The Thom conjecture and the Milnor conjecture

Theorem (Ozsvath-Szabo) Let X be a symplectic 4-mfd. Let $\Sigma \hookrightarrow X$ be a symplectic surface. Then Σ is genus minimizing in its homology class. $[S]=[\Sigma] \Rightarrow g(S) \geq g(\Sigma)$

Set $X=\mathbb{C}P^2$, we have the Thom conjecture:

Theorem (Kronheimer-Mrowka) For $d>0$, we have

$$\min\{g(\Sigma) \mid \Sigma \hookrightarrow X, [\Sigma]=d\cdot P.D.(\mathbb{C}P^1)\} = \frac{(d-1)(d-2)}{2}$$

Proof: Consider $\Sigma_d = \{[x,y] \in \mathbb{C}P^2 \mid \underset{\text{generic, deg-}d}{f}(x,y)=0\}$.

Then Σ_d is a symplectic surface so satisfies the adjunction formula

$$2g(\Sigma_d)-2 = \Sigma_d\cdot\Sigma_d - c_1(T\mathbb{C}P^2)\cdot[\Sigma_d]$$
$$= d^2-3d \qquad \square$$

Corollary (local Thom conjecture) Let $\Sigma \hookrightarrow \mathbb{C}^2$ be an affine, smooth algebraic curve. Then Σ is locally genus minimizing.

(i.e. $\forall D^4 \subset \mathbb{C}^2$, $S \hookrightarrow D^4$ s.t. 1) $\partial D^4 \pitchfork \Sigma = K$
2) $\partial S = K$

we have $g(S) \geq g(\Sigma \cap D^4)$

Proof: Complete $\mathbb{C}^2$ into $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$.

Let $\overline{\Sigma}$ = closure of Σ in $\mathbb{CP}^2$.

We perturbe $\overline{\Sigma}$ s.t. it is smooth.

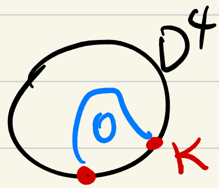
Since Σ is already smooth, this perturbation doesn't change $K = \Sigma \pitchfork \partial D^4$ up to isotopy

Suppose $\exists S \hookrightarrow \partial D^4$ s.t. $\partial S = K$ $g(S) < g(\Sigma \cap D^4)$.

Then $\overline{\Sigma}_S := (\overline{\Sigma} \setminus D^4) \cup_K S$ is homologous to $\overline{\Sigma}$

and $g(\overline{\Sigma}_S) < g(\overline{\Sigma})$. Contradiction. $\square$

- The slice-ribbon conjecture.



Given a knot $K \subset S^3$, we define the

$$g(K) := \min\{\Sigma \hookrightarrow S^3 \mid \partial\Sigma = K\}$$
$$g_s(K) := \min\{\Sigma \hookrightarrow D^4 \mid \partial\Sigma = K\}$$

Smooth, proper embedding

Facts: 1) $g_s(K) \leq g(K)$

2) $g(K) = 0 \Leftrightarrow K = U$.

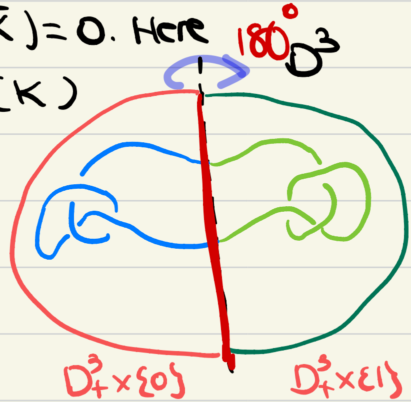
3) $g(K_1 \# K_2) = g(K_1) + g(K_2)$

However, (2) and (3) are not true for g_s.

slice $\Rightarrow$ ribbon

Actually, we have

4) For any K, we have $g_s(K \# \overline{K}) = 0$. Here $\overline{K}$ = mirror of K = reflection(K)



proof: $K \# \overline{K} = K^\circ \cup_{S^0} \overline{K}^\circ \quad K^\circ = K \setminus (K \cap D^3)$

$K^\circ \hookrightarrow D^3_+ = \{(x,y,z) \in D^3 \mid z \geq 0\}$

$D^4 = \underline{D^3} \times [0,1] / (x,y,0,t) \sim (x,y,0,t')$

$K \# \overline{K} \hookrightarrow D^3_+ \times \{0\} \cup_{D^2} D^3_+ \times \{1\} \subset \partial D^4$

bounds a disk $D^2 = K^\circ \times [0,1] / (s,t) \sim (s,t') \quad s \in \partial K^\circ$, $t, t' \in [0,1]$.

$\square$

We say K is $\boxed{\text{slice}}$ if $g_s(K) = 0$

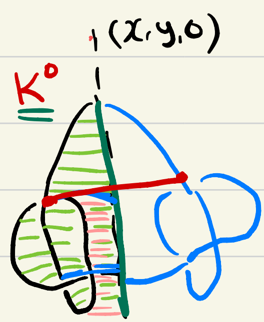
We say K is $\boxed{\text{ribbon}}$ if K bounds immersed $D^2 \subset S^3$ s.t. self intersection of $D^2 = \sqcup$ arcs.

$\partial \text{arcs} \subset K$

ribbon disk

Ribbon $\Rightarrow$ slice, we can push $\mathring{D}^2$ into $\mathring{D}^4$ to avoid self-intersections.

$\boxed{K \# \overline{K}}$ is always ribbon:



$K^\circ \hookrightarrow D^3_+$ bounds immersed disk D^2

s.t. self intersection of D^2

$= \{\text{arcs } \alpha_i \text{ with } \partial \alpha_i \subset K^\circ \cup \{(x,y,0)\}\}$

so $K \# \overline{K} = K^\circ \cup \overline{K}^\circ$ bounds $D^2 \cup \overline{D}^2$

ribbon disk.

Conjecture (The slice-ribbon conjecture)
Any slick knot is ribbon.

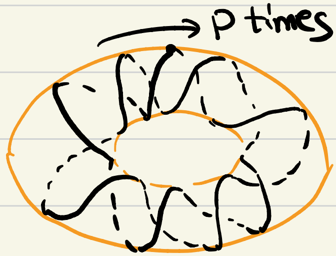
Equivalent version:
Conjecture: Suppose K is slice. Then $\exists\, D^2 \hookrightarrow D^4$ s.t.
(1) $\partial D^2 = K$
$D^4 \cup$
(2) The radius function $r: D^2 \to \mathbb{R}$ is
$(x_0, x_1, x_2, x_3) \longmapsto \sum x_i^2$
a Morse function with no local maximum.

- The Milnor conjecture. (solved by Kronheimer - Mrowka)

$T_{p,q}$ = the (p,q)-torus knot $\hookrightarrow T^2 \hookrightarrow S^3$

p times $\qquad p,q \geq 1$

q-times



$T(2,9)$

Theorem (Kronheimer - Mrowka) $g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2}$

Proof: By the local Thom conjecture, it suffices to find a smooth complex curve $C \hookrightarrow \mathbb{C}^2$ s.t.
1) $C \cap S^3 = T(p,q)$
2) $g(C \cap D^4) = \frac{(p-1)(q-1)}{2}$

To find C, we consider the map $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ $(x,y) \mapsto x^p - y^q$

Claim: $f^{-1}(0) \cap \partial B(\sqrt{2}) = T_{p,q}$.

proof: $f^{-1}(0) = \{(r^q e^{i\theta q}, r^p e^{i\theta p}) \mid r \in [0,+\infty), \theta \in [0,2\pi)\}$

so $f^{-1}(0) \cap \partial B(\sqrt{2}) = \{(e^{iq\theta}, e^{ip\theta}) \mid \theta \in [0,2\pi)\}$

$\subset \{(x,y) \mid |x|=|y|=1\} = T^2$

$\subset \{(x,y) \mid |x|^2+|y|^2=2\} = \partial B(\sqrt{2})$

$f^{-1}(0)$ is not smooth: $\{0\}$ is a singular point.

Consider $\Sigma_\varepsilon = f^{-1}(\varepsilon)$ for small $\varepsilon \in \mathbb{R}$.

called the milnor fiber

Then $\Sigma_\varepsilon \cap \partial B(\sqrt{2}) = T_{p,q}$. $\quad g(\Sigma_\varepsilon \cap B(\sqrt{2})) = ?$

Consider $\boxed{\Sigma_\varepsilon} \to \mathbb{C}$ $\quad (x,y) \mapsto x$. $\quad y^q = x^p - \varepsilon$

This is a q to 1 brached covering with branching set $x^p = \varepsilon$

By Riemann-Hurewicz

$1-2g(\Sigma_\varepsilon \cap B\sqrt{2}) = \chi(\Sigma_\varepsilon \cap B(\sqrt{2})) = q \cdot \chi(D^2) - p(q-1)$

$= q + p - pq$

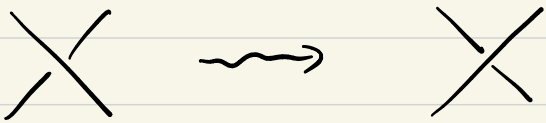
so $g(\Sigma_\varepsilon \cap B(\sqrt{2})) = \dfrac{(p-1)(q-1)}{2}$.

so $g_s(T(p,q)) = \dfrac{(p-1)(q-1)}{2}$ $\quad \square$.

Note that $g(T(p,q))$ also equals $\dfrac{(p-1)(q-1)}{2}$, which is easier to prove (using the Alexander polynomial)

There are some other invariants related to g_s.

Given a knot diagram, we can do a "crossing change" to turn it into another knot diagram.

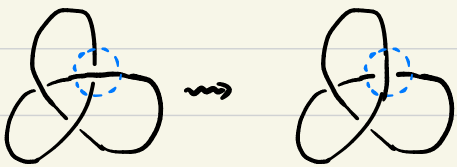


Given a diagram D, we define

$u(D) := \{$minimal number of crossing change to get a diagram of unknot$\}$

Given a knot K, we define the unknotting number

$$u(K) := \min\{u(D) \mid D \text{ is a diagram for } K\}.$$



$u(T_{2,3}) = 1$

We also define the "clasp number"

$c_4(K) := \min\{\#$ double points in S s.t. S is an immersed disk in D^4 bounded by K with only transverse double point$\}$.

Note $c_4(K) \leq u(K)$ since we can do a "crossing change" in $S^3 \times I$, that gives an annulus with a double point.

$\frac{(p-1)(q-1)}{1/2}$

double-point $\rightsquigarrow$ genus

$$g_s(K) \le C_4(K) \le u(K) \le \frac{(p-1)(q-1)}{2}$$

Note $g_s(K) \le C_4(K)$. The local model near a double point is $\{(x,y)\in\mathbb{C}^2 \mid xy=0\} \cap B(1) = D^2 \vee D^2$

$\{(x,y)\in\mathbb{C}^2 \mid xy=\varepsilon\} \cap B(1) = S^1 \times I$.

This "exchange" a double point with a genus.

So $g_s(K) \le C_4(K) \le u(K)$

When $K=T_{p,q}$, it's not hard to turn K into the unknot by $\frac{(p-1)(q-1)}{2}$ crossing change. So

$$g_s(T_{p,q}) = C_4(T_{p,q}) = C(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

- Knots bounding affine algebraic curves.

Q: Given K, when does there exist an affine algebraic curve $S \subset \mathbb{C}^2$ s.t. $S \pitchfork \partial D^4 = K$?

We have seen that $K=T(p,q)$ is OK.

It turns out a complete answer is known. (using braid)

Given a manifold X, the configuration space

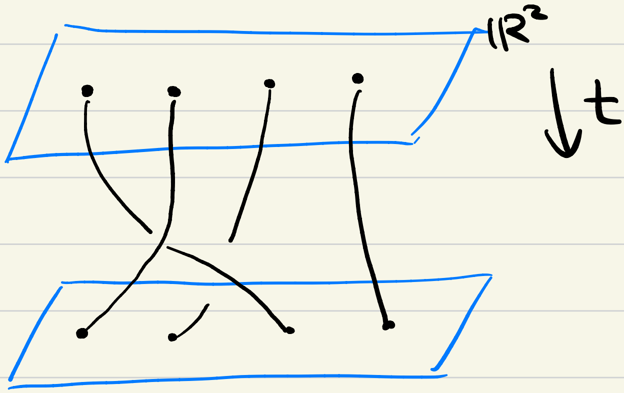
$$\mathrm{Conf}_n(X) = \{A \subset X \mid |A| = n\}$$

so a point in $\mathrm{Conf}_n(X)$ is $\{x_1, \cdots, x_n\}$ $x_i \neq x_j$.

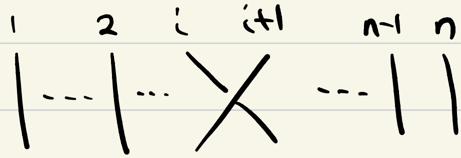
$X=\mathbb{R}^2$. Take the base point $*=\{(0,1),\cdots,(0,n)\}\in \mathrm{Conf}_n(\mathbb{R}^2)$.

An n-braid is a closed loop in $\mathrm{Conf}_n(\mathbb{R}^2)$ based at $*$.

We can represent a braid by a "braid diagram"



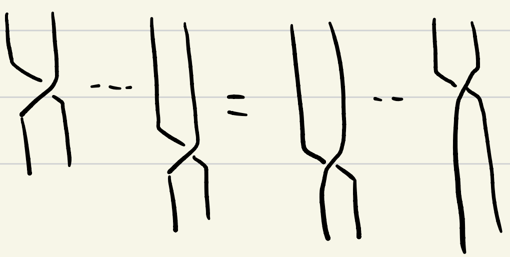
The braid group $B_n=\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$, it is generated by

$\sigma_i =$ 

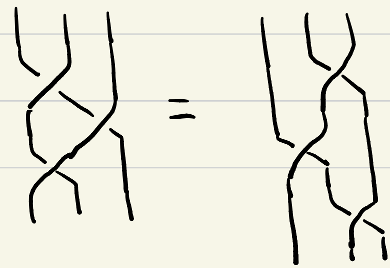
for $1\le i\le n-1$

subject to relation

$\sigma_i\sigma_j=\sigma_j\sigma_i$ if $|j-i|>1$

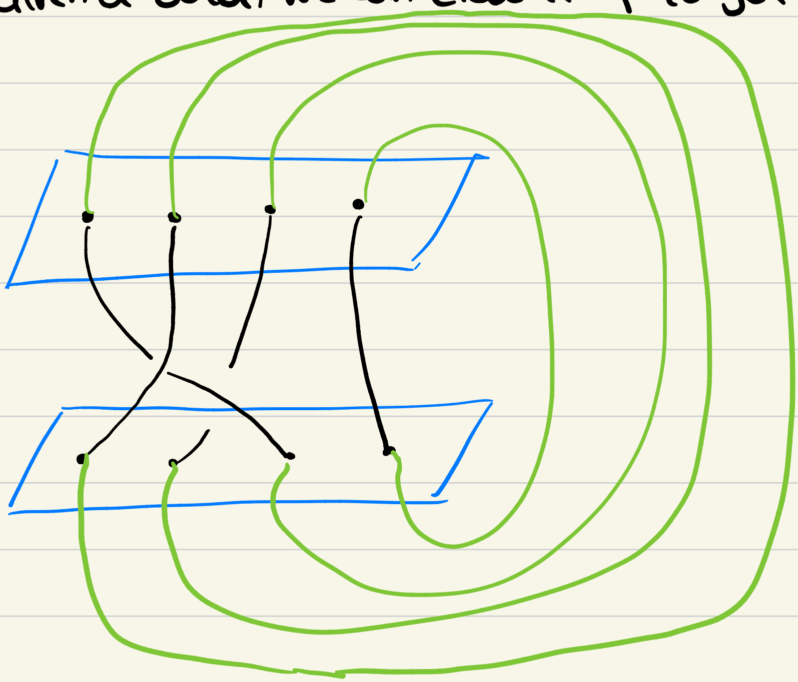


$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$



We say a braid is "positive" if it is a product of σ_is (no σ_i^{-1}s) A braid is "quasi-positive" if it has the form $\prod_{k=1}^{m} w_k \sigma_{i_k} w_k^{-1}$ where w_k is any word in the braid group.

Given a baid, we can close it up to get a link.



called the closure of the braid.

Theorem: Every link is a braid closure.

We say a knot is "braid positive" if it is the closesure of a positive braid.

We say a knot is "quasi-positive" if it is the closure of a quasi-positive braid.

Theorem (Rudolph, Boileau-Orevkov) A knot K arises as $K = S \pitchfork \partial D^4$ for some affine algebraic curve $S \subset \mathbb{C}^2$ iff K is quasi-positive.

Moreover, suppose $K = \bar{b}$ where

$$b = \prod_{k=1}^{m} w_k \sigma_{i_k} w_k^{-1} \in B_n$$

Then we can find S with $g(S \cap D^4) = \frac{m-n+1}{2}$.

Corollary: Suppose K is quasi-positive. Then $g_s(K) = \frac{m-n+1}{2}$.

(e.g. $T_{p,q}$)

For braid positive knot, one can show $u(K) = g_s(K)$

This is not true for quasi-positive knot.

One more open question (Genus versus double points)

Given $\Sigma \hookrightarrow X$, can we always find an immersed $S \looparrowright X$ with only double points s.t.

(1) $[S] = [\Sigma] \in H_2(X)$

(2) # double points in $S = g(\Sigma)$

(I.e., can we alway trade a genus with a double-point?)

f degree 5

$X_f = \{[x_0, x_1, x_2, x_3] \in \mathbb{CP}^3 \mid f(x_i) = 0\}$ quantic surface

$h = [X_f \cap \mathbb{CP}^2] \in H_2(X_5; \mathbb{Z})$ P.D. (i^* P.D. $(\mathbb{CP}^2)$)

(Caporaso - Harris - Mazur)

Conj: $\exists d_0$ s.t. $\forall d \geq d_0$ $\forall f$ with $\deg(f) = 5$ s.t. X_f smooth

$d \cdot h$ can $\boxed{\text{not}}$ be represented by rational curve.