Asymptotics of analytic torsion

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de Rham theorem

- M a compact smooth manifold
- $(F, \nabla^F) \to M$ a flat vector bundle : $(\nabla^F)^2 = 0$

- de Rham complex with coefficient F :

$$(\Omega^*(M,F), d^F) : 0 \to \Omega^0(M,F) \to \cdots \to \Omega^{\dim M}(M,F) \to 0$$

- **de Rham cohomology** :

$$H^q(M,F) \simeq \frac{\ker(d^F|_{\Omega^q(M,F)})}{\mathrm{Im}(d^F|_{\Omega^{q-1}(M,F)})}$$

Hodge theorem

- Given g^{TM}, g^F
- $d^{F,*} : \Omega^*(M,F) \to \Omega^*(M,F)$ adjoint of d^F
- $\Box_q^F = (d^F + d^{F,*})^2 : \Omega^q(M,F) \to \Omega^q(M,F)$
- **Hodge theorem** :

$$H^q(M,F) \simeq \ker(\Box_q^F)$$

Ray-Singer analytic torsion

- For simplicity, we assume $H^q(M,F)=0$ for any $q\geq 0$. Then each $\Box_q^F$ is invertible
- For $s\in\mathbf{C}, \mathrm{Re}(s)>\frac{\dim M}{2}$, set

$$\theta^F(s)=-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{q=0}^{\dim M}(-1)^q\, q\mathrm{Tr}\left[\exp(-t\Box_q^F)\right]dt.$$

- (real) Ray-Singer analytic torsion :

$$T(M,g^{TM},g^F)=\exp\left(\frac{1}{2}\frac{\partial\theta^F}{\partial s}(0)\right).$$

Ray-Singer analytic torsion

- Tautologically

$$T(M, g^{TM}, g^F) = \text{“} \prod_{q=0}^{\dim M} \left(\det\left(\square_q^F\right)\right)^{q(-1)^q/2} \text{”}.$$

- Smooth invariance : If $\dim M$ is odd, then $T(M, g^{TM}, g^F)$ does not depend on (g^{TM}, g^F) !

- Denoted simply by $T^{RS}(M, F)$

Ray-Singer conjecture

- **Ray-Singer conjecture :** If there is g^F such that $\nabla^F g^F = 0$ (orthogonal flat bundle), then

$$T^{RS}(M,F) = T^R(M,F)$$

  where $T^R(M,F)$ is the classical Reidemeister torsion

- Proved by J. Cheeger and W. Müller independenly (1978)

- Both use essentially the surgery method in topology

- Müller (1992) : Case of $\nabla^F(\mathrm{vol}(g^F)) = 0$ (unimodular flat bundle)

Ray-Singer conjecture : the general case

- Bismut-Zhang (1992) : Case of general flat vector bundles : g^F arbitrary

- $f : M \to \mathbf{R}$ a Morse function.
- ∇f the gradient of f, Thom-Smale complex

- One can define Milnor torsion $T^{\mathcal{M}}(M, \nabla f, g^F)$ which may depend on g^F (but equal to $T^R(M, F)$ when $\nabla^F(\mathrm{vol}(g^F)) = 0$)

- ∇^{TM} the Levi-Civita connection of g^{TM}
- Euler form $e(TM, \nabla^{TM})$: $\underline{\chi(M) = \int_M e(TM, \nabla^{TM})}$

Ray-Singer conjecture : the general case

- $\psi(TM, \nabla^{TM})$ the Mathai-Quillen current on TM :

$$d\psi(TM, \nabla^{TM}) = \pi^* e(TM, \nabla^{TM}) - \delta_M$$

- **Bismut-Zhang (1992) :**

$$(1) \quad \log\left(\frac{T^{RS}(M,F)}{T^{\mathcal{M}}(M,\nabla f, g^F)}\right) = -\frac{1}{2}\int_M \operatorname{tr}\left[\left(g^F\right)^{-1}\nabla^F g^F\right](\nabla f)^* \psi(TM, \nabla^{TM}).$$

- Purely analytic proof
  using the Witten deformation $e^{-Tf} d^F e^{Tf} = d^F + T df$

Line bundle on general manifold

- F real line bundle on a closed manifold M
- $\nabla = d + \omega$ with ω real closed 1-form, nowhere zero on M
- F_p with connection $\nabla = d + p\,\omega$
- If $p >> 0$, then $H^*(M, F_p) = 0$.
- Asymptotic formula for analytic torsion :

$$\lim_{p\to+\infty} \frac{T^{RS}(M, F_p)}{p} = \frac{1}{4}\int_M \omega\, e(TM/[\omega])$$

Müller's formula for hyperbolic 3-manifolds

- $M = \Gamma \setminus \mathbf{H}^3$ closed hyperbolic 3-manifold
  $\Gamma \subset SL(2, \mathbf{C})$ discrete, torsion free, cocompact.
- $\rho : SL(2, \mathbf{C}) \to GL(\mathbf{C}^2)$ canonical representation
  induces : $\rho_\Gamma : \Gamma \to GL(\mathbf{C}^2)$
- $\rho_{\Gamma,p} = \mathrm{Sym}^p(\rho_\Gamma)$ p-th symmetric power
  $F_p \to M$ associated (unimodular) flat vector bundle

Müller's formula for hyperbolic 3-manifolds

- For any $p > 0$, $H^*(M, F_p) = 0$
- $T^{RS}(M, F_p)$ equals to the Reidemeister torsion
- **Müller (2010) :**

$$\lim_{p \to +\infty} \frac{\log(T^{RS}(M, F_p))}{p^2} = \frac{\mathrm{vol}(M)}{4\pi}$$

- Müller's proof : Selberg trace formula
- Reidemeister torsion determines hyperbolic volume

Flat vector bundles from complex geometry

- In view of Bismut-Zhang, seeking more geometric inperpretation of Müller's result/proof and try to get generalizations

- Complex geometry enters : F Hermitian vector space, $L_F \to P_{\mathbf{C}}(F)$ canonical line bundle, then

$$H^{0,0}(P_{\mathbf{C}}(F), L_F^p) = \mathrm{Sym}^p(F)$$

- This motivates the following construction

Flat vector bundles from complex geometry

- $L \to N$ positive line bundle L over Kähler manifold N
- $q : P_G \to M$ principal bundle with flat connection, fiber G Lie Group
- G acts holomorphically on (N, L)
- $p >> 0$, get flat vector bundle F_p (via Kodaira vanishing and Hirzebruch-Riemann-Roch) :

$$P_G \times_G H^{0,0}(N, L^p) \to M$$

- Study asymptotic behavior of $T^{RS}(M, F_p)$ as $p >> 0$

Asymptotic torsion formula

- $q : \mathcal{N} = P_G \times_G N \to M$
  $T\mathcal{N} = T^H\mathcal{N} \oplus T^V\mathcal{N}$ with $T^H\mathcal{N} \simeq q^*(TM)$

- $\omega(L, g^L) \in \Gamma(q^*(T^*M))$ defined by
$$\omega(L, g^L)(U) = (g^L)^{-1} L_{q^*U} g^L \quad \text{for} \quad U \in \Gamma(TM)$$

- **Nondegenerate condition :** $\omega(L, g^L)$ nowhere zero

- **Bismut-Ma-Zhang :** Under nondegenerate condition,

(i) $$H^*(M, F_p) = 0, \quad \text{when} \quad p >> 0$$

(ii) $$\lim_{p \to +\infty} \frac{\log(T^{RS}(M, F_p))}{p^{\dim N + 1}} = \int_M W$$

Asymptotic torsion formula

- **Bismut-Ma-Zhang :**

$$\lim_{p\to+\infty} \frac{\log(T^{RS}(M,F_p))}{p^{\dim N+1}} = \int_M W$$

- W locally defined form : let θ be the one form on $\mathcal{N}$,

$$\theta = -\frac{\omega(L,g^L)}{2} \in \Gamma(q^*(T^*M))$$

- View θ as a map $\sigma_\theta : \mathcal{N} \to T^*M \simeq TM$

- **Bismut-Ma-Zhang :** $q : P_G \times_G N \to M$,

$$W = \int_N \theta \wedge \sigma_\theta^* \psi(TM, \nabla^{TM}) \wedge \exp(c_1(L,g^L)) \in \Omega^*(M)$$

($\psi(TM,\nabla^{TM})$ Mathai-Quillen current on TM)

Asymptotic torsion formula

- Method of proof : local index theory
  (more flexible than the Selberg trace formula)

- When apply to locally symmetric space,

$$\int_M W = C \cdot \mathrm{vol}(M)$$

Hyperbolic 3-manifold revisited

- $M = \Gamma \setminus \mathbf{H}^3$, $\rho_\Gamma : \Gamma \to GL(\mathbf{C}^2)$, $\overline{\rho}_\Gamma : \Gamma \to GL(\overline{\mathbf{C}}^2)$
- For any a, $b \geq 0$, $p > 0$,

$$F_p = \mathbf{H}^3 \times_{\rho_\Gamma} \left(\mathrm{Sym}^{pa}(\mathbf{C}^2) \otimes \mathrm{Sym}^{pb}(\overline{\mathbf{C}}^2) \right)$$

- **Bismut-Ma-Zhang :** If $a > b > 0$,

$$\lim_{p \to +\infty} \frac{\log(T^{RS}(M, F_p))}{p^3} = \frac{(3a^2 b - b^3)}{12\pi} \mathrm{vol}(M)$$

- **Müller (2010) :** If $a > 0$, $b = 0$,

$$\lim_{p \to +\infty} \frac{\log(T^{RS}(M, F_p))}{p^2} = \frac{a^2}{4\pi} \mathrm{vol}(M)$$

Outline of proof : Toeplitz operators

- Consider the flat vector bundle $F_p \to M$, g^{F_p}
- $\omega(F_p, g^{F_p}) = \left(g^{F_p}\right)^{-1} \nabla^{F_p} g^{F_p}$
- $\nabla^{F_p,u} = \nabla^{F_p} + \frac{\omega(F_p, g^{F_p})}{2}$, unitary connection on F_p
- $d^{F_p} + d^{F_p,*} = \sum_{i=1}^{\dim M} c(e_i)\nabla^u_{e_i} - \frac{\widehat{c}\left(\omega(F_p, g^{F_p})\right)}{2}$

$$= D^u - \frac{\widehat{c}\left(\omega(F_p, g^{F_p})\right)}{2}$$

(Compare with
the Witten deformation $d_{Tf} + d^*_{Tf} = D + T\widehat{c}(\nabla f)$)

Outline of proof : Toeplitz operators

- $\square^{F_p} = (D^u)^2 - \frac{1}{2}[D^u, \omega(F_p, g^{F_p})] + \frac{1}{4}\widehat{c}(\omega(F_p, g^{F_p}))^2$
- When acting fiberwise, one meets Toeplitz operators

$$P^{L^p}[D^u, \omega(F_p, g^{F_p})]P^{L^p} \quad \text{and} \quad P^{L^p}\widehat{c}(\omega(F_p, g^{F_p}))^2 P^{L^p}$$

with $P^{L^p} : \Omega^{0,*}(N, L^p) \to H^{0,0}(N, L^p)$

Outline of proof : Toeplitz operators

- The analysis of the asymptotics of such Toeplitz operators goes back to
- The Thesis of Gang Tian, which was supervised by Professor Yau
- where one studies the asymptotics of the Bergman kernel of $P^{L^p} : \Omega^{0,*}(N, L^p) \to H^{0,0}(N, L^p)$, as $p >> 0$
- For a modern systematic treatment, see the comprehensive book of Xiaonan Ma and Marinescu : **Bergman Kernels and Holomorphic Morse inequalities**

Outline of proof : Toeplitz operators

- Back to the Toeplitz operators

$$P^{L^p}[D^u, \omega(F_p, g^{F_p})]P^{L^p} \quad \text{and} \quad P^{L^p}\widehat{c}(\omega(F_p, g^{F_p}))^2 P^{L^p}$$

with $P^{L^p} : \Omega^{0,*}(N, L^p) \to H^{0,0}(N, L^p)$

- Easy : $P^{L^p}[D^u, \omega(F_p, g^{F_p})]P^{L^p} = O(p)$ as $p >> 0$

- By nondegenerate condition and analysis of Toeplitz operators, there is $c > 0$ such that

$$P^{L^p}\widehat{c}(\omega(F_p, g^{F_p}))^2 P^{L^p} \geq c\, p^2 \quad \text{when} \quad p >> 0$$

Outline of proof : Toeplitz operators

- When $p >> 0$,

$$\square^{F_p} \geq (D^u)^2 + \frac{c\,p^2}{2} \geq \frac{c\,p^2}{2}$$

- Immediately implies

$$H^*(M, F_p) = 0 \quad \text{when} \quad p >> 0$$

- Local index techniques + Toeplitz operator analysis identify local contributions, leading to the W-invariant

- Q.E.D.

Possible relation with the volume conjecture

- Reidemeister torsion determines hyperbolic volume

- In 3-manifold or knot theory, Reidemeister torsion *is* twisted Alexander polynomial
- (Twisted) Alexander polynomials determine hyperbolic volume

- **Volume conjecture :** Colored Jones polynomials determine hyperbolic volume

- Naive question : any possible relations ?

- Complex geometry will play more roles in knot theory

Happy Birthday to Professor Yau!