Geometry emerging from chaos

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Geometry and Mathematical Physics

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April 2022

Plan:

- Dimer models.
- Remarkable combinatorics.
- Concentration of dimer probability measure for large domains and limit shapes.
- Fluctuations, conformal invariance.
- Other developments.

Dimer models

Def.

- $\Gamma \subset \mathbb{R}^2$ a graph
- $w: E(\Gamma) \to \mathbb{R}_{>0}$ $\quad e \mapsto w(e) > 0$

  weight function
- Dimer configuration is

  a perfect matching on vertices connected by edges, $D \subset E(\Gamma)$

- Probability distribution

$$\mathrm{Prob}(D) = \frac{\prod_{e \in D} w(e)}{Z_\Gamma}$$

- $$Z_\Gamma = \sum_{D \subset \Gamma} \prod_{e \in D} w(e)$$

  Partition function. For the uniform distribution $w(e) = 1$, $\boxed{Z_\Gamma = \#(D)}$

- Physical meaning:

$$w(e) = \exp\left(-\frac{E(e)}{k_B T}\right)$$

$$\prod_{e \in D} w(e) = \exp\left(-\frac{E(D)}{k_B T}\right)$$

$$E(D) = \sum_{e \in D} E(e)$$

it would be a noninteracting system if not for dimer rules.

- Correlation functions

$$\sigma_e(D) = \begin{cases} 1, & e \in D \\ 0, & e \notin D \end{cases}$$

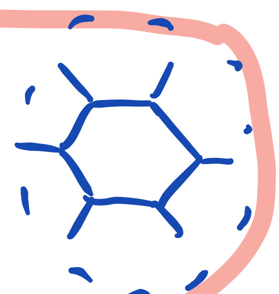
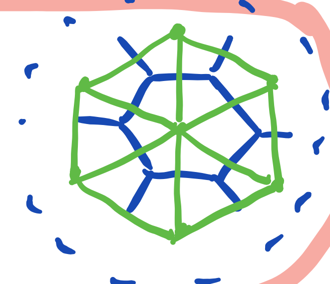
$$\mathbb{E}(\sigma_{e_1} \cdots \sigma_{e_k}) = \sum_{D} \mathrm{Prob}(D) \prod_{i=1}^{k} \sigma_{e_k}(D)$$

$\mathbb{E}(\sigma_{e_1} \cdots \sigma_{e_k})$ = Probability that $\{e_i\}$ are occupied by dimers.

$$\mathbb{E}(\sigma_{e_1}\cdots\sigma_{e_k}) = w(e_1)\cdots w(e_k)\cdot \frac{\partial^k}{\partial w(e_1)\cdots\partial w(e_k)} \ln(Z_\Gamma)$$

<u>Some remarkable combinatorics</u>

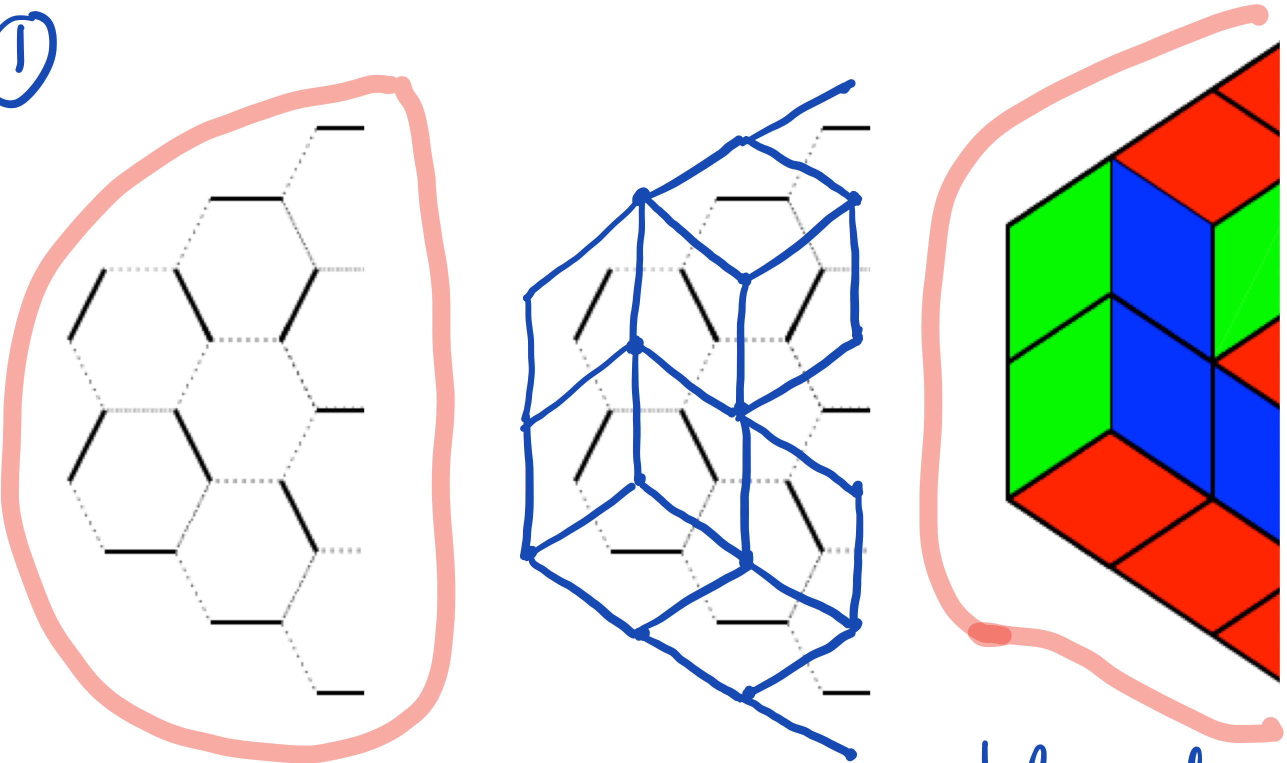
For simplicity focus on Regions of hex. lattice in $\mathbb{R}^2$

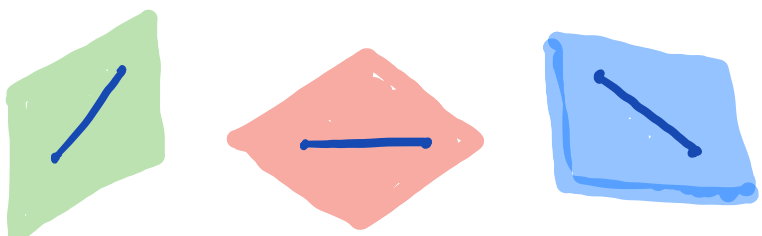
$\Gamma =$  $\subset \mathbb{R}^2$; $\Gamma^\vee =$ 

cell complex in $\mathbb{R}^2$, dual cell complex

<u>Connected, simply connected.</u>

1





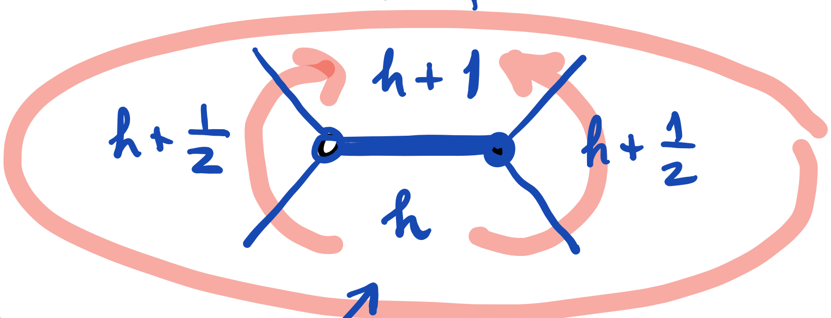
tiling of a triang. lattice by

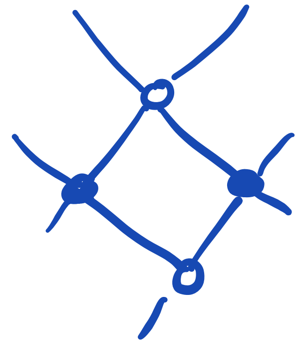
Bijection between dimers on Γ and tilings of $\Gamma^\vee$.

2 Bijection between dimers on Γ and height functions.

Assumes a bipartite structure on Γ.

h : faces of $\Gamma \longrightarrow \mathbb{R}$
vertices of $\Gamma^\vee$

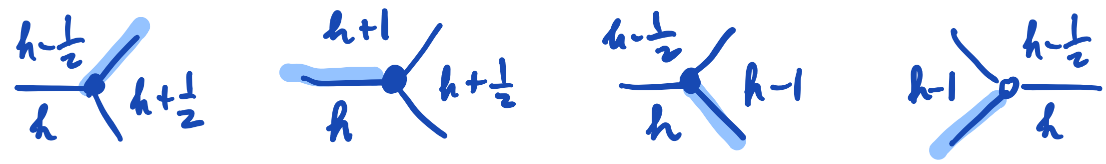


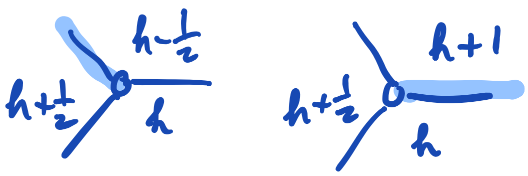


Bijection

Space of height functions

$\mathcal{H}_{\Gamma, f_0} = \{ h : \text{faces of } \Gamma \longrightarrow \tfrac{1}{2}\mathbb{Z} \mid$



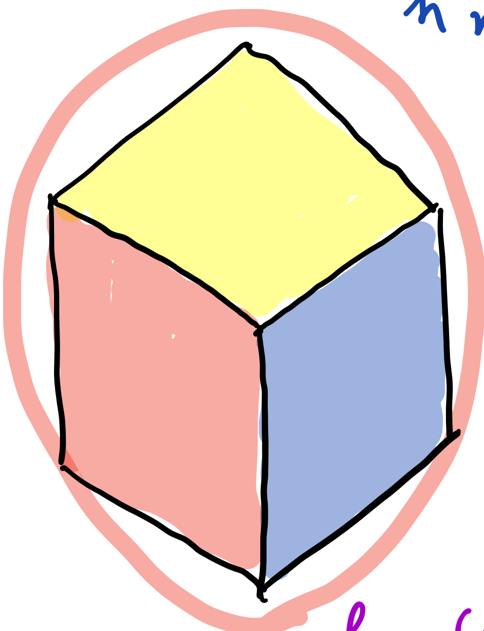


$, \ h(f_0) = 0 \}$ ✓

<u>Theorem</u>. For any $h, h' \in \mathcal{H}_{\Gamma, f_0}$
$$h|_{\partial \Gamma^\vee} = h'|_{\partial \Gamma^\vee}$$

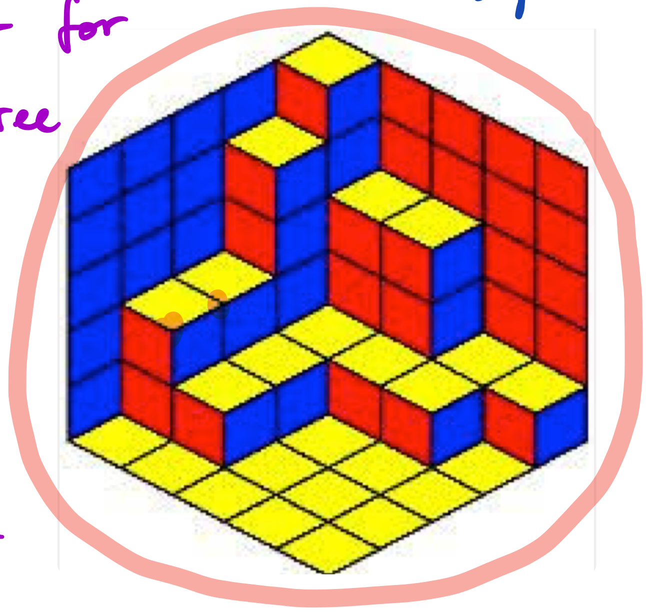
• $\mathcal{H}_{\Gamma, f_0}$ is partially ordered, $\exists\, h_{min}, h_{max}$

h_{max} ("filled room")



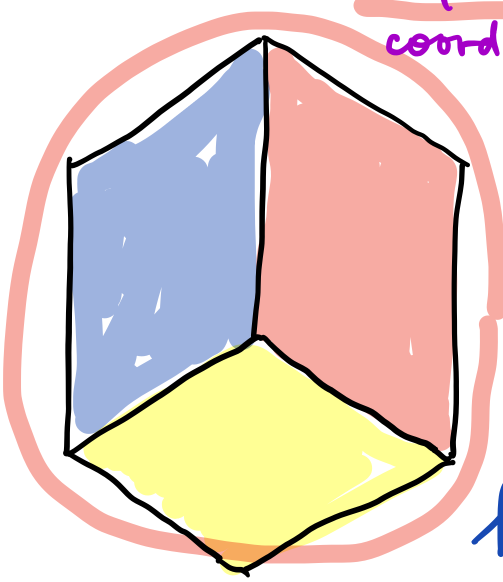
Same value $h|_{\partial\Gamma}$ for all tree

$\Gamma^\vee \subset$ Hex
hexagon



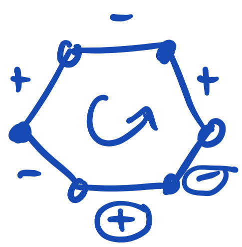
$h = (1,1,1)$
coord of a vertex on the surface of the pile

h ("pile of cubes in a room")



h_{min} ("empty room")

$\mathbb{R}^2 \circlearrowleft$



$$q_f = \prod_{e \in \partial f} w(e)^{\varepsilon(e,\partial f)} \qquad f \in \mathrm{Face}(\Gamma)$$

Thm. $\mathrm{Prob}(D) = \dfrac{\prod_{e \in D} w(e)}{\sum_{D \subset \Gamma} \prod_{e \in D} w(e)} = \dfrac{\prod_f q_f^{h_D(f)}}{\sum_{h \in \mathcal{H}_{\Gamma, f_0}} \prod_f q_f^{h(f)}}$,

local

Thus:

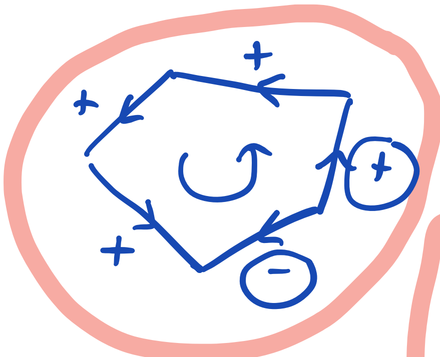
w-weight probability measure on dimer configurations = q-weight measure on height functions

(dimers $\xleftrightarrow[\text{bijection}]{}$ height functions)

Remark Generalizes to $\Gamma \subset \Sigma$.

Kasteleyn solution

- K-orientation of edges of Γ

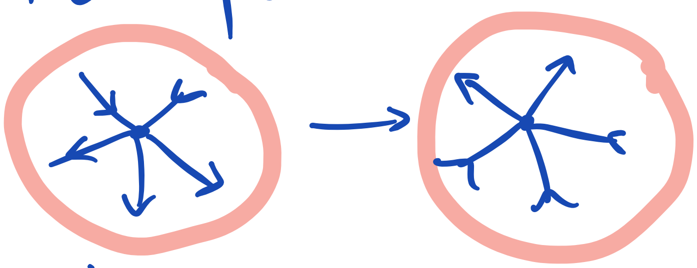


$\circlearrowright$ is the orientation of $\mathbb{R}^2$

$\displaystyle\prod_{e\in\partial f}\varepsilon(e,f) = -1$, $\forall f$

Thm 1) K-orientations exist,

2) The equivalence class $K \to K'$ by a sequence of moves



is unique.

Remark $[K] \longleftrightarrow$ spin structure on $\mathbb{R}^2$

similarly for surfaces.

$[K] \xrightarrow{\sim}$ Spin structures $\Gamma \subset \Sigma$

Def. Kasteleyn matrix:

$$\boxed{A^K_{v,v'} = \begin{cases} w(v,v') , & v \bullet\!\!\longrightarrow\!\!\bullet\, v' \\ -w(v,v') , & v \bullet\!\!\longleftarrow\!\!\bullet\, v' \\ 0 , & v \text{ is not connected to } v' \end{cases}}$$

$$(A^K)^t = -A^K$$

discrete ∂

discrete Dirac

fixed K-orientation

Thm (Kasteleyn, 1964)

$$\boxed{Z_\Gamma = \boxed{\sum_{D \subset \Gamma} \prod_{e \in D} w(e)} = \boxed{|Pf(A^K)|}}$$

$$\boxed{Pf(A) = \frac{1}{2^{n/2}\left(\frac{n}{2}\right)!} \sum_{\sigma \in S_n} (-1)^\sigma A^K_{\sigma_1\sigma_2} A^K_{\sigma_3\sigma_4} \cdots A^K_{\sigma_{n-1}\sigma_n} ,}$$

$$|Pf(A)| = \underline{(Det(A))^{1/2}} \qquad Det(A) > 0$$

Summary:

$$\begin{tikzcd}
\text{Dimers} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{Tilings} \arrow[r, leftrightarrow] \arrow[dl, dashed, leftrightarrow] & \text{Height functions} \arrow[dll, dashed, leftrightarrow] \\
\begin{array}{c}\text{Kastelein solution}\\ \text{(spin structure)}\end{array} & \Rightarrow \quad \Rightarrow & \left(\begin{array}{c}\text{Random}\\ \text{surfaces}\end{array}\right)
\end{tikzcd}$$

Remark

Same for surfaces $\Gamma \subset \Sigma$:

$$Z_\Gamma = \frac{1}{2^g} \sum_{\substack{\text{spin}\\ \text{structures } [K]}} \varepsilon_{[K]} \, \mathrm{Pf}(A_K)$$

$$= \sum_h \prod_{f \subset \Sigma \setminus \Gamma} q_f^{h(f)} \prod_{\substack{\text{fund}\\ \text{cycles}}} z_C^{\Delta h(C)}$$

Discrete Bose-Fermi correspondence.