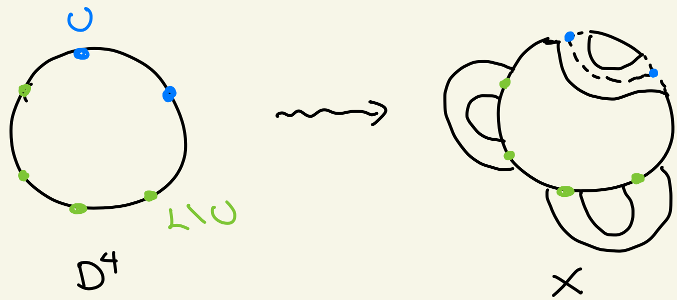
Kirby calculas (II)

Recall: A Kirby diagram D is a link L in S^3. Each component is decorated with • or an integer a_i, s.t. the dotted component form a unlink U.

From D, we can recover a 4-mfd $X = X_0 | X_1 | X_2$ as follows:

1) Let $\sqcup D^2 \hookrightarrow S^3$ be the disks bounded by U.
2) Push $\sqcup \mathring{D}^2$ into interior of D^4. Remove small neighborhoods.
3) Attach 2-handles along the framed link $L \setminus U$ to get

   $X = X_0 | X_1 | X_2$



4) $\partial X =$ Surgery of S^3 along L

$$\text{surgery coefficient} = \begin{cases} 0 & \text{dotted components} \\ a_i & \text{undotted components} \end{cases}$$

If $\partial X \cong \#^n (S^2 \times S^1)$, then $\exists$ unique way to complete X into closed $\tilde{X} = X_0 | X_1 | X_2 | X_3 | X_4$

More generally, given any decomposition $\partial X = Y \# Y'$ with $Y' \cong \#^k (S^2 \times S^1)$

$\exists$ unique way to form $\tilde{X} = X_0 | X_1 | X_2 | X_3$ with $\partial \tilde{X} = Y$.

Classical invariants of X can be directly read from D.

- $\pi_1(X) \cong \pi_1(\tilde{X}) = \pi_1(S^3 \setminus \{\text{dotted components}\}) / \text{undotted components}$

In particular, no dotted component $\Rightarrow$ no 1-handles $\Rightarrow \pi_1(X) = 0$

We say X (or $\tilde{X}$) is geometrically simply connected if it has a handle decomposition without 1-handles.

$\tilde{X}$: simply-connected, smooth 4-mfd.

Conjecture A: $\tilde{X}$ is geometrically simply-connected.

Conjecture B: $\tilde{X}$ admits a Morse function without index 1 and 3 critical points.

(Conjecture B $\Rightarrow$ Conjecture A
$\Rightarrow$ SPC 4)

- $\tilde{H}_*(X;\mathbb{Z}) = H_*(0 \to \mathbb{Z}\langle 2\text{-handles}\rangle \xrightarrow{[\text{incidence numbers}]} \mathbb{Z}\langle 1\text{-handles}\rangle \to 0)$

$= H_*(0 \to \mathbb{Z}\langle L \setminus U\rangle \xrightarrow{[lk(-,-)]} \mathbb{Z}\langle U\rangle \to 0)$

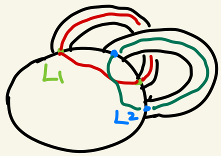
In particular, if no 1-handles, then $H_2(X) = \mathbb{Z}\langle L\rangle$.
$(U = \emptyset)$

These generators can be explicitely described:

$L \supset L_i$ bounds: 1) Seifert surface $F_i \hookrightarrow S^3$

2) core of the attached 2-handle $0 \times D^2 \hookrightarrow H_i^2 = D^2 \times D^2$

Let $S_i = F_i \cup_{L_i} (0 \times D^2) \hookrightarrow X$ Then $H_2(X) = \mathbb{Z}\langle [S_i]\rangle$.



Corollary: $S_i \cdot S_j = lk(L_i, L_j)$

proof: $S_i \cdot S_j = F_i \cdot F_j = lk(L_i, L_j)$ $\square$

For a 4-manifold X with $\partial X \neq \phi$, we define intersection form Q_X as bilinear form on $V = \operatorname{Im}(H_2(X) \xrightarrow{i_*} H_2(X, \partial X)) / \text{torsion}$

$Q_X(i_*(\alpha), i_*(\beta)) = (P.D.(\alpha) \cup \beta) \cdot [X]$.

Fact: • When $b_1(\partial X) = 0$, $V = H_2(X)/\text{torsion}$

• Q_X is nondegenerate over $\mathbb{R}$ (i.e. $\det Q_X \neq 0$)

but not necessary unimodular.

Proposition: Given any symmetric Q with $\det(Q) \neq 0$, $\exists$ a simply connected 4-mfd X s.t. 1) $Q_X = Q$

2) $|\det(Q)| = |H_1(\partial X)|$

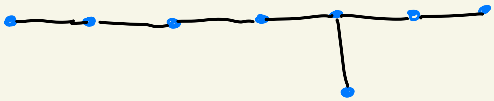
proof: Let $Q = (a_{ij})_{1 \le i,j \le n}$. Then we pick a link $L = L_1 \sqcup \cdots \sqcup L_n$ s.t. $lk(L_i, L_j) = a_{ij}$. We attach 2-handles H_i^2 to the framed links (L_i, a_{ii}) and get X. Then $Q_X = Q$.

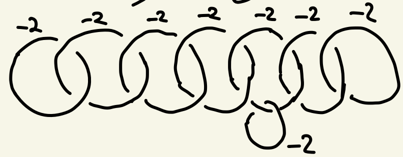
$\partial X = S^3_{(a_{11}, \cdots, a_{nn})}(L_1 \sqcup \cdots \sqcup L_n)$ so

so $H_1(\partial X) = \mathbb{Z}^n / (a_{ij}) \cdot \mathbb{Z}^n \Rightarrow |H_1(\partial X)| = |\det(a_{ij})|$. $\square$

Example $Q = E_8$

Kirby diagram





Note: ∂X is actually the Poincaré homology 3-sphere $S^3_{+1}(T_{2,3})$.

Theorem (Freedman) Any integer homology 3-sphere bounds a contractible topological 4-mfd.

(IHS^3, i.e. $H_1(Y;\mathbb{Z}) = 0$)

Corollary: Any unimodular Q can be realized as $Q_{\tilde{X}}$ for some simply connected, topological $\tilde{X}$.

proof: $Q = Q_X$ for some S.C. smooth X with boundary.

Q unimodular $\Rightarrow \partial X$ is a homology 3-sphere

$\Rightarrow \partial X$ bounds a contractible W

$\tilde{X} = X \cup_{\partial X} W$ $\square$

Remark: Most IHS^3 do not bound smooth, contractible 4-mfds.

E.g. $\Sigma(2,3,5)$ does not bound a contractible smooth 4-mfd.

E.g. (Casson-Harer) $\Sigma(r, rs-1, rs+1)$ r even s odd

$\Sigma(r, rs\pm 1, rs\pm 2)$ r odd

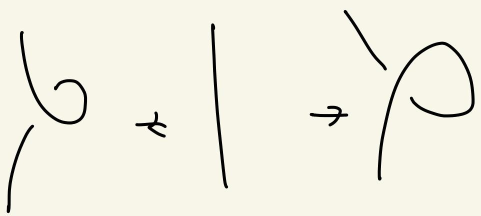
bounds contractible 4-manifold.

Kirby moves

Theorem. Any two Kirby diagrams for X^4 are related by a sequence of the following moves:

1) Isotpies of the link (Reidemeister moves)
2) Handle slides (1-handles, 2-handles)
3) Handle creation and cancellation (between 1 and 2-handles, between 2 and 3 handles)
4) One more move due to the dotted notation (slide 2-handles over 1-handles)

1)



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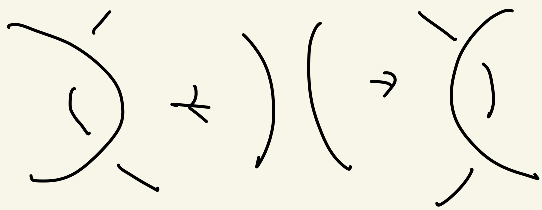

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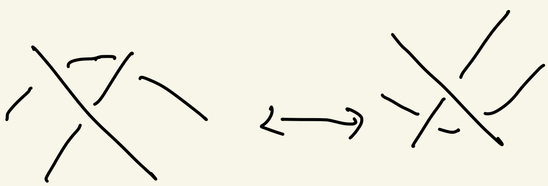

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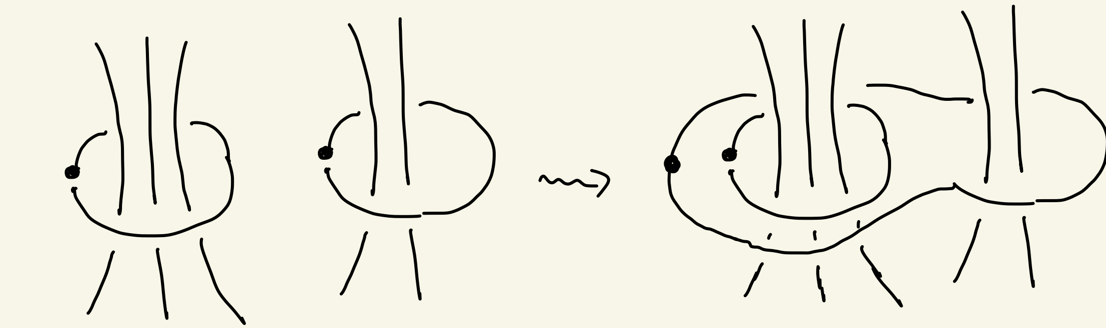
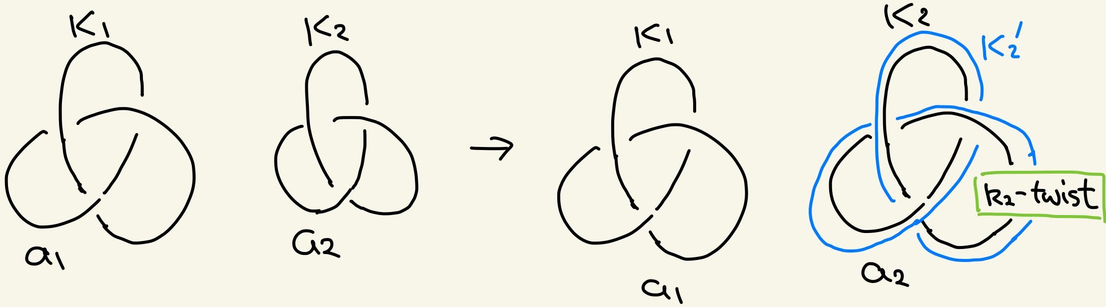
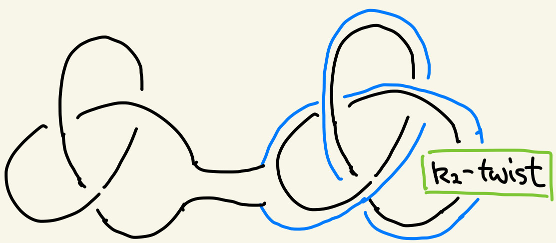
2)



Sliding of 1-handles



$$k_2 = a_2 - \#(+\text{crossing of } K_2) + \#(-\text{crossing of } K_2)$$



$K_1 \mapsto K_1 \# K_2'$

$K_2 \mapsto K_2$

$a_1 \mapsto a_1 + a_2 \pm lk(K_1, K_2)$

wheher # respect orientation

$a_2 \longmapsto a_2$

Sliding of 2-handles.

3)

cancellation of 1-handle U_1 and 2-handle K_1

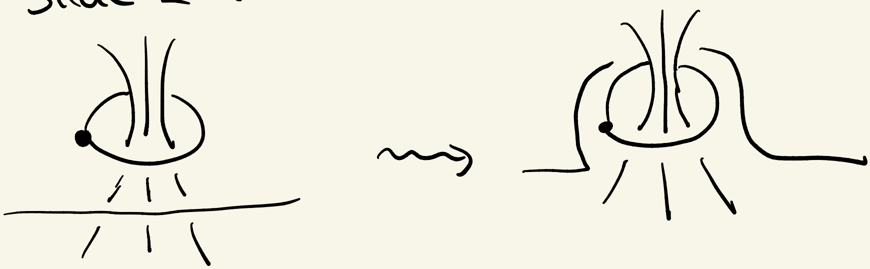
K_1 should be unknotted with other U_i

U_1 should be unknotted with other K_i (can be achieved by sliding over K_1)

cancellationg of 2-handle and 3 handle:

Remove an isolated $\bigcirc\,0$

4) slide 2-handle over 1-handle



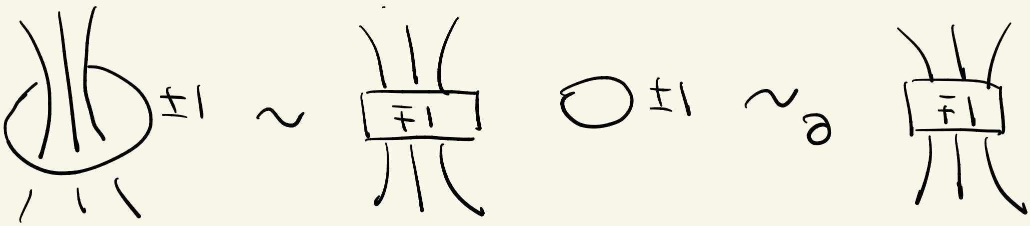
One move that doesn't change ∂X but change X:

5) blow up: add an isolated $\bigcirc\,\pm 1$

$$X \rightsquigarrow X \# \mathbb{CP}^2 \text{ or } X \# \overline{\mathbb{CP}}^2$$

blow down: remove an isolated $\bigcirc\,\pm 1$

Note: we can combine 5) with handle slide



3-dim version:

Theorem (Kirby) Two surgery diagrams represent the same 3-manifold iff they are related by Reidemeister moves, handle slides, blow-up/downs.

Mazur manifold

A smooth 4-manifold X is called a Mazur manifold if

1) M has a handle decomposition with a 0-handle a 1-handle, a 2-handle, and nothing else.
2) $H_1(M)=0$ ($\Leftrightarrow$ M contractible $\Leftrightarrow$ $H_1(\partial M)=0$)

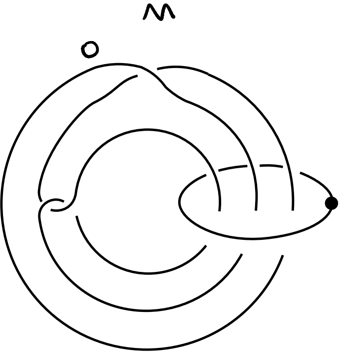
The following are equivalent:

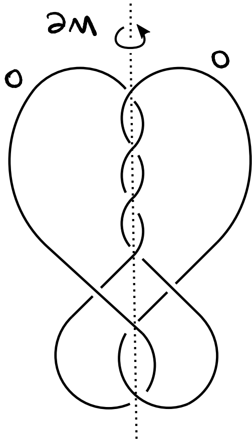
1) M is a Mazur manifold
2) M has a Kirby diagram $U \sqcup K$ (dotted U, undotted K) with $lk(U,K)=\pm 1$
3) M is obtained by attach a 2-handle to $S^1 \times D^3$ along a link ℓ that generates $H_1(S^1$

Theorem (Mazur) $M \cup_{\partial M} M \cong_{diff} S^4$

Corollary: $\exists$ orientation reversing involution $\tau: S^4 \to S^4$ s.t. $Fix(\tau) \neq S^3$

The Akbulut - Mazur cork





Definition: A cork is a contractible 4-mfd M with a diffeomorphism $f: \partial M \to \partial M$ that can not be extended as a diffeomorphism on M.

Note: f can always be extended as homeomorphism by the following theorem:

Theorem (Freedman) Let M, M' be contractible, topological 4-manifolds. Then any diffeomorphism $\partial M \xrightarrow{f} \partial M'$ can be extended as homeomorphism $M \longrightarrow M'$.

Suppose we can embed a cork M into a closed X.

I.e. $X = (X \setminus \mathring{M}) \cup M$. Then we can form a new 4-mfd $X' = (X \setminus \mathring{M}) \cup_f M$. We say X, X' are related by a cork twist. By Freedman's theorem, $X \cong_{top} X'$.

However, in general, a cork twist will change smooth structure.

Example (Akbulut) The Akbulut-Mazur cork $M \hookrightarrow K3 \# \overline{\mathbb{CP}^2}$

$K3 \# \overline{\mathbb{CP}^2} \overset{\text{cork twist}}{\rightsquigarrow} 3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$

But $K3 \# \overline{\mathbb{CP}^2}$ is symplectic so $\nexists_{\text{diff}}\, X_1 \# X_2$ with $b^+(X_i) > 0$.

Theorem (Matveyev, Cutis-Freedman-Hsiang-Stong, Bižaca)
Let X, X' be s.c. smooth 4-mfd s.t. $X \cong_{top} X'$, $X \ncong_{diff} X'$.
Then X, X' are related by a single cork twist.

So all exoticness can be "pushed" into a contractible piece.