Mini course 2 An example and Geometric Satake theory

<u>One case of the Theorem.</u> (Tian-X)

$$\begin{tikzcd}[column sep=small] & E \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ E_0 \arrow[dr, dash, "2"'] & & F = \text{totally real} \arrow[dl, dash, "2"] \\ & \mathbb{Q} & \end{tikzcd}$$

E_0 imag. quad.

$$\begin{tikzcd}[column sep=small] & v \quad \bar{v} \arrow[dl, dash] \arrow[dr, dash] & \\ v \quad \bar{v} \arrow[dr, dash] & & p \arrow[dl, dash, "\text{inert}"] \\ & p & \end{tikzcd}$$

$\mathrm{Hom}(E, \mathbb{C}) = \{\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2\}$

V hermitian space over E of sgn $(1,1)$ at both τ_1, τ_2

V' ———————— $(0,2)$ at τ_1, $(2,0)$ at τ_2

s.t. $V \otimes_F \mathbb{A}_{F,f} \simeq V' \otimes_F \mathbb{A}_{F,f}$

$\rightsquigarrow G = GU(V) = G(U(1,1) \times U(1,1)) \quad G' = GU(V') = G(U(0,2) \times U(2,0))$

Assume that V & V' are unramified at p, i.e. $V_{\mathbb{Q}_p}$ & $V'_{\mathbb{Q}_p}$ admit self-dual lattices

Fix an open compact subgroup $K \subseteq G(\mathbb{A}_f) = G'(\mathbb{A}_f)$

$\rightsquigarrow \underset{\text{dim}=2}{Sh_K(G)} / \mathbb{Z}_{p^2}, \quad \underset{\text{dim}=0}{Sh_K(G')} / \mathbb{Z}_{p^2}$

$\dim Sh_{G(\prod_i U(a_i, n-a_i))} = \sum_i a_i(n-a_i)$

In this case, $V_{\mu^*}^{\mathrm{Tate}} = (\mathrm{std}_2^* \otimes \mathrm{std}_2^*)^{\text{central weights}}$ is 2-dim'l.

So we expect 2 types of cycles.

$\exists$ the following moduli problem

$$\begin{tikzcd} & I_1, I_2 \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\text{locally closed immersion}"] & \\ Sh_K(G') & & Sh_K(G) \end{tikzcd}$$

I_1, I_2 parametrizes isogenies $\alpha: A \to B$ satisfying certain properties

(B, λ_B, η_B^p) — $Sh_K(G')$: discrete Shimura variety

(A, λ_A, η_A^p) — $Sh_K(G)$ $/\mathbb{F}_{p^2}$

At each point $x \in I_1(\bar{\mathbb{F}}_p)$, $A_x \rightsquigarrow A_x[p^\infty] = A_x[v^\infty] \overset{\text{dual}}{\oplus} A_x[\bar{v}^\infty]$ $\longleftrightarrow$ related to the fact that $U(V)_{\mathbb{Q}_p} \simeq GL_{2,\mathbb{Q}_{p^2}}$

$\Big\downarrow$ Taking Dieudonné modules

$\mathcal{D}(A_x) = \text{rank } 2 \text{ over } W(\bar{\mathbb{F}}_p) \otimes \mathcal{O}_{E_v}$

$\mathcal{D}(A_x) = \mathcal{D}(A_x)_{\tau_1} \oplus \mathcal{D}(A_x)_{\tau_2}$, each rank 2 over $W(\bar{\mathbb{F}}_p)$

$W(\bar{\mathbb{F}}_p) \otimes \mathcal{O}_{E_v} \simeq W(\bar{\mathbb{F}}_p) \times W(\bar{\mathbb{F}}_p)$ (via τ_1, τ_2)

Same for B_x

$$\begin{tikzcd}
\mathcal{D}(A_x)_{\tau_2} \arrow[r, "V", "①"'] \arrow[d, "\alpha_{\tau_2}"] & \mathcal{D}(A_x)_{\tau_1} \arrow[r, "V", "①"'] \arrow[d, "\alpha_{\tau_1}"] \arrow[l, bend right, "F"'] & \mathcal{D}(A_x)_{\tau_2} \arrow[d, "\alpha_{\tau_2}"] \arrow[l, bend right, "F"'] \\
\mathcal{D}(B_x)_{\tau_2} \arrow[r, "V", "②"'] & \mathcal{D}(B_x)_{\tau_1} \arrow[r, "V", "⓪"'] & \mathcal{D}(B_x)_{\tau_2}
\end{tikzcd}$$

s.t. $VF = FV = p$.

indicating $\operatorname{coker} \cong \overline{\mathbb{F}}_p^{\oplus 1}$ (from the sign condition)

Option 1: $\alpha_{\tau_2} \cong$, α_{τ_1} has coker $\overline{\mathbb{F}}_p$ $\rightsquigarrow I_1$

(If we want to compute the fiber $I_1 \to Sh_K(G')$, i.e. we have all the knowledge of B_x

$\rightsquigarrow$ The info of A_x amounts to $\overline{\operatorname{Im}\alpha_{\tau_1}} \subseteq \mathcal{D}(B_x)_{\tau_1}/p = \overline{\mathbb{F}}_p^{\oplus 2} \rightsquigarrow \mathbb{P}^1$.)

(Subtlety point: need to remember "$F(\overline{\operatorname{Im}\alpha_{\tau_1}})$" instead; otherwise gets an extra Frobenius twist.)

Option 2: α_{τ_2} has coker $\overline{\mathbb{F}}_p$, $\alpha_{\tau_1} = \times p$. (so coker $= 2$) $\longrightarrow I_2$

(The fiber $I_2 \to Sh_K(G')$ comes from the knowledge of $\overline{\operatorname{Im}\alpha_{\tau_2}}$. $\longrightarrow \mathbb{P}^1$)

Cohomology consequences:

$$\begin{tikzcd}
H^0(Sh_K(G')_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)^{\oplus 2}[\pi_f^p] \arrow[r, "\sim"] & \bigoplus_{i=1}^{2} H^0(I_{i,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)[\pi_f^p] \arrow[dr, "\text{Gysin}"] & \\
& & H^2(Sh_K(G)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))^{\mathrm{Frob}_{p^2}=1}[\pi_f^p] \arrow[dl, "\text{Res}"] \\
H^0(Sh_K(G')_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)^{\oplus 2}[\pi_f^p] \simeq & \bigoplus_{i=1}^{2} H^2(I_{i,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))[\pi_f^p] &
\end{tikzcd}$$

The "intersection matrix" of this is $\begin{pmatrix} -2p & T_p \\ T_p S_p^{-1} & -2p \end{pmatrix}$ "certain central twist; ignore"

Evaluating at π_f^p-isotypical part, $\rho_\pi(\mathrm{Frob}_{p^2}) = \begin{pmatrix} \alpha_\pi & 0 \\ 0 & \beta_\pi \end{pmatrix}$ with $\alpha_\pi \beta_\pi = p^2$

$$\rightsquigarrow \begin{pmatrix} -2p & \alpha_\pi + \beta_\pi \\ \alpha_\pi + \beta_\pi & -2p \end{pmatrix} \rightsquigarrow \det = 4p^2 - (\alpha_\pi + \beta_\pi)^2 = -(\alpha_\pi - \beta_\pi)^2$$

If $\alpha_\pi \neq \beta_\pi$, $H^0(Sh_K(G')_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)^{\oplus 2}[\pi_f^p] \hookrightarrow H^2(Sh_K(G)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))^{\mathrm{Frob}_{p^2}=1}[\pi_f^p]$

Rmk: When $\alpha_\pi = \beta_\pi$, expect image = rank 1 = invariants under Frob_p or $-\mathrm{Frob}_p$.

How to prove such a theorem in general?

$Sh_K(G) \qquad A$

$$\mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\mu} \qquad \mathcal{D}(A),F,V \rightsquigarrow {}^{\sigma}\mathcal{D}(A) \xrightarrow{F} \mathcal{D}(A)$$

$\mathcal{D}(A)$ is a free $\mathbb{Z}_{p^2}\otimes W(\overline{\mathbb{F}}_p)$-module of rank 2.

$\iff$ a $\mathrm{Res}_{\mathbb{Z}_{p^2}/\mathbb{Z}_p} GL_2$-torsor over $W(\overline{\mathbb{F}}_p)$

${}^{\sigma}\mathcal{D}(A) \xrightarrow{F} \mathcal{D}(A)$ a "modification" of $\mathrm{Res}_{\mathbb{Z}_{p^2}/\mathbb{Z}_p} GL_2$-torsors "of type μ"

(an isomorphism over $W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$)

Construct cohomological correspondences

& then lift

$$\begin{tikzcd}
\mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\mu} \arrow[r, phantom, "\square"] & \text{"}\mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\mu|\lambda}\text{"} \arrow[l] \arrow[r] \arrow[r, phantom, "\square"] & \mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\lambda} \\
\mathrm{Sh}_K(G) \arrow[u] & \text{"}\mathrm{Sh}_{G,\mu|\lambda}\text{"} \arrow[l] \arrow[r] \arrow[u] & \mathrm{Sh}_K(G') \arrow[u]
\end{tikzcd}$$

$\leftarrow$ central weight so that $\mathrm{Sh}_K(G')$ is discrete.

Need to study objects $\mathrm{Sht}^{\mathrm{loc}}_{G_{\mathbb{Q}_p},\mu}$ abstractly and locally.

<u>Geometric Satake equivalence</u> (Fix prime p)

$k := \mathbb{F}_p$, $\mathcal{O} := \mathbb{F}_p[\![\varpi]\!]$ or $\mathbb{Z}_p$, $E := \mathbb{F}_p(\!(\varpi)\!)$ or $\mathbb{Q}_p$. (Put $\varpi = p$ when $\mathcal{O} = \mathbb{Z}_p$)

G a reductive group$/\mathcal{O}$ (i.e. G_E is unramified.) $\breve{\mathcal{O}} := \overline{\mathbb{F}}_p[\![\varpi]\!]$ or $\widehat{\mathbb{Z}}_p^{ur}$

For a k-algebra R, define

① $\mathcal{O} = \mathbb{F}_p[\![\varpi]\!] \rightsquigarrow L^+G(R) := G(R[\![\varpi]\!])$, $LG = G(R(\!(\varpi)\!))$, $L^nG(R) = G(R[\![\varpi]\!]/(\varpi^n))$

② $\mathcal{O} = \mathbb{Z}_p$, we require R to be <u>perfect</u>

$\rightsquigarrow L^+G(R) := G(W(R))$, $LG(R) = G(W(R)[\frac{1}{p}])$.

$\uparrow$ ring of Witt vectors

The affine Grassmanian is the presheaf:

$$\mathrm{Gr} = \mathrm{Gr}_G : \mathbb{F}_p\text{-algs}^{(\mathrm{perf})} \longrightarrow \mathit{Sets}$$
$$R \longmapsto LG(R)/L^+G(R).$$

Theorem (Beilinson-Drinfeld? for $\mathcal{O} = \mathbb{F}_p[\![\varpi]\!]$, Zhu, Bhatt-Scholze for $\mathcal{O} = \mathbb{Z}_p$)

Gr is represented by (the perfection of) an ind-projective scheme $/\overline{\mathbb{F}}_p$.

__Example__: When $G = GL_n$ and $\mathcal{O} = \overline{\mathbb{F}}_p[\![\varpi]\!]$

$$Gr(\overline{\mathbb{F}}_p) = \left\{ \begin{array}{l} \mathcal{O}\text{-lattices } \mathcal{E} \subseteq \overline{\mathbb{F}}_p(\!(\varpi)\!)^{\oplus n}, \text{ commensurable with } \mathcal{E}_0 = \overline{\mathbb{F}}_p[\![\varpi]\!]^{\oplus n} \\ \text{write } \mathcal{E} \overset{\beta}{\dashrightarrow} \mathcal{E}_0 \end{array} \right\}$$

$$\updownarrow$$

$$GL_n(\overline{\mathbb{F}}_p(\!(\varpi)\!)) \big/ GL_n(\overline{\mathbb{F}}_p[\![\varpi]\!])$$

Choose another identification $\mathcal{E} \simeq \overline{\mathbb{F}}_p[\![\varpi]\!]^{\oplus n}$

then α gives an element of $GL_n(\overline{\mathbb{F}}_p(\!(\varpi)\!)) / GL_n(\overline{\mathbb{F}}_p[\![\varpi]\!])$

More generally, $Gr(R)$ classifies
- $\mathcal{E}$ a G-torsor over $\operatorname{Spec} R[\![\varpi]\!]$, and
- isom. $\mathcal{E}|_{\operatorname{Spec} R(\!(\varpi)\!)} \xrightarrow[\sim]{\beta} G \times_{\overline{\mathbb{F}}_p[\![\varpi]\!]} R(\!(\varpi)\!) = \mathcal{E}_0|_{\operatorname{Spec} R(\!(\varpi)\!)}$

__Cartan decomposition__

$$Gr = \coprod_{\lambda \in X_*(T)^+} \underbrace{L^+G\,\varpi^\lambda L^+G / L^+G}_{\mathring{Gr}_\lambda}$$

as locally closed subschemes

$\mathring{Gr}_\lambda \leftarrow$ smooth of dim $\langle 2\rho, \lambda\rangle$

ρ = half sum of positive roots

When $G = Gr_n$ $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ and $x \in \mathring{Gr}_\lambda(\overline{\mathbb{F}}_p)$ means

$$\mathcal{E} \big/ \varpi^N \overline{\mathbb{F}}_p[\![\varpi]\!]^{\oplus n} = \frac{\overline{\mathbb{F}}_p[\![\varpi]\!]}{(\varpi^{N+\lambda_1})} \oplus \cdots \oplus \frac{\overline{\mathbb{F}}_p[\![\varpi]\!]}{(\varpi^{N+\lambda_n})}$$

In particular if $\lambda = \omega_a = (\overbrace{1, \cdots, 1}^{a}, 0, \cdots, 0)$, $Gr_{\omega_a} = Gr(n, n-a)$ ← $n-a$ dim'l subspaces.

__Theorem__ ("Absolute" Geometric Satake equivalence)

Let $\hat{G}$ denote the Langlands dual group of G.

There exists an equivalence of tensor categories

$$\operatorname{Perv}_{L^+G}(Gr_{\bar{k}}) \xrightarrow{H^*(Gr_{\bar{k}}, -)} \operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$$

irreducible objects →

$$\operatorname{Sat}(V_\mu) = IC_{Gr_\mu} = j_{\mu!*}\overline{\mathbb{Q}}_\ell[\langle 2\rho, \mu\rangle] \longleftrightarrow V_\mu$$

The tensor structure on LHS is given by

$\mathcal{F}_1 \boxtimes \mathcal{F}_2$ $\quad p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$

$$\begin{tikzcd} Gr \times Gr & LG \times Gr \arrow[l, "p"'] \arrow[r, "q"] & LG \overset{L^+G}{\times} Gr = Gr \tilde{\times} Gr \arrow[d, "m"] \\ & & Gr \end{tikzcd}$$

proper

semismall!

$$\mathcal{E}_2 \dashrightarrow \mathcal{E}_1 \dashrightarrow \mathcal{E}_0 \quad \mapsto \quad \mathcal{E}_2 \dashrightarrow \mathcal{E}_0$$

$$(\mathcal{F}_1, \mathcal{F}_2) \longmapsto m_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$$

Geometric Satake vs. classical Satake

$$P^\circ_{L^+G}(Gr) = \{L^+G\text{-equivariant perverse sheaves, pure of weight } 0\}$$

$\cup$ simple objects: IC^N_μ s.t. $IC^N_\mu|_{\mathring{Gr}_\mu} = \bar{\mathbb{Q}}_\ell[\langle 2\rho,\mu\rangle](\langle \rho,\mu\rangle)$ choice of $\bar{\mathbb{Q}}_\ell(\tfrac{1}{2})$

if $\sigma(\mu)=\mu$, otherwise, take $\bigoplus_{\mu' \in \langle\sigma\rangle\cdot\mu} IC^N_{\mu'}$.

Then Satake equivalence upgrades to

$$P^\circ_{L^+G}(Gr) \xrightarrow{\sim} Rep_{\bar{\mathbb{Q}}_\ell}({}^LG)$$

Theorem The following diagram commutes (ϕ for geometric Frobenius)

$$\begin{tikzcd} {[\mathcal{F}] \in} & K_0(P^\circ_{L^+G}(Gr)) \arrow[r, "Sat"] \arrow[d, "\text{Sheaf-function dictionary}"] & K_0(Rep_{\bar{\mathbb{Q}}_\ell}({}^LG)) \arrow[r] & \bar{\mathbb{Q}}_\ell[\hat{G}\phi/Ad\hat{G}] \simeq \bar{\mathbb{Q}}_\ell[\hat{T}\phi/\hat{T}]^W = \bar{\mathbb{Q}}_\ell[X^\cdot(\hat{T})^\phi]^W \arrow[d, equal] \\ & Hk_G = C_c^\infty(G(\mathcal{O})\backslash \underbrace{G(E)/G(\mathcal{O})}_{= Gr(k)}, \bar{\mathbb{Q}}_\ell) \arrow[rr] & & C_c^\infty(T(\mathcal{O})\backslash T(E), \bar{\mathbb{Q}}_\ell)^{W_0} = \bar{\mathbb{Q}}_\ell[X_\cdot(T)^\phi]^{W_0} \end{tikzcd}$$

$[\mathcal{F}] \mapsto \left(x \mapsto Tr(\phi, [H^\cdot(\mathcal{F}_x)])\right)$

$[V] \longmapsto \chi_V|_{G\phi}$

A subtle remark: There are two Satake isomorphisms ϕ for geom. Frob, $\sigma = \phi^{-1}$ for arithmetic

$$\begin{tikzcd} & \bar{\mathbb{Q}}_\ell[\hat{G}\phi/Ad\hat{G}] \arrow[r, "\simeq"] \arrow[d, "g\phi \mapsto (g\phi)^{-1} = \sigma(g^{-1})\sigma"] & \bar{\mathbb{Q}}_\ell[X^\cdot(\hat{T})^\phi]^W \arrow[d, equal, "\lambda \mapsto -\lambda"] \\ K_0(Rep_{\bar{\mathbb{Q}}_\ell}({}^LG)) \arrow[ur] \arrow[dr] \arrow[urr, bend left, "Sat^{cl}"] \arrow[drr, bend right] & & \\ & \bar{\mathbb{Q}}_\ell[\hat{G}\sigma/Ad\hat{G}] \arrow[r, "\simeq"] & \bar{\mathbb{Q}}_\ell[X^\cdot(\hat{T})^\sigma]^W \end{tikzcd}$$

will use this later

$\mathrm{Sat}^{cl'}$

$$\mathrm{Sat}^{cl}(V) = \mathrm{Sat}^{cl'}(V^*)$$

Back to the "absolute" geometric Satake.

For later purpose: For $\gamma \in \mathrm{Gal}(\bar{k}/k)$, γ acts on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$

For a rep'n V of $\hat{G}$, define γV to be the rep'n $\hat{G} \xrightarrow{\gamma^{-1}} \hat{G} \longrightarrow GL(V)$

then $\gamma^* \mathrm{Sat}(V) \cong \mathrm{Sat}(\gamma V)$

Meaning:

$$\begin{tikzcd}
m^{-1}(\varpi^\nu) \arrow[d] & \in & \mathrm{Gr}_\mu \tilde{\times} \mathrm{Gr}_\lambda \arrow[d, "m"] \\
\varpi^\nu & \in & \mathrm{Gr}_{\mu+\lambda}
\end{tikzcd}$$

$$m_!\left(IC_\mu \tilde{\otimes} IC_\lambda\right) \cong \bigoplus_{\nu \in X^\cdot(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(V_\nu, V_\mu \otimes V_\lambda) \otimes IC_\nu$$

Compare to $V_\mu \otimes V_\lambda \cong \bigoplus_{\nu \in X^\cdot(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(V_\nu, V_\mu \otimes V_\lambda) \otimes V_\nu$

Note: $\mathring{\mathrm{Gr}}_\nu = G(\mathcal{O})$-orbit of ϖ^ν.