- $[s, s']_\circ = -(-1)^{qq'} [s', s]_\circ$
- $(-1)^{qq''} [s, [s', s'']] + (-1)^{qq'} [s', [s'', s]] + (-1)^{q'q''} [s'', [s, s']] = 0$.

The center of $\Gamma \mathcal{W} \otimes \Lambda^* M$ is $\Gamma \Lambda^* M [[\nu]]$

We go to the def of connection:

Exercise 1: Infinitesimal action of $sp(T_xM, \omega_x)$ on W_x.

For $A \in sp(T_xM, \omega_x)$ ie $\omega_{ik} A^k_j$ is symmetric in i, j.

One defines $\overline{A} := \frac{1}{2} \omega_{ik} A^k_j y^i y^j \in W_x$

Then $\rho_*(A) \underset{\substack{\uparrow \\ W_x}}{a(y,\nu)} = \frac{1}{\nu} [\overline{A}, a]$.

Def 9: The symplectic connection ∇ induces a connection on the bundle ΓW

$$\partial : \Gamma W \longrightarrow \Gamma W \otimes \Lambda^1 M$$

$$a \longmapsto \partial a \underset{\text{locally}}{:=} da + \frac{1}{\nu} [\overline{\Gamma}, a].$$

for $\overline{\Gamma} = \frac{1}{2} \underbrace{\omega_{lk} \underbrace{\Gamma^k_{ij}}_{\text{christoffel symbols of } \nabla}}_{\text{symm. in } l, j} y^l y^j dx^i$

∂ extends to a covariant exterior derivative by

$$\partial : \Gamma W \otimes \Lambda^q M \longrightarrow \Gamma W \otimes \Lambda^{q+1} M$$

$$\partial(\underset{\substack{\uparrow \\ \Gamma W}}{a} \otimes \underset{\substack{\uparrow \\ \Gamma \Lambda^q M}}{\alpha}) = \partial a \wedge \alpha + a \otimes d\alpha.$$

The curvature of ∂ is the 2-form with values in $\mathrm{End}(\Gamma W)$

$$\underset{\substack{\uparrow\\ \text{composition of } \partial}}{\partial \circ \partial} : \Gamma W \otimes \Lambda^q M \longrightarrow \Gamma W \otimes \Lambda^{q+2} M$$

<u>Prop 1</u>: The curvature $\partial \circ \partial$ is given by the formula:

$$\partial \circ \partial a = \frac{1}{\nu} [\overline{R}, a]_\circ$$

with $\overline{R} := \frac{1}{4} \omega_{ir} R^r_{jkl} \, y^i y^j \, dx^k \wedge dx^l$

$$R^r_{jkl} = [R(\partial_k, \partial_l)\partial_j]^r \qquad \partial_k = \frac{\partial}{\partial x^k}$$

<u>Proof</u>: Let $a \in \Gamma W$, we work locally on Darboux chart:

$$\partial \circ \partial a = \left(d + \tfrac{1}{\nu}[\overline{\Gamma}, \cdot]_\circ\right) \circ \left(d + \tfrac{1}{\nu}[\overline{\Gamma}, \cdot]_\circ\right) a$$

$$= \frac{1}{\nu} d[\overline{\Gamma}, a]_\circ + \frac{1}{\nu}[\overline{\Gamma}, da]_\circ + \frac{1}{\nu^2}[\overline{\Gamma}, [\overline{\Gamma}, a]_\circ]_\circ$$

graded Jacobi.

$$= \frac{1}{\nu}[d\overline{\Gamma}, a] + \frac{1}{\nu^2}\left[\frac{1}{2}[\overline{\Gamma}, \overline{\Gamma}]_\circ, a\right]_\circ$$

Let me compute

$$[\overline{\Gamma}, \overline{\Gamma}]_\circ = \frac{1}{4} \omega_{lk} \Gamma^l_{ij} \, \omega_{pq} \Gamma^q_{rs} [y^l y^j, y^p y^s]_\circ \, dx^i \wedge dx^r$$

$$\left(\overline{\Gamma} = \tfrac{1}{2} \omega_{lk} \Gamma^k_{ij} \, y^l y^j dx^i\right)$$

$$[y^l y^j, y^p y^s]_\circ = \left[\exp\left(\tfrac{\nu}{2} \Lambda^{ab} \partial_{y^a} \partial_{z^b}\right)(y^l y^j z^p z^s - y^p y^s z^l z^j)\right]\Big|_{y=z}$$

$$= \nu\left(\Lambda^{lp} y^j y^s + \Lambda^{jp} y^l y^s + \Lambda^{ls} y^j y^p + \Lambda^{js} y^l y^p\right)$$

(No contributions in ν^0, ν^2) because it is a commutator)

$$\frac{1}{2}[\overline{\Gamma}, \overline{\Gamma}]_\circ = \frac{\nu}{4} \omega_{qk} \left(\Gamma^k_{ij} \Gamma^q_{rs} - \Gamma^k_{rj} \Gamma^q_{is}\right) y^j y^s \, dx^i \wedge dx^r$$

$$R^k_{sir} = \partial_i \Gamma^k_{rs} - \partial_r \Gamma^k_{is} + \Gamma^k_{iq} \Gamma^q_{rs} - \Gamma^k_{rq} \Gamma^q_{is}$$

c) Flat connections on ΓW

Idea (Fedosov): lift $C^\infty(M)[[\nu]]$ to flat sections of ΓW

Fedosov proposes to look at more general connection of the form:

$$d + \frac{1}{\nu}[\tilde{r}, \cdot]_\circ \quad \text{for } \tilde{r} \in \Gamma W \otimes \Lambda^1 M$$

∂ is already of this form, but need to be modified to become flat!

One main ingredient is an Hodge decomposition of $W_x \otimes \Lambda^*_x M$

Introduce $\delta : W^*_x \otimes \Lambda^*_q M \longrightarrow W^{*-1}_x \otimes \Lambda^{*+1}_x M$

defined by $\delta a = dx^i \wedge \partial_{y^i} a$

and $\delta^{-1} : W^*_x \otimes \Lambda^*_q M \longrightarrow W^{*+1}_x \otimes \Lambda^{*-1}_x M$

defined by: $\delta^{-1} a_{pq} = \begin{cases} \frac{1}{p+q} y^i \iota\left(\frac{\partial}{\partial x^i}\right) a_{pq} & \text{if } p+q \neq 0 \\ 0 & \text{if } p = q = 0. \end{cases}$

a_{pq} is of $\begin{cases} \text{degree } p \text{ in } W_x \\ \text{degree } q \text{ in } \Lambda^* M \end{cases}$

- δ, δ^{-1} acts on $\Gamma W \otimes \Lambda^* M$ fiberwisely.

Prop 2: $\delta^2 = 0$, $(\delta^{-1})^2 = 0$ and

for any $a \in \Gamma W \otimes \Lambda^* M$, a_{00} is the $\left|\begin{array}{l} W\text{-degree } 0 \text{ part of} \\ \Lambda\text{-degree } 0 \text{ part of} \end{array}\right|$ a.

$$\delta^{-1}\delta\, a + \delta\delta^{-1} a = a - a_{00}$$

Exercise 2: Show that for any $a \in \Gamma W \otimes \Lambda^* M$

- $\delta a = -\frac{1}{\nu}[\omega_{ij} y^i dx^j, a]$
- $\partial\delta + \delta\partial = 0$.

Fedosov looked at connections of the form

$$Da := \partial a - \delta a + \frac{1}{\nu}[r, a]_\circ \,, \quad r \in \Gamma W \otimes \Lambda^1 M$$

to be determined to make D flat.

Compute

$$D^2 a = \left(\partial - \delta a + \frac{1}{\nu}[r, \cdot\,]_\circ\right)\left(\partial - \delta + \frac{1}{\nu}[r, \cdot\,]_\circ\right) a \,, \quad a \in \Gamma W \otimes \Lambda^* M$$

$$= \frac{1}{\nu}\Big[\underbrace{\bar{R} - \partial r + \delta r + \frac{1}{\nu} r \circ r}, a\Big]_\circ$$

using graded Jacobi $[r, r]_\circ = 2\, r \circ r$.

Flat D means: $\bar{R} - \partial r + \delta r + \frac{1}{\nu} r \circ r = \Omega$

for some $\Omega \in \nu \Gamma \Lambda^2 M[[\nu]]$ ie an element in the center

$$\Omega = \sum_{r \geq 1} \nu^r \underbrace{\Omega_r}_{\in\, \Gamma\Lambda^2 M}.$$

Theorem 1: (Fedosov)

For any given series of closed 2-forms on M, $\Omega = \sum_{k\geq 1} \nu^k \Omega_k$,

the equation

$$\overline{R} - \partial r + \underline{\delta r} + \frac{1}{\nu} r \circ r = \Omega \qquad (*)$$

has a unique solution $r \in \Gamma W \otimes \Lambda^1 M$ st

- $\delta^{-1} r = 0$
- r has W-degree at least 3.

r depends on Ω and D.

Proof: Assume $r \in \Gamma W^3 \otimes \Lambda^1 M$ satisfying $(*)$ and $\delta^{-1} r = 0$

We use Hodge decomposition:

$$\delta\delta^{-1} r + \delta^{-1}\delta r = r - r_{00}$$

$$0 \qquad\qquad\qquad\qquad 0$$

Then: r must satisfy: $r = \delta^{-1}\left(\Omega - \overline{R} + \partial r - \frac{1}{\nu} r \circ r\right)$ $(**)$

Fact: Equation $(**)$ has a unique solution.

- δ^{-1} raises the W-degree by 1 $\left(\delta^{-1} = y^i \iota\left(\frac{\partial}{\partial x^i}\right)\right)$
- ∂ preserves the W-degree
- Denoting by $a^{(\ell)}$ the terms of W-deg ℓ in some $a \in \Gamma W \otimes \Lambda^* M$

$$\Rightarrow \left(\frac{1}{\nu} r \circ r\right)^{(m)} \text{ depends on } r^{(\ell)} \text{ with } \ell \leq m-1.$$

So equation $(**)$ determines uniquely r by induction.

Next, we show a solution r of $(**)$ satisfies $(*)$ and $\delta^{-1} r = 0$.

- $\delta^{-1} r = 0$ is clear because $(\delta^{-1})^2 = 0$.
- Set $A := \overline{R} - \Omega - \partial r + \delta r + \frac{1}{\nu} r \circ r$ with r the solution of $(**)$

We want to show $A = 0$.

First, $\delta^{-1}(A) = \underbrace{\delta^{-1}\left(\overline{R} - \Omega - \partial r + \frac{1}{\nu} r \circ r\right)}_{\overset{(**)}{=} -r} + \underbrace{\delta^{-1}\delta r}_{= r} = 0.$

After that, one checks that:

$$\partial A + \frac{1}{\nu}[r, A]_\circ = \delta A. \qquad \text{(exercise)}$$

So that: $A = \delta^{-1}\delta A = \delta^{-1}\left(\partial A + \frac{1}{\nu}[r, A]_\circ\right)$ $(***)$

But:
- δ^{-1} raises the W-degree by 1.
- ∂A preserves the W-degree
- $\left(\frac{1}{\nu}[r, A]_\circ\right)^{(m)}$ depends on $A^{(\ell)}$ with $\ell \leq m-1$.

Then $(***)$ has a unique solution by induction

$\Longrightarrow A = 0$ as 0 is indeed a solution $(***)$

It means r given by $(**)$ is indeed a solution of $(*)$. $\square$

Exercise 3: Equation (**) makes r computable, $r = \delta^{-1}(\Omega - \bar{R} + \partial r - \frac{1}{\nu} r \circ r)$

$$\Omega = \sum_{r \geq 1} \nu^r \Omega_r$$

Check that:

- $r^{(3)} = \delta^{-1}(\Omega_1 - \bar{R})$

$$= \frac{1}{8} \omega_{kr} R^r_{lij} y^k y^l y^i \otimes dx^j - \frac{\nu}{2} (\Omega_1)_{ij} y^i \otimes dx^j \quad \text{if } \Omega_1 = (\Omega_1)_{ij}\, dx^i \wedge dx^j$$

- Compute $r^{(4)}$ (long)

Remark: $r^{(5)}$ is (too) long to be computed

- $r^{(m)}$ depends on Ω_i only for $2i + 1 \leq m$

Th 1 means that flat connections of the form

$$D = \partial - \delta + \frac{1}{\nu}[r, \cdot]_\circ$$

do exist and can be determined by $\nabla \in \mathcal{E}(M, \omega)$ and $\Omega \in \nu \Gamma \Lambda^2 M[[\nu]]$.

D) Fedosov star products

Define the space of flat sections

$$\Gamma \mathcal{W}_D = \{ f \in \Gamma \mathcal{W} \mid Df = 0 \}$$

It is an algebra because D is actually a derivation

Define the symbol map
$$\sigma : \Gamma \mathcal{W} \longrightarrow C^\infty(M)[[\nu]]$$
$$s \longmapsto s_{00} = s|_{y=0}$$

Theorem 2: (Fedosov)

$\forall F \in C^\infty(M)[[\nu]] : \exists ! f \in \Gamma \mathcal{W}_D$ with $\sigma(f) = F$.

f is given recursively by $f = F + \delta^{-1}(\partial f + \frac{i}{\nu}[r, f]_\circ)$

Theorem 2 means that the symbol map has an inverse (when restricted to $\Gamma \mathcal{W}_D$)

We denote it by
$$Q : C^\infty(M)[[\nu]] \longrightarrow \Gamma \mathcal{W}_D$$
$$F \longmapsto Q(F)$$
st $\sigma(Q(F)) = F$.

Def 10: The Fedosov star product $*$ determined by $D \in \mathcal{E}(M, \omega)$ and $\Omega \in \nu \Gamma \Lambda^2 M[[\nu]]$ a series of closed 2-forms is: for $F, G \in C^\infty(M)[[\nu]]$

$$F * G = \sigma\big(Q(F) \circ Q(G)\big).$$