Calabi-Yau period motives in quantum field theory and general relativity

at Yau's 75th Birthday & the 15th Anniversary of YMSC

Albrecht Klemm, BCTP/HCM Bonn University

Tsinghua University April 5th 2024

universitätbonn

Based on work with

Kilian Bönisch, Claude Duhr, Fabian Fischbach, Florian Loebbert, Christoph Nega, Jan Plefka, Franzika Porkert, Reza Safari, Benjamin Sauer, Lorenzo Tancredi

[1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1,
[3]=arXiv:2108.05310, in JHEP
[4]=arXiv:2209.05291 in PRL and [5]=arXiv: 2212.09550 in JHEP,
[6]= arXiv:2310.08625 acc. JHEP, [7]= arXiv:2402.xxxxx,
[8]= arXiv:2401.07899 sub. PRL

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III. Post Minkowskian (PM) Worldline Quantum Field Theory approximation to General Relativity to predict the gravitational wave forms in black hole scattering/mergers detected by LIGO,. . ..

Introduction perturbative QFT

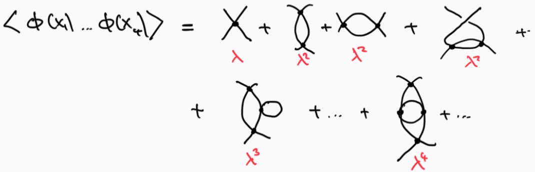
$$Z[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int \mathrm{d}^D x(\mathcal{L} + J\phi)\right] \ .$$

E.g. with $\mathcal{L} = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_u \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$.

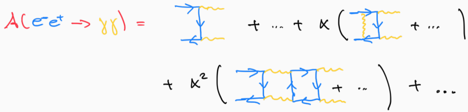
All physical correlators are of the form

$$\langle\phi(x_1)..\phi(x_n)\rangle = Z[J]^{-1}\left(\frac{\delta}{\delta J(x_1)}\right)..\left(\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}$$

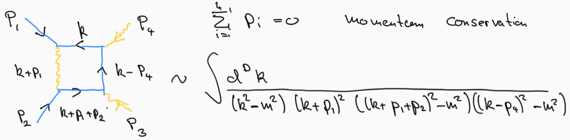
In interacting theories $\lambda \neq 0$ this is expanded asymptotically in Feynman graphs



Realistic theories: Probability for e^- e^+ to annihilate to two photons $P(e^- e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^- e^+ \to \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$



Scalar part e.g. for e.g. the box integral I: Propagators $\frac{1}{q^2 - m^2 + i \cdot 0}$



$D = D_{cr} - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta that we need to know!

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varieties

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
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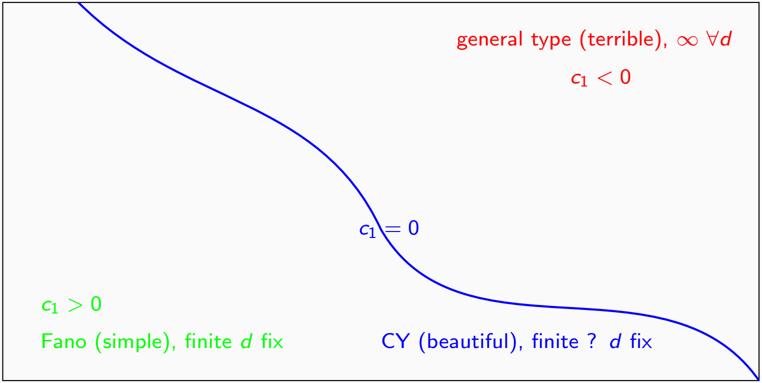
Kodaira map of algebraic varieties



$d = 1$

$d = 2$

$d = 3$

$\vdots \quad \vdots$

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$l = 0 \quad l = 1 \quad l = 2 \quad l = 3 \quad \cdots$

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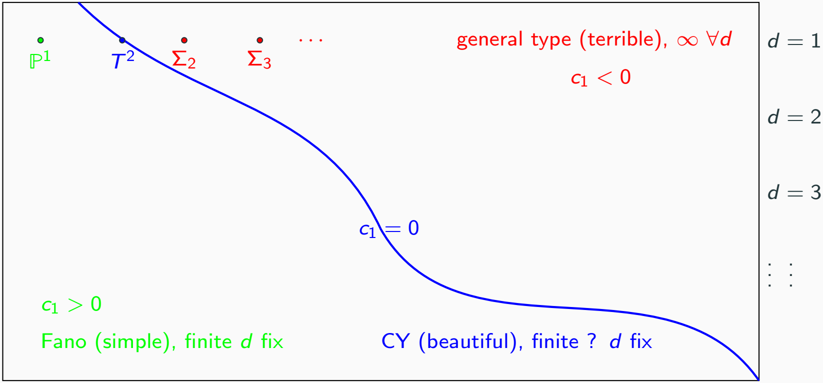

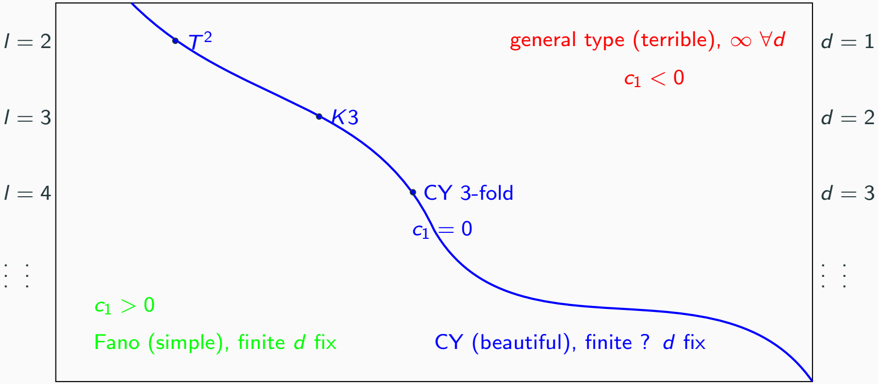

$l = 2$
$l = 3$
$l = 4$
$\vdots \ \vdots$
T^2
$K3$
CY 3-fold
$c_1 = 0$
general type (terrible), $\infty\ \forall d$
$c_1 < 0$
$c_1 > 0$
Fano (simple), finite d fix
CY (beautiful), finite ? d fix
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Dictionary Feynman graphs/amplitudes and geometry

Perturbative QFT	Geometry X	Differential eq.	Arithmetic Geometry
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ-deformed)	Homogeneous Gauss Manin $(d - A(z))\underline{\Pi} = 0$	Motive defined by l-adic coh $H^k_{et}(\overline{X}, \mathbb{Q}_l)$
	$\circlearrowleft$ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?		$\circlearrowleft$ Galois group $\mathrm{Gal}(\overline{K}/K)$ irreducible ?
actual Feynman integral	Chain integral ϵ-deformed	Inhomogeneous Gauss Manin connection $(d - A(z))\underline{\Pi} = B(z)$	Extended motive

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The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
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Remark: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.

Construction of Calabi-Yau n-folds in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$:

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2 H^2} + \dots$$

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CICYs: Complete intersections: $P_k = 0$, $k = 1, \ldots, r$ in $\mathbb{P} = \otimes_{l=1}^{m} \mathbb{P}_l^{n_l}$ define a CY $(\sum n_l - r)$-fold if $\sum_{k=1}^{r} d_{kl} = n_l + 1$, $\forall l = 1, \ldots, m$, with d_{kl} degrees of the k'th polynomial in the l'th factor: 2d n-1 loop bananas.

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BCs: Branched covers: $\mathbb{P}$ an n-dimensional Fano variety $c_1(\mathbb{P}) > 0$, then a d-fold cover branched at $\frac{d}{d-1}K(\mathbb{P})$ is a CY n-variety: $\mathbb{P} = (\mathbb{P}^1)^n$ and $d = 2, 3$ are relevant for 2d n-loop fishnets .

(Relative) Calabi-Yau periods in the Symanzik representation

For certain graphs the Calabi-Yau geometry appears directly in the Symanzik representation of the Feyman integral.

In this representations contribution of the l-loop integral yields a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

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Feyman graphs and (relative) Calabi-Yau periods

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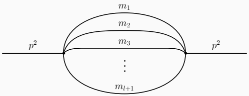

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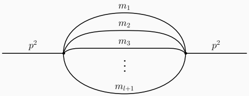

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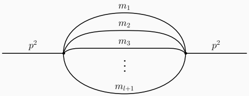

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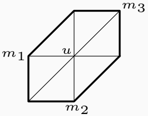

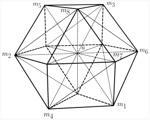

A better CY motive

The problem with the toric hyper-surface representation of the CY $(n = l - 1)$-fold is that the number of its complex moduli grow with $h^{l-2,1} \sim l^2$, while the number of physical parameter in the l loop Banana integral is only $l + 1$: $z_i = p^2/m_i^2$, $i = 1, \ldots, l + 1$.

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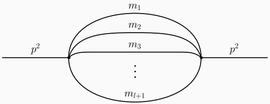
$$\mathbb{P}_{l+1} := \otimes_{i=1}^{l+1} \mathbb{P}^1_{(i)} \,.$$

Such a CICY is denoted for short as

$$M^{\mathrm{CI}}_{l-1} = \left(\begin{array}{c} \mathbb{P}^1_{(1)} \\ \vdots \\ \mathbb{P}^1_{(l+1)} \end{array} \middle\| \begin{array}{cc} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{array} \right) \quad \subset \quad \left(\begin{array}{c} \mathbb{P}^1_{(1)} \\ \vdots \\ \mathbb{P}^1_{(l+1)} \end{array} \middle\| \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) =: F_l \subset \mathbb{P}_{l+1} \,.$$

A better CY motive

In Hosono, Theisen, Yau & A.K. hep-th/9406055 we considered the vertical quantum cohomology of M^{CI}_{l-1} using mirror symmetry:



$$\longleftrightarrow \quad M^{CI}_{l-1} = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 & 1 \end{array} \right)$$

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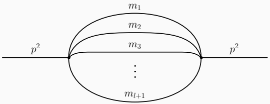

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In this realisation the complexified (large volume) Kähler parameters t^k of the $l+1$ rational curves $\mathbb{P}^1_k$ are identified with the physical parameters m_k^2/p^2

$$t^k \simeq \frac{1}{2\pi i}\int_{\mathbb{P}^1_k}(i\omega - b) + \mathcal{O}(e^{-t^k}) = \frac{\log\left(\frac{m_k^2}{p^2}\right)}{2\pi i} = \frac{\log(z_k)}{2\pi i}$$

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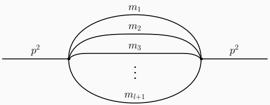

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Away from the limit the associated GKZ system provides the exact answer in terms of its periods.

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The discriminant components, where is vanishes correspond to the physical Landau poles.

Gauss Manin connection and sub sectors

A more systematic way to get the Gauss-Manin connection and the inhomogeneous term is to use the integration by by parts relations IBP relation between so called master integrals. Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{1}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the q_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r. We defined $\epsilon := \frac{D_{cr} - D}{2}$.

Master Integrals and integration by parts relations

The Feynman integral depends besides D (ϵ) on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{w} = (w_1, \ldots, w_N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$ and dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i, we chose

$$x_k := \frac{w_k}{w_N} \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{x}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

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The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^{\mu}} \left(q_l^{\mu} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

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There is a finite region in the lattice that contains all non-vanishing master integrals. In a basis of master integrals one can express derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

Master Integrals and integration by parts relations

The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

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- A complete set of IBP relations corresponds to the complete Picard Fuchs ideal of Gauss-Manin connection for the period integrals.

Among the elements in the lattice $\mathbb{Z}^p$ and, in particular, for the master integrals one can define sectors and a semi-ordering on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1\leq j\leq p} \ .$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\vartheta}(\underline{\nu}) \leq \underline{\vartheta}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

IBP relation summary:

The IBP relations characterise a suitable finite set of master integrals

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \ ,$$

with $D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \dots, p$ propagators and $(\underline{\nu}, D)$ in a finite region in $\mathbb{Z}^{p+1}$, by a first order Gauss Manin connection

$$\boxed{d\underline{I}(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon)\underline{I}(\underline{x}, \epsilon)}$$

$\epsilon = (D_{cr} - D)/2.$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

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$$\left[d_z - \epsilon \begin{pmatrix} & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\ * & \dots & * & A^1_{11} & \cdots & A^1_{1r_1} & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \mathbf{0} & \\ * & \dots & * & A^1_{r_1} & \cdots & A^1_{r_1 r_1} & & & \\ & \vdots & & & \ddots & & & & \\ * & \dots & * & & & & A^n_{11} & \cdots & A^n_{1r_1} \\ \vdots & \ddots & \vdots & & \mathbf{0} & & \vdots & \ddots & \vdots \\ * & \dots & * & & & & A^n_{r_2} & \cdots & A^n_{r_n r_n} \end{pmatrix} \right] \begin{pmatrix} I^{sub} \\ \Pi^1_1 \\ \vdots \\ \Pi^1_{r_1} \\ \vdots \\ \Pi_{n_1} \\ \vdots \\ \Pi_{n_r} \end{pmatrix}^{best} = 0$$

The blocks

Here $A_{ij}^k(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z and we typically have a situation, where the l-loop block in this improved IBP first order flat connection above is described by period integrals in the sense of Kontsevich and Zagier fullfilling the Gauss-Manin flat connection of a geometry X, which is typically a (non-smooth) Calabi-Yau manifold.

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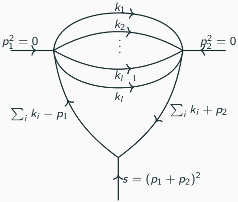
Example: From the (l+1)-loop ice-cone graph



it is clear that it contains l-loop banana graph as block(s).

Dictionary for the blocks

	$l = (n+1)$-**loop in block integrals in** D_{cr} **dimensions**	**Calabi-Yau (CY) geometry**
1	Maximal cut integrals in D_{cr} dimensions	$(n, 0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integration-by-parts (IBP) reduction	Griffiths reduction method
4	Integrand-basis for maximal cuts of of master integrals in D_{cr}	Middle (hyper) cohomology $H^n(M_n)$ M_n
5	Complete set of differential operators annihilating a given maximal cut in D_{cr} dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection

Periods on Calabi-Yau n-folds

Periods integrals

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a non-degenerate pairing between between (middle) homology and (middle) cohomology well defined by the theorem of Stokes:

$$\Pi : H_n(M_n, \mathbb{K}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C} \,.$$

It is possible and natural to have $\mathbb{K}$ to be $\mathbb{Z}$. There is an intersection pairing

$$\Sigma : H_n(M_n, \mathbb{K}) \times H_n(M_n, \mathbb{K}) \to \mathbb{K},$$

that can be made in particular integral. If n is odd Σ is antisymmetric and can be made symplectic. If n is even Σ is a symmetric on the even self dual lattice $H_n(M_n, \mathbb{K})$. E.g. for $K3$ $b_2 = 22$ and $\sigma = b_2^+ - b_2^- = \frac{1}{3}\int_{M_2} c_1^2 - 2c_2 = -16$ hence b_2 has signature $(3, 19)$ and is $E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

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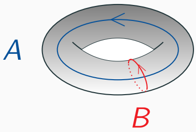
Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



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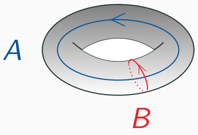

$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}, \; \partial_z \Omega(z) \sim \frac{xdx}{y}$$

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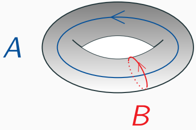

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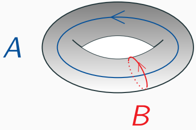

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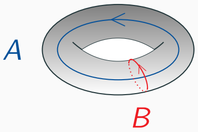

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$$\mathcal{L} \int_\Gamma \Omega = \left[(1-z)\partial_z^2 + (1-2z)\partial_z - \frac{1}{4} \right] \int_\Gamma \Omega = 0 \ .$$

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \qquad (2)$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

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$$\int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{I_n}(z) \in \mathbb{Q}[z] & \text{if } k = n \; . \end{cases} \tag{3}$$

Here $\underline{\partial}^k_{I_k} \Omega = \partial_{z_{I_1}} \ldots \partial_{z_{I_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^{k} H^{n-p,p}(W)$ are arbitrary combinations of derivatives w.r.t. to the z_i, $i = 1, \ldots, r$.

Advantages of the geometric representation

1.) Griffith-transversality (3) implies

a.) The Inverse of the Wronskian is up rational factors linear in the periods $W^{-1} = \Sigma W^T Z^{-1}$

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{c_2}{c_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{c_2}{c_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

b.) The Gauss-Manin connection can be brought into a canonical form

$$\partial_{t_*^i} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_j \\ \mathcal{V}^j \\ \mathcal{V}^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_k \\ \mathcal{V}^k \\ \mathcal{V}^0 \end{pmatrix} .$$

2.) a.) Implies that that in the "variation of constant" procedure the inhomogeneous solution is an iterated integral of the periods $\partial_n^k \Pi$ modulo rational functions. b.) implies that the higher terms in the ϵ expansion can be similar written as iterated integrals.

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The periods of Calabi-Yau varieties and their extensions (inhomogeneous solutions) evaluate Feynman integrals. Different Calabi-Yau varieties have the same periods structures. Therefore we abstract the essential data of the period $\Pi(z)$ into the notion of a Calabi-Yau period motive with the properties

(a) $\Pi(z)$ is restricted by the real Griffiths bilinear relations defining positivity of volumes and the holomorphic Griffiths transversality conditions.

(b) $\vec{\Pi}(z)$ is a flat section of the Hodge bundle over the moduli space and fulfils a first-order homogeneous differential equation $\nabla_{GM}\Pi = (\partial_z - N(z))\Pi = 0$, or equivalently a set of higher order homogenous differential equations $\mathcal{L}^{(k)}\vec{\Pi} = 0$. The higher-order operators $\mathcal{L}^{(k)}(z, \partial_z)$ generate the Picard-Fuchs differential ideal.

(c) There is a $\mathbb{Z}[\alpha]$-integer intersection form Σ with entries $\Sigma_{ab} = \Gamma_a \cap \Gamma_b$, which is anti-symmetric and symplectic for n odd with signature $\left(\frac{b_n}{2}, \frac{b_n}{2}\right)$, and for n even it is symmetric of a signature $(m, b_n - m)$ determined by the Hirzebruch signature index.

(d) Flat sections of the Hodge bundle are determined by their monodromies M_γ for loops γ around special divisors of $\mathcal{M}_{cs}(M)$, that for a choice of basis $\Gamma_a \in H_n(M, \mathbb{Z}[\alpha])$ generate the monodromy group $\Gamma_M \subset \mathrm{Sp}(b_n, \mathbb{Z}[\alpha])$ for n odd and $\Gamma_M \subset O(\Sigma, \mathbb{Z}[\alpha])$ for n even. In particular, $\vec{\Pi}(z)$ defines a representation of Γ_M.

N=4 Super-Yang-Mills and integrablity

Driving question: Which symmetries allow to solve n.t. QFT's.

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Integrable Deformations: Marginal β deformations Leigh, Strassler (95) Maldacena Luni (05). Here most relevant the supersymmetry breaking γ_i, $i = 1, 2, 3$ deformations in the double scaling limit $g \to 0$, $\gamma_3 \to i\infty$ with $\xi^2 = g^2 N_c e^{-i\gamma_3}$ fixed Gürdoğan, Kazakov (16), with Caetano (18) and the bi-scalar model χFT Kazakov, Olivucci (18) leading to holographic dual pairs of integrable fishnet and fishchain theories in D dimensions.

Orginal Fishnet Lagrangians

These bi-"scalar" fishnet theories in D dimensions have a Lagrangian with quartic interaction $V = 4$

$$\mathcal{L}_{\text{quad}}^{\omega D} = N_{\text{c}} \text{tr}[-X(-\partial_\mu \partial^\mu)^\omega \bar{X} - Z(-\partial_\mu \partial^\mu)^{\frac{D}{2}-\omega} \bar{Z} + \xi^2 XZ\bar{X}\bar{Z}] \; .$$

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Most importantly this theory exhibit as symmetry the Yangian extension of the bosonic conformal symmetry.

Hexagonal Fishnets Lagrangian

A generalization with analogous symmetry properties are Fishnet theories with cubic interaction $V = 3$ Kazakov, Olivucci (23) and Lagrangian

$$\begin{aligned}\mathcal{L}^{D}_{\rm cub} = \quad & N_{\rm c}{\rm tr}\big[-X(-\partial_\mu\partial^\mu)^{\omega_1}\bar{X} - Y(-\partial_\mu\partial^\mu)^{\omega_2}\bar{Y} - Z(-\partial_\mu\partial^\mu)^{\omega_3}\bar{Z} \\ & +\xi_1^2\bar{X}YZ + \xi_2^2 X\bar{Y}\bar{Z}\big]\,,\end{aligned}$$

with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$.

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with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$. Scalar field have conformal dimension $\Delta_\phi = (D-2)/2$ and conformal interactions have to have valency $V = 2D/(D-2)$, i.e. $D = 6, 4, 3$ enforce $V = 3, 4, 6$ respectively.

Regular tilings and Calabi-Yau motives

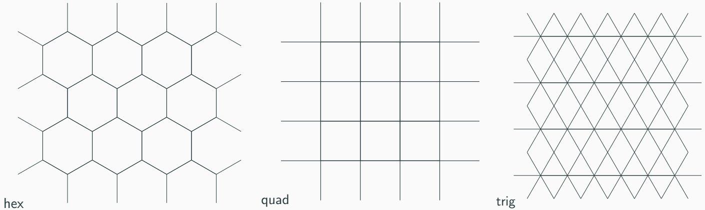

Figure 1: The three regular tilings of the plan with vertices of valence $\nu = 3, 4, 6$ respectively.

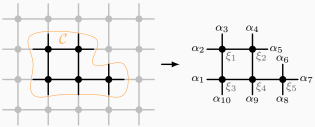

Figure 2: Ten-point five-loop fishnet integral cut out of a square tiling of the plane.

Regular tilings and Calabi-Yau motives

To obtain a graph G consider a convex closed oriented curve $\mathcal{C}$ that cuts edges of the tiling and does not pass to vertices. To each vertex inside the curve $\mathcal{C}$ we associate a $\mathbb{P}^1$ with homogeneous coordinates $[x_i : u_i]$, $i = 1, \ldots, l$ over which we want to integrate with the measure

$$\mathrm{d}\mu_i = u_i \mathrm{d}x_i - x_i \mathrm{d}u_i \ . \tag{4}$$

To the end point of each cut edge outside $\mathcal{C}$ we associate a parameter $a_j \in \mathbb{C}$, $j = 1, \ldots, r$. The graph is constructed by the l vertices with propagators

$$P^I_{ij} = \frac{1}{(x_i - x_j)^{w_{ij}}}, \qquad P^E_{ij} = \frac{1}{(x_i - a_j)^{w_{ij}}} \ . \tag{5}$$

To be conformal in D dimension the weights of propagators incident to each vertex V_i has to fullfill

$$\sum w_{ij} = D \tag{6}$$

Regular tilings and Calabi-Yau motives

We deal mainly with $D = 2$ and choose the propagator weights all equal $w_{ij} = w = 2/\nu(V)$, where $\nu(V)$ is the valence of the vertices, i.e. for the hexagonal tiling we have $w = \frac{2}{3}$, for the quartic tiling $w = \frac{1}{4}$ and for the trigonal tiling $w = \frac{1}{3}$.

To the hexagonal and the quartic lattice we can associate an in general singular l-dimensional Calabi-Yau variety M_l as the $d = 3$ or $d = 2$ fold cover

$$W = \frac{y^d}{d} - P([\underline{x} : \underline{u}]; \underline{a}) = 0 \tag{7}$$

over the base $B = (\mathbb{P}^1)^l$ branched at

$$P([\underline{x} : \underline{w}]; \underline{a}) = \prod_{ij}(u_j x_i - x_j u_i) \prod_{ij}(x_i - a_j u_i) = 0 \ , \tag{8}$$

respectively. The orders of the covering automorpishm exchanging the sheets will play a crucial role in the following geometric analysis.

Regular tilings and Calabi-Yau motives

Note that (7) defines a Calabi-Yau manifold, because the canonical class of the base is with H_i the hyperplane class of the i'th $\mathbb{P}^1$ given by

$$K_B = 2 \bigoplus_{i=1} H_i, \tag{9}$$

and the Calabi-Yau condition ensuring $K_{M_l} = 0$

$$\frac{d}{d-1} K_B = [P([\underline{x} : \underline{u}]; \underline{a})] = \nu \bigoplus_{i=1} H_i \tag{10}$$

is true with $d = 3, 2$ as $\nu = 3, 4$ for graphs from the hexagonal and the quartic tiling, respectively.

Another way of stating this is that the periods over the unique holomorphic (ℓ,0)-form, given by the Griffiths residuum form Ω

$$\Pi_G = \int_C \Omega = \int_C \frac{1}{2\pi i} \oint_\gamma \frac{dy \prod_{i=1}^{l} d\mu_i}{W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{\partial_y W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{P^{\frac{d-1}{d}}} = \int_C \prod_{ij} P_{ij}^{l} \prod_{ij} P_{ij}^{E} \prod_{i=1}^{l} d\mu_i \,, \tag{11}$$

are well defined. The significance for the application is that these period integrals over cycles $C \in H_l(M_l, \mathbb{Z})$ are building blocks for the amplitudes.

$$I_G = \int_C \Omega = \int \sqrt{\left| \prod_{ij} P_{ij}^{l} \prod_{ij} P_{ij}^{E} \right|^2} \prod_{i=1}^{l} d\mu_i \wedge d\bar{\mu}_i \,, \tag{12}$$

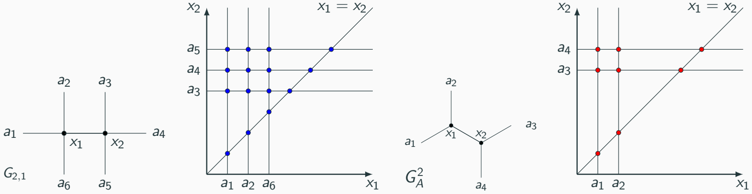

Figure 3: Singularities of the $K3$ denoted for the valence 4 graph $M_{G_{1,2}}$ and the valence 3 graph $M_{G_A^2}$. Note that 3 of the a_i can be set to $0, 1, \infty$ by a diagonal $PSL(2, \mathbb{C})$ acting on the projective plane in which the a_i lie

Regular tilings and Calabi-Yau motives

Claim 1: To each graph G we can associate a Calabi-Yau variety M_G whose periods determine I.

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Remark: Each I_G integral is an amplitude in the CFNT, i.e. $I_G(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm in $D=4$ or in the $D=2$ case e^{-K}.

The Yangian symmetry:

To each semi simple finite Lie Algebra g one can associate a Yangian extension $Y(g)$. E.g. for the conformal group in $D = 2$ is $S(3,1)$ and the Yangian algebra splits:

$$Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}.$$

The holomorphic Yangian is generated by the algebra

$$\begin{aligned} P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu \partial_{a_j,\nu} + \Delta_j) + ia_j^2 \partial_{a_j}^\mu \\ L_j^{\mu\nu} &= i(a_j^\mu \partial_{a_j}^\nu - a_j^\nu \partial_{a_j}^\mu), & D_j &= -i(a_j^\mu \partial_{a_j,\mu}), \end{aligned}$$

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in differentials w.r.t. to the external position, generates together with the permutation symmetries of the latter a differential ideal Y_G that annihilates the amplitude $I_G(\underline{z})$

Regular tilings and Calabi-Yau motives

Claim 3: Y_G is *equivalent* to the Picard-Fuchs differential ideal that describes the variation of the Hodge structure in the middle cohomology of M_G and annihilates the periods of Ω.

Remark 1: The factorisation of the amplitudes of the integrable system subject to the Yangian symmetry implies non-trivial relations for the periods of the of the M_G. Denoting the one parameter specialisation of $n \times m$ box graphs $M^{(n,m)}$ then the periods of $M^{(n,m)}$ are $(m \times m)$ minors of the periods $M^{(1,m+m)}$ ...

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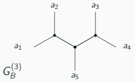

Figure 4: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

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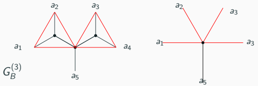

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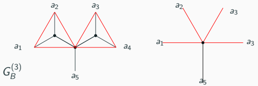
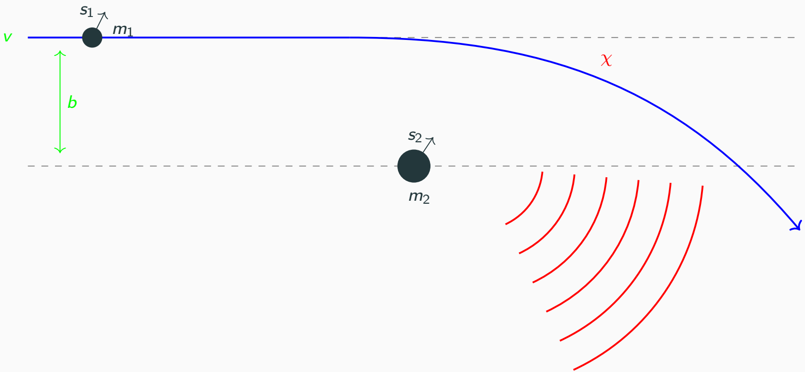


Figure 4: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

It connects singular Calabi-Yau motives with motives of Picard Varieties.

Worldline Quantum Field Theory approach to General Relativity

Scattering of two black holes (BH) as starter to the description of BH mergers as the main sources for gravitational waves detected at LIGO, ...



Worldline Quantum Field Theory approach to General Relativity

The action for the scattering process

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^{\mu} \dot{x}_i^{\nu}\right] + S_{\mathrm{EH}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^{\mu} = b_i^{\mu} + v_i^{\mu} \tau + z_i^{\mu}(\tau)\,, \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}(x)\ .$$

Worldline Quantum Field Theory approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int \mathrm{d}\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\,\mu}(x) \; .$$

At each order the contributions $\Delta p^{(n)\,\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

Worldline Quantum Field Theory approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector

(K3.0)

involve bilinear of elliptic function which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector

(CY3)

Worldline Quantum Field Theory approach to General Relativity

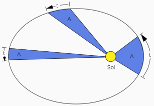
The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

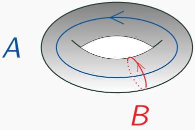
$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. The periods of the above $K3$ and CY threefold determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ in the 1SF sector.

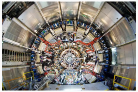
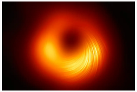
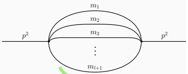
In the 5PM 2SF further different CY and K3 appear.

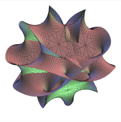
Conclusion and Outlook





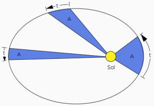
Conclusion and Outlook

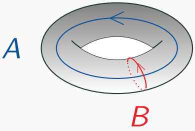






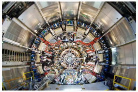
Physics

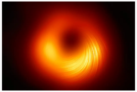
Mathematics





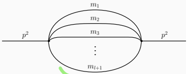
Conclusion and Outlook

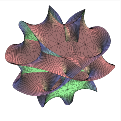




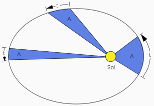
String Theory

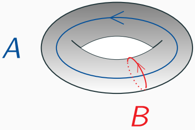
Physics





Mathematics



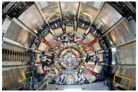


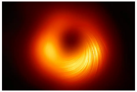
Enumerative Geometry

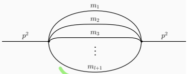
Conclusion and Outlook

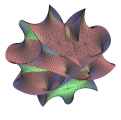
String Theory

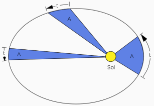
Black Hole Physics





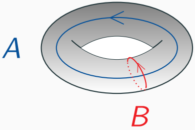






Physics

Mathematics



Enumerative Geometry

Arithmetic Geometry