PART II. Deligne–Lusztig theory for finite reductive groups

Chapter 5. Review of algebraic groups

Chapter 6. $\mathbb{F}_q$-structures on alg. groups

Chapter 7. Harish-Chandra theory

Chapter 8. Deligne–Lusztig theory

Chapter 9. Application, open problems

Schedule. LAST LECTURE

→ Friday, Dec. 23

Chapter 8. Deligne–Lusztig theory

- G connected reductive algebraic group
- $F : G \longrightarrow G$ Frobenius endomorphism $/\mathbb{F}_q$
- $G^F = \{ g \in G \mid F(g) = g \}$ (finite reductive group)
- B_0 F-stable Borel subgroup
  $\cup$
  T_0 F-stable maximal torus
- $W = N_G(T_0)/T_0$: Weyl group
- $S = \{ s \in W \mid \dim B_0 s B_0 - \dim B_0 = 1 \}$
- $W = \langle S \rangle$ and $s^2 = 1$ for all $s \in S$.

Example. $G = GL_n(\mathbb{F})$

8.A. Deligne-Lusztig varieties

$$P = L \ltimes V \,, \qquad V = R_u(P)$$
$$F(L) = L \qquad \cancel{(F(P) = P)}$$

$$G^F \curvearrowright Y_P = \{ gV \in G/V \mid g^{-1}F(g) \in V \cdot F(V) \} \curvearrowleft L^F$$

8.B. Deligne-Lusztig induction and restriction

- $$\mathcal{R}^G_{L\subset P} : KL^F\text{-grmod} \longrightarrow KG^F\text{-grmod}$$
  $$M \longmapsto H^\bullet_c(Y_P) \otimes_{KL^F} M$$

- $$^*\mathcal{R}^G_{L\subset P} : KG^F\text{-grmod} \longrightarrow KL^F\text{-grmod}$$
  $$N \longmapsto H^\bullet_c(Y_P)^* \otimes_{KG^F} N$$

- $$\chi^{eu}_M = \sum_{i\geq 0} (-1)^i \chi_{M_i} \in \mathbb{Z}\,\mathrm{Irr}\, L^F$$

$$\begin{tikzcd} \mathrm{Class}(L^F) \arrow[r, bend left, "R^G_{L\subset P}"] & \mathrm{Class}(G^F) \arrow[l, bend left, "^*R^G_{L\subset P}"] \end{tikzcd}$$

$$(8.4) \begin{cases} R^G_{L\subset P}(\chi^{eu}_M) = \chi^{eu}_{\mathcal{R}^G_{L\subset P}(M)} \\ ^*R^G_{L\subset P}(\chi^{eu}_N) = \chi^{eu}_{^*\mathcal{R}^G_{L\subset P}(N)} \end{cases}$$

Proposition 8.5 (adjunction).

$$\langle \gamma, R^G_{L\subset P}\lambda \rangle_{G^F} = \langle {}^*R^G_{L\subset P}\gamma, \lambda \rangle_{L^F}$$

Transitivity 8.7. $(L', P') \subset L$

$$\begin{cases} \mathcal{R}^G_{L\subset P} \circ \mathcal{R}^L_{L'\subset P'} = \mathcal{R}^G_{L'\subset P'V} \\ ^*\mathcal{R}^L_{L'\subset P'} \circ {}^*\mathcal{R}^G_{L\subset P} = {}^*\mathcal{R}^G_{L'\subset P'V} \end{cases}$$

Corollary 8.8.

$$\begin{cases} R^G_{L\subset P} \circ R^L_{L'\subset P'} = R^G_{L'\subset P'V} \\ ^*R^L_{L'\subset P'} \circ {}^*R^G_{L'\subset P'V} = {}^*R^G_{L'\subset P'V} \end{cases}$$

Example 8.9. ${}^*\mathcal{R}^G_{L\subset P}(K_{G^F}[0])$?

Note the following property of ℓ-adic cohomologie I have forgotten

(*) If $f : Y \longrightarrow X$ is a flat morphism such that $f^{-1}(x) \simeq \mathbb{A}^d(\mathbb{F})$ for all $x \in X$, then $H^i_c(Y) \simeq H^{i-2d}_c(X)$.

(i.e. $H^\bullet_c(Y) = H^\bullet_c(X)[-2d]$)

See Hartshorne, chapter III, section 9

(for instance, $\mathcal{L}^{-1}(V\cdot F(V)) \to \mathcal{L}^{-1}(V\cdot F(V))/V$ satisfies the hypotheses of (*) $\quad \mathcal{L}^{-1}(V\cdot F(V))/V = Y_P$)

Let $d = \dim V$ and $e = \dim Y_P$ (so that $\dim V\cdot F(V) = d+e$). Then

$$
\begin{aligned}
{}^*\mathcal{R}^G_{L\subset P}(K_{G^F}) &= H^\bullet_c(Y_P)^* \otimes_{KG^F} K_{G^F} \\
&= H^\bullet_c(\mathcal{L}^{-1}(V\cdot F(V)))[2d] \otimes_{KG^F} K_{G^F} \quad \text{(see (*))} \\
&= H^\bullet_c(\mathcal{L}^{-1}(V\cdot F(V))/G^F)[2d] \quad \text{(see 3.7(i))} \\
&= H^\bullet_c(V\cdot F(V))[2d] \quad \text{(see section 6.C; "Galois unramified covering")}
\end{aligned}
$$

(L^F acts by conjugacy on $V\cdot F(V)$).

But $V\cdot F(V)$ is an affine space so

$$H^\bullet_c(V\cdot F(V)) = K[-2d-2e]$$

Therefore

$${}^*\mathcal{R}^G_{L\subset P} K_{G^F} = K[-2e]$$

What is the action of L^F? The action of L^F on $V\cdot F(V)$ is the restriction of the action of the connected group L, so L^F acts trivially on the cohomology (see 3.7(j)). Conclusion:

$$
\begin{cases}
{}^*\mathcal{R}^G_{L\subset P} K_{G^F} = K_{L^F}[-2e] \\
{}^*\mathcal{R}^G_{L\subset P}(1_{G^F}) = 1_{L^F}
\end{cases}
\qquad \blacksquare
$$

8.C. The Mackey formula.

Mackey formulas for Harish-Chandra induction and restriction (7.6 and 7.7)

$$\begin{cases} {}^*\mathcal{R}^G_{L\subset P} \circ \mathcal{R}^G_{L'\subset P'}(M) \simeq \bigoplus_{g\in L^F\backslash S(L,L')^F/L'^F} \mathcal{R}^L_{L\cap {}^gL'\subset L\cap {}^gP'} \circ {}^*\mathcal{R}^{{}^gL'}_{L\cap {}^gL'\subset P\cap {}^gL'}({}^gM) \\ \langle R^G_{L\subset P}\lambda, R^G_{L'\subset P'}\lambda'\rangle_{G^F} = \sum_{g\in L^F\backslash S(L,L')^F/L'^F} \langle {}^*R^L_{L\cap {}^gL'\subset L\cap {}^gP'}\lambda, {}^*R^{{}^gL'}_{L\cap {}^gL'\subset P\cap {}^gL'}{}^g\lambda'\rangle_{L^F\cap {}^gL'^F} \end{cases}$$

Mackey formula for Deligne-Lusztig induction and restriction (CONJECTURE)

$$(\mathcal{M}^G_{L,P,L',P'}) \qquad \langle R^G_{L\subset P}\lambda, R^G_{L'\subset P'}\lambda'\rangle_{G^F} = \sum_{g\in L^F\backslash S(L,L')^F/L'^F} \langle {}^*R^L_{L\cap {}^gL'\subset L\cap {}^gP'}\lambda, {}^*R^{{}^gL'}_{L\cap {}^gL'\subset P\cap {}^gL'}{}^g\lambda'\rangle_{L^F\cap {}^gL'^F}$$

Theorem 8.10. Mackey formula $(\mathcal{M}^G_{L,P,L',P'})$ holds in the following cases:

(1) If P and P' are F-stable (Deligne, 1979)

(2) If L or L' is a maximal torus (Deligne-Lusztig, 1976; Lusztig 1979; Digne-Michel 1991)

(3) If $q > 2$ } (B.-Michel, 2011)

(4) If G is classical }

(classical means type A, B, C, D; i.e GL_n, SL_n, PGL_n, Sp_{2n}, SO_n, $Spin_n$, ...)

Corollary 8.11. Assume $q > 2$, or L is a maximal torus, or G is classical.

Then $R^G_{L\subset P}$ does not depend on P.

Ideas of the proof of the Mackey formula if L or L' is a maximal torus. For $l \in L^F$,

$$\cdot\ {}^*R^G_{L\subset P}\, R^G_{L'\subset P'}(f)(l) = \frac{1}{|L'^F|}\sum_{l'\in L'^F}\left(\frac{1}{|G^F|}\sum_{g\in G^F} Tr^*_{Y_P^G}(g,l)\, Tr^*_{Y_{P'}^G}(g,l')\right) f(l'^{-1})$$

$$= \frac{1}{|G^F|}\sum_{g\in G^F} Tr^*_{Y_P^G\times Y_{P'}^G}(g,l,l') \underset{\uparrow}{=} Tr^*_{(Y_P^G\times Y_{P'}^G)/G^F}(l,l')$$

(Künneth ; 3.7(e)) Exercise 4.11 + 3.7(i).

COMPARE THIS PROOF WITH THE $SL_2(\mathbb{F}_q)$ CASE

Let
$$\begin{aligned} \varphi : Y_P^G\times Y_{P'}^G &\longrightarrow Z = \{(x,v,v')\in G\times V\cdot F(V)\times V'\cdot F(V') \mid vF(x) = xv'\}\\ (g,h) &\longmapsto (g^{-1}h,\ g^{-1}F(g),\ h^{-1}F(h)) \end{aligned}$$

First step. Prove that φ induces an isomorphism $\bar\varphi : (Y_P^G\times Y_{P'}^G)/G^F \xrightarrow{\sim} Z$ (exercise*)

So $${}^*R^G_{L\subset P}\, R^G_{L'\subset P'}(f)(l) = \frac{1}{|L'^F|}\sum_{l'\in L'^F} Tr^*_Z(l,l')\, f(l'^{-1})$$

Second step. Decompose $Z = \bigcup\limits^{\cdot}_{g\in L\backslash S(L,L')/L'} Z_g$ where $Z_g = \{(x,v,v')\in Z \mid x\in PgP'\}$

$S(L,L') = \{g\in G \mid L\cap {}^gL' \text{ contains a maximal torus of } G\}$

So $${}^*R^G_{L\subset P}\, R^G_{L'\subset P'}(f)(l) = \sum_{g\in L\backslash S(L,L')/L'}\left(\frac{1}{|L'^F|}\sum_{l'\in L'^F} Tr^*_{Z_g}(l,l')\, f(l'^{-1})\right) \qquad (3.10(a))$$

Third step. Find an action of a torus S_g on Z_g, commuting with the action of $L^F\times L'^F$ such that:

$$Z_g^{S_g} \simeq \begin{cases} \emptyset & \text{if } F(LgL')\neq LgL' \\ \left(Y^L_{L\cap {}^gP'}\times Y^{{}^gL'}_{P\cap {}^gL'}\right)/L^F\cap {}^gL'^F & \text{if } F(g)=g \end{cases}$$

ONLY POSSIBLE IF $L\cap {}^gL' = L$ or ${}^gL'$

This concludes the proof (3.10). ∎

8.D. Maximal tori.

Let $\nabla(G,F)$ be the set of pairs (T,θ) such that T is an F-stable maximal torus and $\theta \in \operatorname{Irr} T^F = \operatorname{Hom}_{gp}(T^F, K^\times)$. The group G^F acts on $\nabla(G,F)$ by conjugacy.

$(T,\theta) \in \nabla(G,F) \implies R_T^G(\theta) = R_{T\subset B}^G(\theta)$ for some (or any) Borel subgroup B containing T.

(8.12) $$\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle_{G^F} = \begin{cases} 0 & \text{if } (T,\theta) \not\sim_{G^F} (T',\theta') \\ |N_{G^F}(T,\theta)/T^F| & \text{if } (T,\theta) = (T',\theta') \end{cases}$$

Corollary 8.13. If θ is regular (i.e. if $N_{G^F}(T,\theta) = T^F$), then

$$R_T^G(\theta) \in \pm \operatorname{Irr} G^F$$

Corollary 8.14. If θ is regular and T is not contained in an F-stable Levi subgroup of an F-stable parabolic subgroup $P \subsetneq G$, then $R_T^G(\theta) \in \pm \operatorname{Irr}_{cus}(G^F)$.

Proof. ${}^*R_{L\subset P}^G\, R_T^G(\theta) \underset{8.10(2)}{=} 0$ because $S(L,T)^F = \emptyset$ (if $P \neq G$). ∎

Let $\varepsilon_T = (-1)^{\dim\{t\in T \mid F(t)=t^q\}}$

Example. If $G = GL_n(\mathbb{F})$ and $F: (a_{ij}) \mapsto (a_{ij}^q)$, then $\varepsilon_{T_0} = (-1)^n$. ∎

We set $\varepsilon_G = \varepsilon_{T_0}$ $(= \varepsilon_{G,F})$

(8.15) $$R_T^G(\theta)(1) = \varepsilon_T\, \varepsilon_G \frac{|G^F|_{p'}}{|T^F|}$$

Proof of the fact that $R_T^G(\theta)(1)$ does not depend on θ:

$$R_T^G(\theta)(1) = \frac{1}{|T^F|} \sum_{t\in T^F} \operatorname{Tr}^*_{Y_B}(1,t)\, \theta(t^{-1})$$

$(B \supset T)$

But, by 3.10(b),

$$\operatorname{Tr}^*_{Y_B}(1,t) = \operatorname{Tr}^*_{Y_B^t}(1)$$

But $Y_P \subset G/V$, where $V = R_u(B)$ and T^F acts by right translation: this action is free. So $Y_B^t = \emptyset$ if $t \neq 1$.

So $R_T^G(\theta)(1) = \frac{1}{|T^F|} \operatorname{Tr}^*_{Y_B}(1)$. ∎

Corollary 8.16. If θ is regular, then

$$\varepsilon_G \varepsilon_T R_T^G(\theta) \in \operatorname{Irr} G^F.$$

Example 8.17. Assume that $G^F = SL_2(\mathbb{F}_q)$.

Then we have seen in example 8.3 that there is an F-stable maximal torus T' which is not contained in an F-stable Borel subgroup.

$$R_{T'}^G(1_{T'^F}) = -St_{G^F} + 1_{G^F}$$

$$R_{T_0}^G(1_{T_0^F}) = St_{G^F} + 1_{G^F}$$

So $\langle R_{T_0}^G(1_{T_0^F}), R_{T'}^G(1_{T'^F}) \rangle = 0$

but they have common irreducible components. ∎

Definition 8.18. Let (T,θ) and (T',θ') in $\nabla(G,F)$. We say that (T,θ) and (T',θ') are geometrically conjugate (and we denote $(T,\theta) \equiv (T',\theta')$) if there exists $n \in \mathbb{Z}_{\geq 1}$ such that the pairs $(T, \theta \circ N_{F^n/F})$ and $(T', \theta' \circ N_{F^n/F})$ are conjugate in G^{F^n}.

Here $N_{F^n/F} : T^{F^n} \longrightarrow T^F$

$t \longmapsto t \cdot F(t) \cdots F^{n-1}(t)$

Exercise. Prove that $N_{F^n/F}$ is surjective (Hint: use Lang Theorem!!). ∎

Theorem 8.19 (consequence of the hidden part of the proof of the Mackey formula)

Let (T,θ) and (T',θ') in $\nabla(G,F)$ and let B and B' be two Borel subgroups such that $T \subset B$ and $T' \subset B'$. Assume that $H_c^i(Y_B) \otimes_{KT^F} K_\theta$ and $H_c^j(Y_{B'}) \otimes_{KT'^F} K_{\theta'}$ have a common irreducible component. Then $(T,\theta) \equiv (T',\theta')$.

If $\mathcal{X} \in \nabla(G,F)/\equiv$, we denote by $\mathcal{E}(G^F, \mathcal{X})$ the set of irreducible characters of G^F which occur in some $R_T^G(\theta)$, for $(T,\theta) \in \mathcal{X}$.

Theorem 8.20 (Deligne - Lusztig 1976)

$$\operatorname{Irr} G^F = \dot{\bigcup_{\mathcal{X} \in \nabla(G,F)/\equiv}} \mathcal{E}(G^F, \mathcal{X})$$

Lusztig series

For $G^F = SL_2(\mathbb{F}_q)$, you can check that $(T_0, \alpha_0) \equiv (T', \theta_0)$

$$R(\alpha_0) = R(\alpha_0)^+ + R(\alpha_0)^-$$

$$R'(\theta_0) = R'(\theta_0)^+ + R'(\theta_0)^-$$