# Lecture notes for Tsingua Workshop

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## Lecture 3: Hyperbolic attractors

Our goal in this lecture is to describe a class of systems for which physical measures exist, and to sketch proofs of this property. Although these results are interesting, the techniques used in their proofs are perhaps more so: the concept of *stable holonomy*, defined below, is a crucial component in modern approaches to decay of correlations and other statistical properties for deterministic dynamical systems with some hyperbolicity.

### 0.1 Basic properties

**Definition 1.** Let $f : U \to \mathbb{R}^d$ be a diffeomorphism onto its image, where $U \subset \mathbb{R}^d$ is open. We say that an open $V \subset U$ is a *absorbing set* if $\overline{f(V)} \subset V$.

**Lemma 2.** *Let $V$ be an absorbing set and assume $\overline{V}$ is compact. Then,*

$$\mathcal{A} = \bigcap_{n \geq 0} \overline{f^n(V)}$$

*is compact, nonempty, $f$-invariant (i.e. $f(\mathcal{A}) = \mathcal{A}$) and has the property that for all $x \in V$,*

$$\mathrm{dist}(f^n x, \mathcal{A}) \to 0 \quad \text{as } n \to \infty\,.$$

*Proof.* Compactness and nonemptyness of $\mathcal{A}$ follows from the finite-intersection property and the fact that $\overline{f^n(V)}$ is a decreasing sequence of compact sets.

For $f$-invariance:

$$f(\mathcal{A}) = \bigcap_{n \geq 0} \overline{f^{n+1}(V)} = \bigcap_{n \geq 1} \overline{f^n(V)} = \bigcap_{n \geq 0} \overline{f^n(V)} = \mathcal{A}\,.$$

Finally, let $x \in V$ and assume there exists a sequence $n_k \to \infty$ so that $\mathrm{dist}(f^{n_k} x, \mathcal{A}) \geq \epsilon > 0$ for some $\epsilon > 0$. By a compactness argument, refine $\{n_k\}$ so that $f^{n_k} x$ converges to some $x^* \in \overline{V}$. Then, $x^* \notin \mathcal{A}$ by our contradiction hypothesis, and yet $x^* \in \mathcal{A}$ must hold by definition of $\mathcal{A}$. $\square$

**Exercise 3.** Recall the Smale solenoid map defined in Example **??**. Show that the domain $X = S^1 \times D$ is an absorbing set. The corresponding attractor is itself often called the *Smale solenoid*.

**Definition 4.** If $\Lambda$ is the attractor corresponding to some absorbing set $V$ and is also uniformly hyperbolic for $f$, then we call $\Lambda$ a *uniformly hyperbolic attractor*.

The following important lemma describes the local structure of a uniformly hyperbolic attractor as a union of unstable leaves.

**Lemma 5.** *Let $\Lambda$ be a uniformly hyperbolic attractor and let $x \in \Lambda$. Then, the global unstable leaf $W^u_x$ is contained in $\Lambda$.*

*Proof.* Take $\epsilon > 0$ small enough so $\hat{W}^u_{x,r} \subset V$ for all $x \in \Lambda$. Since $\mathcal{A} = \cap_n f^n(V)$ and $W^u_x = \cup_{n\geq 0} f^n W^u_{f^{-n}x,r}$ is an increasing union, it follows that $W^u_x \subset \Lambda$. □

Likewise, $V \setminus \Lambda$ is foliated by stable manifolds through points of $\Lambda$.

**Lemma 6.** *We have* $V \subset \cup_{x\in\Lambda} W^s_x$.

*Proof.* This follows from the fact that there is a neighborhood $\tilde{V} \supset \Lambda$ for which $\tilde{V} \subset \cup_{x\in\Lambda} W^s_x$. To check this, one uses the fact that (i) $W^u_{x,\epsilon} \subset \Lambda$ for all $x$, and that $x \mapsto W^s_{x,\epsilon}$ varies continuously in $x \in \Lambda$. Might add more. □

## 0.2 Physical measures for hyperbolic attractors

**Theorem 7.** *Let* $f : U \to M$ *be a diffeomorphism and let* $\Lambda \subset U$ *be a uniformly hyperbolic attractor for* $f$ *with absorbing set* $V \subset U$*. Assume* $f|_\Lambda$ *is topologically transitive (i.e., for all open* $U_1, U_2 \subset \Lambda$*, there is some* $n \geq 0$ *so that* $f^{-n}(U_1) \cap U_2 \neq \emptyset$*). Then, there exists an* $f$*-invariant, ergodic measure* $\mu$ *for which*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \phi \circ f^i(x) = \int \phi d\mu$$

*for all continuous* $\phi : M \to \mathbb{R}$ *and for Leb-almost every* $x \in V$*.*

That is, uniformly hyperbolic attractors admit physical measures. The physical measure itself is characterized by a geometric property called the SRB property (for Sinai, Ruelle and Bowen, who discovered these measures) or alternatively Gibbs u-states.

Before defining the SRB property, let us motivate it with a heuristic discussion. Imagine that we are trying to build a physical measure. It would be very natural to start with a Lebesgue 'blob' $\nu$ supported on a small open set $U$ contained in the absorbing set $V$, and to take a weak limit of the measures $\frac{1}{N}\sum_{n=1}^{N} f^n_* \nu$. As we push forward, the measures $f^n_*\nu$ become concentrated near the attractor $\Lambda$, and hyperbolicity kicks in: we see stretching along unstable directions and contracting along stable directions. Expansion has a 'smoothing' effect on the density, smearing in a controlled and uniform way along $E^u$ directions, while in $E^s$ directions the density can become more irregular and 'bunched up'. Whatever the limiting measure is, then, should have some 'smoothness' along unstable directions / leaves, while being quite irregular and 'lumpy' along stable directions.

**Remark 8.** Although we will not pursue this tack, an argument of this kind was made rigorous by Ruelle.

**Exercise 9.** In this exercise, we will get a grip on how expansion can 'smooth out' a density, while contraction leads to 'bunching up'.

(a) Consider the doubling map $f : S^1 \to S^1, fx = 2x \mod 1$. Let $\rho : S^1 \to \mathbb{R}_{\geq 0}$ be a Lipschitz continuous density, i.e., $\int \rho(x)dx = 1$. Let $\mathcal{L}_f$ denote the *transfer operator*, defined so that $f_*(\rho dx) = (\mathcal{L}_f \rho)dx$.[1]

   – Show that
$$\mathcal{L}_f \rho(x) = \sum_{y\in f^{-1}x} \frac{1}{2}\rho(y)\,.$$

[1] Given a measure $\nu$ and a map $f$, we write $f_*\nu = \nu \circ f^{-1}$ for the pushforward of $\nu$ by $f$.

– Show that $\mathcal{L}_f$ is a contraction in the Lipschitz seminorm $[\cdot]_{\mathrm{Lip}}$ defined by

$$[\rho]_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|\rho(x) - \rho(y)|}{d(x,y)} .$$

– Conclude that $f_*^n(\rho dx)$ converges weakly to uniform Lebesgue measure on $S^1$ as $n \to \infty$.

– Generalize this to the setting where $f : S^1 \to S^1$ is an *expanding map of degree* $d$, i.e., $|f'(x)| \geq c > 1$ for all $x \in S^1$ and $\# f^{-1}\{x\} = d$ for all $x \in S^1$. What is the form for $\mathcal{L}_f \rho$?

(b) Consider the *halving* map $f : \mathbb{R} \to \mathbb{R}$, $fx = \frac{1}{2}x$. Let $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a Lipschitz density.

– Show that with $\mathcal{L}_f$ denoting the corresponding transfer operator that

$$\mathcal{L}_f \rho(x) = 2\rho(f^{-1}x) ,$$

hence $[\mathcal{L}_f \rho]_{\mathrm{Lip}} = 2[\rho]_{\mathrm{Lip}}$.

– Conclude that the limit $\mathcal{L}_f^n \rho$ does not exist in the Lip norm. What is the weak limit of $f_*^n(\rho dx)$?

– Generalize this to the setting where $f : \mathbb{R} \to \mathbb{R}$ has a sink at 0 and the density $\rho$ is supported in the basin of attraction of 0.

## 0.3 Description of geometric SRB property:

Let us now set about making rigorous this idea of 'smoothness' along unstable directions.

**Proposition 10** (Local structure of hyperbolic attractors)**.** *Let $\Lambda$ be a uniformly hyperbolic attractor for $f$.*

- *$\Lambda$ can be covered by a finite union of open sets $\mathcal{N}$ (in the subspace topology) which are homeomorphic to*

$$\Sigma \times B^u ,$$

*where $\Sigma$ is a compact metric space, and $B^u$ is the unit ball in $\mathbb{R}^{\dim E^u}$. Write $\Psi : \Sigma \times B^u \to \mathcal{N}$ for this homeomorphism.*

- *For each fixed $\sigma \in \Sigma$, the set $\{\Psi(\sigma, v)\}_{v \in B^u}$ is a relatively open subset of a $W^u$-leaf. Moreover, $\sigma \mapsto \Psi(\sigma, B^u)$ varies continuously in the $d_H$ (Hausdorff) metric. Lastly, for $\sigma \in \Sigma$ fixed, the mapping $v \mapsto \Psi(\sigma, v)$ is a $C^r$ diffeomorphism with $C^r$-norm bounded independently of $\sigma$.*

### An aside: disintegration measures

Let $\mathcal{X}$ be a Polish space, $\mathfrak{m}$ a finite measure on $\mathcal{X}$, and let $\mathfrak{P}$ be a partition of $\mathcal{X}$. For $x \in \mathcal{X}$, define $\mathfrak{P}(x)$ to be the atom of $\mathfrak{P}$ containing $x$.

**Definition 11.** We say that $\mathfrak{P}$ is a *measurable partition* if

$$\mathfrak{P} = \bigvee_n \mathfrak{P}_n$$

modulo $\mathfrak{m}$-null sets, where $\mathfrak{P}_n$ is an increasing sequence ($\mathfrak{P}_n$-atoms are unions of $\mathfrak{P}_{n+1}$ atoms, i.e., $\mathfrak{P}_n \leq \mathfrak{P}_{n+1}$) of finite partitions, $\bigvee$ denotes the join of partitions (for partitions $\zeta, \eta$, we define $\zeta \vee \eta = \{C \cap D : C \in \zeta, D \in \eta\}$).

**Lemma 12.** *There is a measurable*[2] *assignment $x \mapsto \mathfrak{m}_x$ to each $x \in \mathcal{X}$ of a Borel probability measure $\mathfrak{m}_x$ on $\mathcal{X}$, with $\mathfrak{m}_x(\mathfrak{P}(x)) = 1$, such that for all Borel $K \subset \mathcal{X}$, we have that*

$$\mathfrak{m}(K) = \int_{\mathcal{X}} \mathfrak{m}_x(K)\, d\mathfrak{m}(x)\,.$$

**Remark 13.** To wit, the measures $(\mathfrak{m}_x)_{x\in\mathcal{X}}$ are a version of the conditional probability measures $\mathfrak{m}(\cdot|\sigma(\mathfrak{P}))$, where $\sigma(\mathfrak{P})$ is the smallest $\sigma$-algebra containing the atoms of $\mathfrak{P}$, with the property that $\mathfrak{m}_x$ actually charges $\mathfrak{P}(x)$ with probability 1. It is not always possible to construct disintegration measures $(\mathfrak{m}_x)$– see ?? for counterexamples.

**Lemma 14.** *For any Borel set $K \subset \mathcal{X}$ we have that*

$$\mathfrak{m}_x = \lim_{n\to\infty} \frac{\mathfrak{m}(K \cap \mathfrak{B}_n(x))}{\mathfrak{B}_n(x)} = \lim_{n\to\infty} \mathfrak{m}(K|\sigma(\mathfrak{B}_n))(x)$$

*for $\mathfrak{m}$-almost every $x \in \mathcal{X}$. In particular, $x \mapsto \mathfrak{m}(\cdot|\sigma(\mathfrak{B}_n))(x)$ converges weakly to $x \mapsto \mathfrak{m}_x(\cdot)$ for $\mathfrak{m}$-almost every $x \in \mathcal{X}$.*

The proof is a nice exercise using the Martingale Convergence Theorem (for fixed Borel $K$, the sequence of functions $x \mapsto \mathfrak{m}(K|\sigma(\mathfrak{B}_n))(x)$ is a martingale with respect to the filtration $\sigma(\mathfrak{B}_n)$). See the notes of Viana, "Disintegration into conditional measures: Rokhlin's theorem" for a proof.

### Definition of the SRB property

Given a neighborhood $\mathcal{N} \subset \Lambda$ as above, let us write $\Xi = \Xi^{\mathcal{N}}$ for the partition of $\mathcal{N}$ into $W^u$-leaves of the form $\Psi(\sigma, B^u)$. This partition is measurable, in the sense that it can be given as

$$\Xi = \bigvee_{n=1}^{\infty} \Xi_n\,, \tag{1}$$

pointwise where $(\Xi_n)$ is a sequence of finite measurable partitions of $\mathcal{N}$, and $\vee$ denotes the join of partitions.

**Exercise 15.** Prove that (1) holds using the continuity of $\sigma \mapsto \Psi(\sigma, B^u)$ in the Hausdorff ($d_H$) metric, where each finite partition $\Xi_n$ consists of full $\xi$-leaves.

**Definition 16.** Let $\mu$ be an invariant measure for $f|_\Lambda$. We say that $\mu$ has the SRB property (or that $\mu$ is an SRB measure, or Gibbs u-state for some authors) if for any neighborhood $\mathcal{N}$ as above, we have the following:

Define $\mu^{\mathcal{N}} = \mu|_{\mathcal{N}}$. Then, for $\mu^{\mathcal{N}}$-almost every $x \in \mathcal{N}$,
we have that $(\mu^{\mathcal{N}})_x$ is equivalent to Lebesgue measure
$\nu_x$ on the $u$-leaf $\xi(x)$ containing $x$.

## 0.4 Construction of SRB measures for uniformly hyperbolic attractors

**Theorem 17.** *Assume $f$ is $C^2$. Let $\Lambda$ be a uniformly hyperbolic attractor for $f$.*

*(a) There exists an SRB measure $\mu$ for $f$ supported on $\Lambda$.*

[2] Borel measurable with respect to the weak* topology on probability measures on $\mathcal{X}$

*(b) If $f$ is topologically transitive, then the SRB measure for $f|_\Lambda$ is unique. In particular, it is ergodic.*

Let us first establish existence as in part (a). Earlier we considered a heuristic argument (which can be made rigorous) for constructing SRB measures by pushing forward Lebesgue. Here we present a somewhat simpler argument.

**Proposition 18.** *Let $L = W^u_{x,\epsilon}$ for some $x \in \Lambda$. Let $m_L$ denote normalized Lebesgue measure on $L$. Then, any weak* limit $\mu$ of the sequence $m_N := \frac{1}{N}\sum_{n=1}^N f^n_* m_L$ is an $f$-invariant SRB measure supported on $\Lambda$.*

**Exercise 19.** Assume $X$ is a compact metric space and $F : X \to X$ is a continuous mapping. Let $m$ be an arbitrary Borel probability measure on $X$. By Prokhorov's theorem, the sequence $\{\frac{1}{N}\sum_{n=1}^N f^n_* m\}$ is weak* compact. Show that any weak* limit $\mu$ is an $F$-invariant measure[3].

**Proof of Proposition 18**

The primary tool is the following *distortion estimate* along unstable leaves.

**Lemma 20** (Distortion estimate)**.** *There exists a constant $C > 0$ with the following property. For all $\epsilon > 0$ sufficiently small and all $x \in \Lambda$, we have that for any $n \geq 1$ and $y, y' \in W^u_{x,\epsilon}$ that*

$$C^{-1} \leq \frac{\det(D_{f^{-n}y} f^n|_{E^u_{f^{-n}y}})}{\det(D_{f^{-n}y'} f^n|_{E^u_{f^{-n}y'}})} \leq C$$

*Proof.* Observe that $T_z W^u_x = E^u_z$ for $z \in \Lambda$. Define $J^u_n(z) = \det(D_z f^n|_{E^u_z})$, $J^u = J^u_1$. Then,

$$(*) = \frac{\det(D_{f^{-n}y} f^n|_{E^u_{f^{-n}y}})}{\det(D_{f^{-n}y'} f^n|_{E^u_{f^{-n}y'}})} = \frac{J^u_n(f^{-n}y)}{J^u_n(f^{-n}y')} = \prod_{i=1}^n \frac{J^u(f^{-i}y)}{J^u(f^{-i}y')}$$

and

$$|\log(*)| = \sum_{i=1}^n \left|\log J^u(f^{-i}y) - \log J^u(f^{-i}y')\right|$$

Now, since $W^u_{z,\epsilon}$ are $C^1$ embedded disks and $f \in C^2$, it follows that $z \mapsto J^u(z)$ is Lipschitz along fixed local $W^u$-manifolds. Thus,

$$|\log(*)| \leq C \sum_{i=1}^n \operatorname{dist}(f^{-i}y, f^{-i}y') \tag{2}$$

$$\leq C' \sum_{i=1}^n \lambda^i \operatorname{dist}(y, y') \leq C'' \operatorname{dist}(y, y') \tag{3}$$

where in the second line we use the backwards-time contraction estimate along $W^u$-leaves as in Theorem **??**. □

To continue, let $\mu$ be a weak* cluster point of the sequence $m_N$, i.e.,

$$\mu = * - \lim_{k\to\infty} m_{N_k}$$

[3]This technique is called the Krylov-Bogoliubov method for producing invariant measures of a dynamical system.

for some subsequence $N_k \to \infty$. Fix an open set $\mathcal{N} \subset \Lambda$ and a homeomorphism $\Psi : \Sigma \times B^u \to \mathcal{N}$ be fixed as in Proposition 10. Let $\Xi$ be the partition into u-leaves $\Psi(\sigma, B^u), \sigma \in \Sigma$. Let $\Xi_n$ be an increasing sequence of finite partitions for which $\Xi = \vee_n \Xi_n$.

We will check the SRB property for $\mu^{\mathcal{N}} = \mu|_{\mathcal{N}}$. Without loss, we can assume

(A1) $\mu(\mathcal{N}) > 0$

(A2) $\mu(\mathcal{N}) = \lim_k m_{N_k}(\mathcal{N})$

(A3) For all $C \in \Xi_n, n \geq 1$, we have $\mu^{\mathcal{N}}(\partial C) = 0$[4]; in particular,

$$\mu^{\mathcal{N}}(C) = \lim_{k\to\infty} m_{N_k}(C)$$

**Exercise 21.** Let $m_n \to m$ we a weakly convergent sequence of Borel probabilities on a Polish space. Prove that if $m(\partial C) = 0$ for some Borel $C$, then $\lim_n m_n(C) = m(C)$.

**Exercise 22.** Let $X$ be a metric space and $m$ a Borel measure on $X$. Show that for any $x \in X$ there is a sequence $r_n \to 0$ so that $m(\partial B_{r_n}(x)) = 0$ for all $n$.

**Exercise 23.** Using Exercises 21 and 22, prove that we can always cover $\Lambda$ by open sets $\mathcal{N}$ equipped with finite approximating partitions $(\Xi_n) = (\Xi_n^{\mathcal{N}})$ for which (A1) – (A3) hold.

**Lemma 24.** *For $n \geq 1$ define $L_n = L_n^{\mathcal{N}} \subset f^n(L)$ to be the union over all $\xi \in \Xi$ such that $\xi \subset f^n(L)$. Then,*

$$\mu^{\mathcal{N}} = \lim_{k\to\infty} m_{N_k}|_{\mathcal{N}} = \lim_{k\to\infty} m_{N_k}|_{\mathcal{N}\cap L_n}$$

*Proof.* When $\dim E^u = 1$, $L \setminus f^{-n} L_n$ consists of at most two sub-arcs of length $\lesssim \lambda^n$. □

**Exercise 25.** Complete the proof of Lemma 24 when $\dim E^u > 1$. Hint: show that if $\xi \in \Xi$ and $\emptyset \subsetneq \xi \cap L \subsetneq \xi$, then $f^{-n}\xi$ is contained in a very small neighborhood of $\partial L$.

In the following lemma, for $K \subset B^u$ we define $C_K = \Psi(\Sigma, K)$. Such sets are 'cylindrical plugs' cutting transversally through the $W^u$-leaves comprising $\mathcal{N}$.

**Lemma 26.** *There is a constant $C > 0$ such that the following holds. Let $n \geq 1$, $\xi \in \Xi$, and assume $\xi \subset L_n$. Then,*

$$C^{-1} \operatorname{Leb}_{B^u}(K) \leq \frac{m_n(\xi \cap C_K)}{m_n(\xi)} \leq C \operatorname{Leb}_{B^u}(K)$$

*for all Borel $K \subset B^u$.*

*Completing the proof of Proposition 18.* By Lemma 26, we have for Borel $K \subset B^u$ that

$$C^{-1} \operatorname{Leb}_{B^u}(K) \leq m_N(C_K|\Xi_n) \leq C \operatorname{Leb}_{B^u}(K)$$

pointwise on $\mathcal{N}$ for all $N$, where $m_N(\cdot|\Xi_n)$ is the conditional measure with respect to the (finite) $\sigma$-algebra $\sigma(\Xi_n)$ generated by the (finite) partition $\Xi_n$.

Assume now that $\mu(\partial C_K) = 0$. With $n$ fixed and taking $N = N_k \to \infty$, we conclude

$$C^{-1} \operatorname{Leb}_{B^u}(K) \leq \mu(C_K|\Xi_n) \leq C \operatorname{Leb}_{B^u}(K)$$

[4]We write $\partial C = \overline{C} \setminus C^\circ$, where $\overline{C}$ is the closure of $C$ and $C^\circ$ the interior; all topology is in the subspace topology of $\Lambda \subset U$.

pointwise on $x \in \mathcal{N}$, having used assumptions (A1) – (A3) (see also Exercise 21). Taking now $n \to \infty$, by Lemma 14 we conclude for such $K$ that

$$C^{-1} \operatorname{Leb}_{B^u}(K) \leq \mu_x(C_K) \leq C \operatorname{Leb}_{B^u}(K).$$

Note that since $u \mapsto \Psi(\sigma, u)$ is a $C^r$ embedding for all $\sigma \in \Sigma$ fixed, we have

$$\operatorname{Leb}_{B^u}(K) \approx \operatorname{Leb}_{\xi(x)}(C_K),$$

and so

$$\mu_x(C_K) \approx \operatorname{Leb}_{\xi(x)}(C_K) \tag{4}$$

holds for $\mu$-a.e. $x \in \mathcal{N}$ whenever $\mu(\partial C_K) = 0$.

To complete the proof, fix $x$ as above. Let $A \subset \xi(x)$ have Lebesgue measure zero, and consider the uncountable family $B_r(A) = \{y \in \xi(x) : \operatorname{dist}_{\xi(x)}(y, A) < r\}, r > 0$, observing that $\operatorname{Leb}_{\xi(x)}(B_r(A)) \to 0$ as $r \to 0$. Let $K_r \subset B^u$ be such that $C_{K_r} \cap \xi(x) = B_r(A)$. Fix $\{r_n\}, r_n \to 0$ for which $\mu(\partial C_{K_{r_n}}) = 0$ for all $n$ (c.f. Exercise 22).

By (4), we conclude

$$\mu_x(A) = \lim_{r_n \to 0} \mu_x(B_{r_n}(A)) = \lim_{r_n \to 0} \operatorname{Leb}_{\xi(x)}(C_{K_{r_n}}) = 0$$

hence $\mu_x \ll \operatorname{Leb}_{\xi(x)}$. The converse follows similarly. □

### 0.4.1 Remarks

In the definition of SRB measure, one might wonder whether one can replace 'equivalence' with Lebesgue measure on unstable leaves with 'absolute continuity'. A priori this might yield a new class of invariant measures corresponding to attractors with 'gaps' along unstable leaves. Somewhat surprisingly, this is not possible.

**Proposition 27.** *Let $\mu$ have the property that its conditional measures on unstable leaves are almost-surely absolutely continuous with respect to Lebesgue measure. Then, these conditional measures are actually equivalent to Lebesgue almost surely. In fact, fix a $\mu$-typical $x \in \Lambda$ and let $\rho_x : W^u_{x,\epsilon} \to \mathbb{R}_{\geq 0}$ denote the density function for the disintegration measure $\mu_x = \mu^{\mathcal{N}}_x$. Then,*

$$\frac{\rho_x(y_1)}{\rho_x(y_2)} = \lim_{n \to \infty} \frac{J^u_n(f^{-n} y_2)}{J^u_n(f^{-n} y_1)} > 0$$

*for all $y_1, y_2 \in W^u_{x,\epsilon}$.*

Note that in particular, the density $y \mapsto \rho_x(y)$ is Lipschitz continuous.

## 0.5 Local ergodicity and of SRB measures

Note that ergodicity was not assumed for SRB measures. Indeed, one can imagine simple scenarios where SRB measures fail to be ergodic (take e.g. a disjoint union of uniformly hyperbolic attractors). We address these issues here.

**Proposition 28** (Local ergodicity of SRB measures)**.** *Let $\mu$ be an SRB measure supported on a uniformly hyperbolic attractor. Then, $\Lambda$ is covered by open sets $\mathcal{N}$ with the property that $\mu$-almost all $x \in \mathcal{N}$ are future generic to the same invariant ergodic measure $\nu$.*

**Corollary 29.** *Assume $f|_\Lambda$ is topologically transitive. Then, $\Lambda$ supports a unique SRB measure.*

*Proof.* Topological transitivity and local ergodicity imply that $\mu$-almost all $x \in \Lambda$ are future generic to the same ergodic invariant measure $\nu$. The Birkhoff theorem implies $\mu = \nu$, and so $\mu$-almost all $x \in X$ are future generic to $\mu$. Exercise: show that this implies ergodicity of $\mu$ by applying the Birkhoff ergodic theorem to $\phi = \chi_A$, where $A \subset \Lambda$ is an $f$-invariant set. □

**Proof of local ergodicity: the Hopf argument**

We start with the following corollary of the Birkhoff ergodic theorem for non-ergodic measures. Let $T : X \to X$ be a measurable transformation of a space $X$ preserving a probability $m$.

**Definition 30.** We say that $x \in X$ is *future generic* to a probability measure $\nu$ on $X$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(T^n x) = \int \phi d\nu$$

for all bounded measurable $\phi : X \to \mathbb{R}$. If $T$ inverts measurably, we say that $x$ is *past generic* to $\nu$ if it is future generic to $\nu$ with respect to $T^{-1}$.

**Corollary 31.** *Let $T : X \to X$ be a measurable transformation of a space $X$ preserving a probability $m$ on $X$. Assume $T$ inverts measurably. Then, we have that $m$-a.e. point $x \in X$ is future and past generic with respect to some ergodic invariant probability $\nu_x$ on $X$.*

We are entitled to apply this corollary to $f|_\Lambda : \Lambda \to \Lambda$ since $f_\Lambda$ is automatically a homeomorphism, hence measurably invertible (Exercise: check this!). Applied to the SRB measure $\mu$ of a uniformly hyperbolic attractor $\Lambda$ for $f$, we derive the following:

**Lemma 32.** *Let $\mu$ be an SRB measure. Then, for $\mu$-almost all $x \in \Lambda$, there is an ergodic invariant measure $\nu = \nu_x$ with the property that $\mathrm{Leb}_{W^u_{x,\epsilon}}$-almost every $y \in W^u_{x,\epsilon}$ is future and past generic to $\nu$ for $\epsilon > 0$ sufficiently small.*

*Proof.* Let $\Lambda' \subset \Lambda$ denote the full $\mu$-measure set such that for each $y \in \Lambda'$, the conclusions of Corollary 31 hold. Let $\mathcal{N} = \Phi(\Sigma \times B^u)$ be as in Proposition 10 and assume $\mu(\mathcal{N}) > 0$. It follows that for $\mu$-a.e. $x \in \mathcal{N}$, we have that $\mu_x$-almost every $y \in \Xi(x)$ belongs to $\Lambda'$, hence $\mathrm{Leb}_{W^u_x}$-almost every $y \in \Xi(x)$ belongs to $\Lambda'$.

The key now is to show that $\nu_y = \nu_x$ for Leb-almost all $y \in \Xi(x)$. For this, we note that all points in $W^u_{x,\epsilon}$ share the same past, and so must be past generic to the same measure. More precisely, if $\phi : \Lambda \to \mathbb{R}$ is continuous, then

$$\left| \frac{1}{N} \sum_{n=1}^{N} \phi(f^{-n} x) - \frac{1}{N} \sum_{n=1}^{N} \phi(f^{-n} y) \right| \le \frac{n_0}{N} \|\phi\|_{L^\infty} + \frac{N - n_0}{N} \epsilon \,.$$

Here, $\epsilon > 0$ is fixed and arbitrary, and $\delta > 0$ is such that $|\phi(x) - \phi(y)| < \epsilon$ if $d(x, y) < \delta$. We have taken $n_0 = n_0(\delta)$ sufficiently large so that $d(f^{-n}x, f^{-n}y) \le \hat{C}\hat{\lambda}^n \le \delta$ for all $n > n_0$. Passing $N \to \infty$, $\epsilon \to 0$ and applying proves that

$$\int \phi d\nu_x = \int \phi d\nu_y \tag{5}$$

for all continuous $\phi : \Lambda \to \mathbb{R}$.

**Exercise 33.** Complete the proof of Lemma 32 by checking that (5) implies $\nu_x = \nu_y$.

□

## 0.6 Holonomy across stable leaves

We have seen that points connected by an 'typical' unstable manifold are past and future generic with respect to the same ergodic measure $\nu$. It is not hard to show that the same is true along stable leaves.

**Lemma 34.** *Let $x \in \Lambda$ and assume $x$ is future generic to a measure $\nu_x$. Then, for any $y \in W^s_x$, we have that $y$ is also future generic to $\nu_x$.*

The proof is a time-reversal of that for Lemma 32 and is omitted.

This idea is at the heart of what is now called the *Hopf chain argument*. What it says is that if a pair of points $x, y \in \Lambda$ can be connected by a 'chain' of stable and unstable manifolds intersecting each other, then $x, y$ are future generic to the same measure.

Let us make use of this idea.

**Definition 35.** Let $W \subset U$ be a $\dim E^u$-dimensional embedded disk. Recall that $U$ is foliated by $W^s$-leaves and so each point $x \in W$ is contained in a unique $W^s$ leaf $W^s_x$. We say that $W$ is *transversal* to the $W^s$ foliation if

$$T_x M = T_x W \oplus T_x W^s_x$$

for all $x \in W$.

**Definition 36.** Let $W, W' \subset U$ be embedded $\dim E^u$ disks which are transversal to the $W^s$ foliation. A *stable holonomy* between $W$ and $W'$ is a continuous map $\pi : W \to W'$, if defined, with the property that for each $x \in W$, we have $\pi(x) \in W' \cap W^s_x$.

**Lemma 37.** *For any two transversals $W, W'$ to the $W^s$ foliation which are $C^1$ sufficiently close, there is a relatively open set $\hat{W} \subset W$ for which a stable holonomy $\pi : \hat{W} \to W'$ is defined.*

A crucial property of stable holonomies is *absolute continuity* with respect to Lebesgue measure.

**Theorem 38.** *Assume $W, W' \subset U$ are $W^s$-transversals for which a stable holonomy $\pi : W \to W'$ is defined. Then, $\pi_*(\mathrm{Leb}_W)$ is equivalent to $\mathrm{Leb}_{W'}$.*

**Remark 39.** You might be thinking that Theorem 38 is 'easy', since it appears to follow from Fubini's theorem applied to the $W^s$ foliation. The problem here is that at this level of generality, the $W^s$ foliation can only be assumed to vary Holder continuously, while the Fubini argument would require the foliation to vary in a $C^1$ way. Note that this is not about the regularity of individual leaves, which are just as smooth as the map $f$, but rather about the way that the laves are packed together in space.

Hopf's original argument was to prove ergodicity of Liouville measure for geodesic flow on surfaces of constant negative curvature. There, the stable foliation is in fact $C^1$ and therefore the stable holonomies, when defined, are absolutely continuous. The failure of the $C^1$ property explains the nearly 30 year gap between this and the proof of ergodicity geodesic flow on a surface of variable negative curvature, for which $E^s$, hence $W^s$, is possibly strictly Holder.

## 0.7 Completing the proof of Proposition 28

For the rest of the proof of Proposition 28, let us fix an $x \in \Lambda$ for which $\mathrm{Leb}_{W^u_{x,\epsilon}}$-almost all points are future and past generic to $\nu_x$ as in Lemma 32. Let $y \in \Lambda$ be close enough to $x$ so that $W^u_{x,\epsilon}$ and $W^u_{y,\epsilon}$ are $C^1$ close, and assume that Leb-a.e. point in $W^u_{y,\epsilon}$ are future and past generic to some

$\nu_y$. Applying Lemma 37, there is a defined stable holonomy $\pi : \hat{W} \to W^u_{y,\epsilon}$ for some relatively open $\hat{W} \subset W^u_{x,\epsilon}$.

By Theorem 38, the set $S_x$ of points in $\hat{W}$ which are future and past generic to $\nu_x$ is carried to a positive Lebesgue measure set $\pi(S_x)$ by the stable holonomy. On the other hand, the set $S_y$ of points in $W^u_{y,\epsilon}$ future and past generic to $\nu_y$ has full Leb-measure, and so $\pi(S_x) \cap S_y \neq \emptyset$. By Lemma 34, points in $\pi(S_x) \cap S_y$ are future generic to the same measure, and so we conclude that $\nu_x = \nu_y$.

## 0.8 Physicality of SRB measures for uniformly hyperbolic attractors

The proof of physicality of SRB measures as in Theorem 7 follows from similar arguments, which we now summarize.

It suffices to show that for $\mu$-almost all $x \in \Lambda$, there is a small neighborhood $V_x \subset U$ for which Leb-almost all $y \in V_x$ are future generic to the unique SRB measure $\mu$. To start, by Theorem 17, the SRB measure $\mu$ for $f|_\Lambda$ is ergodic, and so $\mu$-almost every $x$ has the property that $\text{Leb}_{W^u_x}$-almost every $z \in W^u_{x,\epsilon}$ is future (and past) generic to $\mu$.

**Claim 40.** Let $\epsilon > 0$ be sufficeintly small. Then, there is an open neighborhood $V_x \subset U$ foliated by $W^s$-transversals $W_\alpha, \alpha \in A$ (packed together in a $C^1$ way) with the property that for each $\alpha$, a stable holonomy $\pi_\alpha : W^u_{x,\epsilon} \to W_\alpha$ is defined.

The claim is left as an exercise. If, for instance $U \subset \mathbb{R}^d$, we can set $A = E^s_x \cap B_\delta(0), \delta > 0$ sufficiently small, and $W_\alpha = \alpha + W^u_{x,\epsilon}$. We complete the proof by noting that from Theorem 38, the full Lebesgue measure set $S_x$ of points in $W^u_{x,\epsilon}$ future generic to $\mu$ is carried by $\pi_\alpha$ to a set $\pi_\alpha(S_x)$ of full Lebesgue measure on each $W_\alpha$. We conclude that Leb-a.e. point in $V_x$ is future generic to $\mu$ by Fubini's theorem.