Learning and Optimization in Multiagent Decision-Making Systems
Lecture Notes: Multiagent Opinion Dynamics
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Hegselmann-Krause Opinion Dynamics

In the Hegselmann-Krause model opinion dynamics, a finite number of agents frequently update their opinions based on the possible interactions among them. The opinion of each agent in this model is captured by a scalar quantity in one dimension or a vector in Euclidean space $\mathbb{R}^{d>1}$ in higher dimensions. In fact, because of the conservative nature of social entities, each agent in this model communicates only with those whose opinions are closer to him and lie within a certain level of his confidence (bound of confidence), where the distance between agents' opinions is measured by the Euclidian norm in the ambient space.

Let us assume that we have a set of n agents $[n]=\{1,\dots,n\}$ and we want to model the interactions among their opinions. It is assumed that at each time $t=0,1,2,\dots$, the opinion of agent $i\in[n]$ can be represented by a vector $x_i(t)\in\mathbb{R}^d$ for some $d\geq 1$. According to that model, the evolution of opinion vectors can be modeled by the following discrete-time dynamics:

$$x(t+1)=A(t,x(t),\vec{\epsilon})x(t), \tag{24}$$

where $A(t,x(t),\vec{\epsilon})$ is an $n\times n$ row-stochastic matrix and $x(t)$ is the $n\times d$ matrix such that its ith row contains the opinion of the ith agent at time $t=0,1,2,\dots$, i.e., it is equal to $x_i(t)$. We refer to $x(t)$ as the *opinion profile* at time t. The entries of $A(t,x(t),\vec{\epsilon})$ are functions of time step t, current profile $x(t)$, confidence vector $\vec{\epsilon}=(\epsilon_1,\epsilon_2,\dots,\epsilon_n)>0$ and an updating scheme. The parameters $\epsilon_i, i\in[n]$ are referred to as the *confidence bounds*. In the homogeneous case of the dynamics, we assume that $\epsilon_i=\epsilon,\ \forall i\in[n]$ for some $\epsilon>0$, while in the heterogeneous model, different agents may have different bounds of confidence. For the sake of simplicity of notation and for a fixed $x(0)\in\mathbb{R}^{n\times d}$, we drop the dependency of $A(t,x(t),\vec{\epsilon})$ on $x(t)$ and ϵ and simply write $A(t)$.

In the Hegselmann-Krause model, each agent i updates its value at time $t=0,1,2,\dots$, by averaging its own value and the values of all the other agents that are in its ϵ-neighborhood at time t. To be more specific, given a profile $x(t)$ at time t, define the matrix $A(t)$ in (24) by:

$$A_{ij}(t)=\begin{cases}\frac{1}{|\mathbb{N}_i(t)|} & \text{if } j\in\mathbb{N}_i(t),\\ 0 & \text{else,}\end{cases} \tag{25}$$

where $\mathbb{N}_i(t)$ is the set of agents in the ϵ-neighborhood of agent i, i.e.,

$$\mathbb{N}_i(t)=\{j\in[n]\,|\,\|x_i(t)-x_j(t)\|\leq\epsilon\}.$$

Definition 75. *We say that a time instance t is a merging time for the dynamics if two agents with different opinions move to the same place.*

Based on that definition, we can see that if two agents i and j merge at time instant t, then they will have the same opinion at time $t+1$ and onward, while their common opinion may vary with time. Moreover, prior to the termination time of the dynamics, we cannot have more than n merging times, since there are n agents in the model. In what follows next, we define the notions of termination time and communication graphs.

Definition 76. *For every set of $n \geq 1$ agents we define the termination time T_n of the Hegselmann-Krause dynamics to be the maximum number of iterations before steady state is reached over all the initial profiles.*

Definition 77. *Given an opinion profile at time t, we associate with that opinion profile an undirected graph $\mathscr{G}(t) = ([n], \mathscr{E}(t))$ where the edge $(i, j) \in \mathscr{E}(t)$ if and only if $i \in \mathbb{N}_j(t)$. We refer to such a graph as the communication graph or communication topology of the dynamics at time step t. Furthermore, a connected component of the communication graph is called δ-trivial for some $\delta > 0$, if all the agents in that component lie within a distance of at most δ from each other.*

Remark 18. *From Definition 77, it is not hard to see that for any $\delta < \epsilon$, a δ-trivial component forms a complete component (clique) in the communication topology of the dynamics. In particular, if there is such a δ-trivial component at some time t, then in the next time step, all the agents in that component will merge to the same opinion.*

Lemma 78. *Let $V(t) = \sum_{i,j \in [n]} \min\{\|x_i(t) - x_j(t)\|^2, \epsilon^2\}$. Then V is non-increasing along the trajectory of the Hegselmann-Krause dynamics. In particular, we have*

$$V(t) - V(t+1) \geq 4 \sum_{\ell=1}^{n} \|x_\ell(t+1) - x_\ell(t)\|^2.$$

In the following theorem, we provide a lower bound for the amount of decrease of the above Lyapunov function as long as there exists one non-ϵ-trivial component in the dynamics, which in turn allows us to bound the termination time of the Hegslemann-Krause dynamics.

Lemma 79 (Rayleigh-Quotient). *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a connected undirected graph and $\mathscr{L}$ be the Laplacian of $\mathscr{G}$, i.e., the diagonal entries of $\mathscr{L}$ equal to the degrees of the corresponding nodes, and $\mathscr{L}_{ij} = -1$ if $\{i, j\} \in \mathscr{E}$, and $\mathscr{L}_{ij} = 0$, otherwise. Then, the smallest eigenvalue of $\mathscr{L}$ is $\lambda_1 = 0$ with multiplicity one and the corresponding eigenvector $v_1 = 1$. Moreover, the second smallest eigenvalue of $\mathscr{L}$ is strictly positive and is given by*

$$\lambda_2(\mathscr{L}) = \min_{\substack{\|x\|=1 \\ x \perp \mathbf{1}}} x' \mathscr{L} x,$$

where $x \perp \mathbf{1}$ refers to all the vectors that are orthogonal to the vector of all ones, i.e., $x'\mathbf{1} = 0$.

Theorem 80. *The termination time of the Hegselmann-Krause dynamics in arbitrary finite dimensions is bounded from above by $T_n \leq n^8 + n$.*

Proof: Let us assume that the opinion profile $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))'$ is not an equilibrium point of the dynamics and that time t is not a merging time. Without loss of generality, we may assume that the communication graph at time t is connected with a non-ϵ-trivial component; otherwise, we can restrict ourselves to one of the non-ϵ-trivial components. (Note that such a non-ϵ-trivial component exists, because of Remark 18 and the fact that t is not a merging time.) By projecting each individual column of $x(t)$ to the consensus vector $\mathbf{1}$ (i.e., vector of all ones) we can write

$$x(t) = \left[c_1\mathbf{1} | c_2\mathbf{1} | \ldots | c_d\mathbf{1}\right] + \left[\bar{c}_1 \mathbf{1}^{\perp(1)} | \bar{c}_2 \mathbf{1}^{\perp(2)} | \ldots | \bar{c}_d \mathbf{1}^{\perp(d)}\right], \tag{26}$$

where $\mathbf{1}^{\perp(k)}, k=1,\ldots,d$ are column vectors of unit size that are orthogonal to the consensus vector, i.e., $\mathbf{1}'\mathbf{1}^{\perp(k)}=0$, and $c_k, \bar{c}_k, k=1,\ldots,d$ are coefficients of projection of the kth column of $x(t)$ on $\mathbf{1}$ and $\mathbf{1}^{\perp(k)}$, respectively.

Now we claim that $\sum_{k=1}^{d}\bar{c}_k^2 > \frac{\epsilon^2}{4}$. Otherwise, we show that every two agents $x_i(t)$ and $x_j(t)$ must lie within a distance of at most ϵ from each other, which is in contrast with the assumption that the component is a non-ϵ-trivial component. In fact, if $\sum_{k=1}^{d}\bar{c}_k^2 \leq \frac{\epsilon^2}{4}$, we can write,

$$\begin{aligned}\|x_i(t)-x_j(t)\|^2 &= \sum_{k=1}^{d}\bar{c}_k^2\big(\mathbf{1}_i^{\perp(k)}-\mathbf{1}_j^{\perp(k)}\big)^2 \leq 2\sum_{k=1}^{d}\bar{c}_k^2\big((\mathbf{1}_i^{\perp(k)})^2+(\mathbf{1}_j^{\perp(k)})^2\big)\\ &\leq 2\sum_{k=1}^{d}\bar{c}_k^2\big(\|\mathbf{1}^{\perp(k)}\|^2+\|\mathbf{1}^{\perp(k)}\|^2\big) = 4\sum_{k=1}^{d}\bar{c}_k^2 \leq \epsilon^2,\end{aligned} \tag{27}$$

where the first equality is due to the decomposition given in (26) and the second equality is valid since the vectors $\mathbf{1}^{\perp(k)}, k=1\ldots,d$, are of unit size. The contradiction shows that $\sum_{k=1}^{d}\bar{c}_k^2 > \frac{\epsilon^2}{4}$.

Next, we notice that $x(t+1)=A(t)x(t)$, where $A(t)$ is the stochastic matrix defined in (25). Using (26) we can write,

$$x(t)-x(t+1) = (I-A(t))x(t) = \Big[\bar{c}_1(I-A(t))\mathbf{1}^{\perp(1)}|\ldots|\bar{c}_d(I-A(t))\mathbf{1}^{\perp(d)}\Big], \tag{28}$$

where the equality holds since $\mathbf{1}$ belongs to the null space of $I-A(t)$. In particular, we have,

$$\begin{aligned}\sum_{\ell=1}^{n}\|x_\ell(t)-x_\ell(t+1)\|^2 &= \sum_{\ell=1}^{n}\sum_{k=1}^{d}\big(x_{\ell k}(t)-x_{\ell k}(t+1)\big)^2\\ &= \sum_{k=1}^{d}\Big(\sum_{\ell=1}^{n}\big(x_{\ell k}(t)-x_{\ell k}(t+1)\big)^2\Big)\\ &= \sum_{k=1}^{d}\bar{c}_k^2\|(I-A(t))\mathbf{1}^{\perp(k)}\|^2,\end{aligned} \tag{29}$$

where in the last equality we have used (28). Let us assume that $Q(t)=(I-A(t))'(I-A(t))$. It is not hard to see that $Q(t)$ is a positive semidefinite matrix. Moreover, 0 is an eigenvalue of Q with multiplicity one, corresponding to the eigenvector $\mathbf{1}$. To see that, let us assume that there exists another vector v, such that $Q(t)v=0$. Multiplying that equality from the left by v', we get $\|(I-A(t))v\|^2=0$, and hence $(I-A(t))v=0$. Since by Lemma 79, $\mathbf{1}$ is the only unit eigenvector of $I-A(t)$ corresponding to eigenvalue 0, we conclude that $v=\alpha\mathbf{1}$ for some $\alpha\in\mathbb{R}$. In other words, $\mathbf{1}$ is the only unit eigenvector of $Q(t)$ corresponding to eigenvalue 0. Moreover, $Q(t)$ is a symmetric real-valued matrix and, hence, diagonalizable, where $\mathbf{1}$ is its only eigenvector corresponding to eigenvalue 0. That shows that the multiplicity of the eigenvalue 0 in $Q(t)$ is exactly one.

Let us use $\lambda_2(Q(t))$ to denote the second smallest eigenvalue of $Q(t)$. By the above argument, it must be strictly positive. Using Lemma 79, we get $\lambda_2(Q(t))=\min_{\|y\|=1,y\perp\mathbf{1}}y'Q(t)y$. Now for every $k=1,\ldots,d$, we can write

$$\begin{aligned}\|(I-A(t))\mathbf{1}^{\perp(k)}\|^2 &= (\mathbf{1}^{\perp(k)})'(I-A(t))'(I-A(t))\mathbf{1}^{\perp(k)}\\ &= (\mathbf{1}^{\perp(k)})'Q(t)\mathbf{1}^{\perp(k)} \geq \min_{\substack{\|y\|=1\\ y\perp\mathbf{1}}} y'Q(t)y = \lambda_2(Q(t)),\end{aligned} \tag{30}$$

where the inequality holds, since $\mathbf{1}'\tilde{\mathbf{1}}^{\perp(k)} = 0$ and $\|\tilde{\mathbf{1}}^{\perp(k)}\| = 1$. Substituting (30) in (29) we get

$$\sum_{\ell=1}^{n} \|x_\ell(t) - x_\ell(t+1)\|^2 \geq \sum_{k=1}^{d} \lambda_2(Q(t))\bar{c}_k^2 \geq \lambda_2(Q(t))\frac{\epsilon^2}{4}. \tag{31}$$

Henceforth, we bound $\lambda_2(Q(t))$ from below based on a function of n. For that purpose, let us assume that $D(t) = diag\big(1 + d_1(t), 1 + d_2(t), \ldots, 1 + d_n(t)\big)$, i.e., $D(t)$ is a diagonal matrix with $D_{kk}(t) = 1 + d_k(t), k \in [n]$. Moreover, let $\mathscr{L}(t)$ denote the Laplacian matrix of the communication graph at time step t. By entry wise comparison of both sides, it is not hard to see that $I - A(t) = D(t)^{-1}\mathscr{L}(t)$. Now we can write,

$$\lambda_2(Q(t)) = \lambda_2((D(t)^{-1}\mathscr{L}(t))'(D(t)^{-1}\mathscr{L}(t))) = \lambda_2(\mathscr{L}(t)D(t)^{-2}\mathscr{L}(t)), \tag{32}$$

where the last equality is due to the fact that $\mathscr{L}(t)$ and $D(t)$ are both symmetric matrices. Next, using the same argument as above, we notice that since $\mathscr{L}(t)D(t)^{-2}\mathscr{L}(t)$ is a symmetric and real-valued matrix, it is diagonalizable, and its zero eigenvalue corresponding to eigenvector $\mathbf{1}$ has multiplicity one. Therefore, using Lemma 79, we can write,

$$\begin{aligned}
\lambda_2\big(\mathscr{L}(t)D(t)^{-2}\mathscr{L}(t)\big) &= \min_{\substack{\|y\|=1 \\ y \perp \mathbf{1}}} y'\mathscr{L}(t)D(t)^{-2}\mathscr{L}(t)y \\
&\geq \min_{\substack{\|y\|=1 \\ y \perp \mathbf{1}}} y'\mathscr{L}(t)(\frac{1}{n^2}I)\mathscr{L}(t)y \\
&= \lambda_2\Big(\mathscr{L}(t)(\frac{1}{n^2}I)\mathscr{L}(t)\Big) \\
&= \frac{1}{n^2}\lambda_2\big(\mathscr{L}^2(t)\big) = \frac{1}{n^2}\lambda_2^2\big(\mathscr{L}(t)\big),
\end{aligned} \tag{33}$$

where the last equality is due to the fact that $\mathscr{L}$ is diagonalizable (it is a symmetric and real-valued matrix) with an eigenvalue 0 of multiplicity 1. Substituting (33) in (32) we get $\lambda_2(Q(t)) \geq \frac{1}{n^2}\lambda_2^2\big(\mathscr{L}(t)\big)$. Finally, since $\mathscr{L}(t)$ is the Laplacian of a connected graph, it is known that $\lambda_2\big(\mathscr{L}(t)\big)$ from below by $\frac{2}{n^2}$. Putting it all together, we have,

$$\lambda_2(Q(t)) \geq \frac{1}{n^2}\lambda_2^2\big(\mathscr{L}(t)\big) \geq \frac{4}{n^6}. \tag{34}$$

Finally, combining (34) with (31), we conclude that the amount of decrease in the quadratic Lyapunov function if there is a non-ϵ-trivial component is at least $\frac{\epsilon^2}{n^6}$. In other words, if t is not a merging time, we have $V(t) - V(t+1) \geq \frac{\epsilon^2}{n^6}$. Since by definition $V(\cdot)$ is always a nonnegative quantity with $V(0) \leq \epsilon^2 n^2$ and the number of merging times can be at most n, we conclude that the termination time is bounded from above by $n^8 + n$. □