Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 8. Lyapunov exponents of the Teichmüller geodesic flow and of the Hodge bundle: general facts

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Generalities concerning Lyapunov exponents

Multiplicative ergodic theorem

Theorem (Oseledets) *Let a smooth map $F : X^n \to X^n$ be ergodic with respect to a finite measure. Then, there exists a collection of numbers*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_k,$$

such that for almost any point $x \in X$ there is an equivariant filtration

$$\mathbb{R}^n \simeq T_x X^n = \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_k \supset \mathcal{L}_{k+1} = \{0\}$$

in the fiber $T_x X^n$ of the tangent bundle at x with the following property. For every $\vec{v} \in \mathcal{L}_j - \mathcal{L}_{j+1}$, $j = 1, \ldots, k$, one has

$$\lim_{N \to +\infty} \frac{1}{N} \log \|(DF^N)_x(\vec{v})\| = \lambda_j$$

Remark A norm used to define Lyapunov exponents is irrelevant: any other reasonable norm will give the same result.

Versions of the multiplicative ergodic theorem

Multiplicative ergodic theorem has natural generalizations. Note that

$$D_{x_0}(F^N) = D_{F^{N-1}(x_0)}F \circ \cdots \circ D_{F(x_0)}F \circ D_{x_0}F$$

Actually, matrices $D_x F$ are not distinguished by any special property. One can consider any matrix-valued function $A : X^n \to GL(m, \mathbb{R})$ and study products of matrices

$$\mathcal{A}(x, N) := A(F^{N-1}(x)) \cdot \cdots \cdot A(F(x)) \cdot A(x)$$

along trajectories $x, F(x), \dots, F^{N-1}(x)$ of F. A statement completely analogous to the above Theorem is valid in this more general case provided the matrix-valued function $A(x)$ satisfy a very moderate requirement of *integrability*:

$$\int_{X^n} \log_+ \|A(x)\| \, d\mu < +\infty, \quad \text{where } \log_+(y) = \max(\log(y), 0)$$

In this case one says that the matrix-valued function $A(x)$ defines an *(integrable) multiplicative cocycle* $\mathcal{A}(x, N)$.

Versions of the multiplicative ergodic theorem

One can formulate a “continuous-time” version of multiplicative ergodic theorem when instead of a map $F : X^n \to X^n$ one has a flow $f_t : X^n \to X^n$ which preserves a finite measure on X^n. Corresponding Lyapunov exponents coincide with Lyapunov exponents of a discrete map $F = f_1$ obtained as an action of the flow for unit time.

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Let $Y^{n-1} \subset X^n$ be a section to the flow f_t and $G : Y^{n-1} \to Y^{n-1}$ be the induced first return map. Suppose that the induced G-invariant measure on Y^{n-1} is finite. A multiplicative cocycle $a(x,t)$ related to the flow f_t induces a multiplicative cocycle $\mathcal{A}(y,N)$ for the map G. Namely, for a point $y \in Y^{n-1}$ define $t(y)$ to be the time of the first return of a trajectory emitted from y to the section Y^{n-1}. Then

$$\mathcal{A}(y,1) = A(y) := a(x, t(y))$$

Lyapunov exponents of cocycles $\mathcal{A}$ and a are proportional with a factor $t_{mean} = \int_{Y^{n-1}} t(y)\, d\eta,$ where η is the probability measure on Y^{n-1}.

Geometric interpretation

Consider a vector bundle over X^n endowed with a connection. Having a flow on the base we can take a fiber of the vector bundle and transport it along a very long piece of trajectory of the flow using the connection. When the trajectory comes close to the starting point we identify the fibers using the connection and we study the resulting linear transformation of the fiber.

The multiplicative ergodic theorem says that for almost every starting point one can define (up to a conjugation) a *“matrix of mean holonomy”* of the bundle along the flow. When the flow is ergodic this “mean holonomy” is the same for almost all starting points.

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Linear algebra of multiplicative cocycles

From now on assume that we have a flow f_t which preserves a finite ergodic measure on the base X of a finite-dimensional vector bundle, and that this vector bundle is endowed with a connection so that we can transport fibers of the bundle along trajectories of the flow.

Let $\mathcal{K}, \mathcal{L}$ be such vector bundles of dimensions k and l correspondingly. Let $\varkappa_1 \geq \cdots \geq \varkappa_k$ and $\lambda_1 \geq \cdots \geq \lambda_l$ be the related Lyapunov exponents.

- The Lyapunov exponents of the vector bundle $\mathcal{K} \oplus \mathcal{L}$ can be obtained as a union of the two initial collections of Lyapunov exponents.

- The Lyapunov exponents of the vector bundle $\wedge^j \mathcal{L}$ are $\lambda_1 + \lambda_2 + \cdots + \lambda_j \geq \cdots \geq \lambda_{l-j+1} + \cdots + \lambda_l$.

- If matrices $a(x,t)$ of the cocycle are symplectic, then the Lyapunov exponents are symmetric:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$

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Two more remarks on multiplicative cocycles

If we compose our cocycle with a uniform expansion $b(x,t) = e^t \cdot a(x,t)$, then the Lyapunov exponents of the new cocycle $b(x,t)$ are obtained from the Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_l$ of $a(x,t)$ by shifting them by one: $1 + \lambda_1 \geq \cdots \geq 1 + \lambda_l$. Similarly, composing the cocycle with a uniform contraction e^{-t} we shift Lyapunov exponents as $-1 + \lambda_1 \geq \cdots \geq -1 + \lambda_l$.

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Suppose that the vector bundle $\mathcal{L}$ contains a 1-dimensional subbundle $\mathcal{K} \subset \mathcal{L}$ equivariant with respect to the action of the cocycle $a(x,t)$. Let λ be the single Lyapunov exponent of $\mathcal{K}$. Then λ is present in the spectrum of Lyapunov exponents of $\mathcal{L}$.

In this particular case (and only!) it is easy to compute λ. It is an average of derivative of dilatation of a vector $\vec{v} \in \mathcal{K}$ along the flow:

$$\lambda = \frac{1}{\mu(X)} \int_{X^n} k(x)\, d\mu\,, \quad \text{where } k(x) := \log \frac{\|\dot{a\vec{v}}\|}{\|\vec{v}\|},\ \vec{v} \in \mathcal{K}_x \setminus \{\vec{0}\}\,.$$

Lyapunov exponents of the Teichmüller geodesic flow

Gauss — Manin connection

Consider a stratum $\mathcal{H}(d_1, \ldots, d_n)$ of holomorphic 1-forms with n zeroes of orders $d_1, \ldots, d_n$. "Points" of this stratum correspond to pairs (S, ω), where S is a Riemann surface and ω is a holomorphic 1-form.

Consider a natural vector bundle over the stratum with a fiber $H^1(S, \{zeroes\}; \mathbb{C})$ over a "point" (S, ω). This vector bundle carries a canonical flat connection called *Gauss—Manin connection*: we have a lattice $H^1(S, \{zeroes\}; \mathbb{Z} \oplus i\mathbb{Z})$ in each fiber, which tells us how we can locally identify the fibers. Thus, Teichmüller flow on $\mathcal{H}(d_1, \ldots, d_n)$ defines a multiplicative cocycle acting on fibers of this bundle.

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The vector bundle $H^1(S, \{zeroes\}; \mathbb{C})$ is a direct sum of two copies of identical vector bundles; for each fiber we have:

$$H^1(S, \{zeroes\}; \mathbb{C}) = H^1(S, \{zeroes\}; \mathbb{R}) \oplus H^1(S, \{zeroes\}; i\mathbb{R}).$$

Lyapunov exponents as spectrum of mean monodromy

Consider a vector bundle endowed with a flat connection over a manifold X^n. Having a flow on the base we can take a fiber of the vector bundle and transport it along a trajectory of the flow. When the trajectory comes close to the starting point we identify the fibers using the connection and we get a linear transformation $\mathcal{A}(x,1)$ of the fiber; the next time we get a matrix $\mathcal{A}(x,2)$, etc.

The multiplicative ergodic theorem says that when the flow is ergodic a *"matrix of mean monodromy"* along the flow

$$A_{mean} := \lim_{N\to\infty} \left(\mathcal{A}^*(x,N)\cdot\mathcal{A}(x,N)\right)^{\frac{1}{2N}}$$

is well-defined and constant for almost every starting point.

Lyapunov exponents correspond to logarithms of eigenvalues of this "matrix of mean monodromy".

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Natural vector bundles over a stratum...

The spectrum of Lyapunov exponents of the former cocycle $H^1(S, \{zeroes\}; \mathbb{C})$ can be represented as two copies of the spectrum $\nu_1 \geq \nu_2 \geq \dots \nu_{2g+n-1}$ of the cocycle $H^1(S, \{zeroes\}; \mathbb{R})$.

The multiplicative cocycle acting on the fibers $H^1(S, \{zeroes\}; \mathbb{R})$ has an invariant subspace: the $(n-1)$-dimensional image of the map $H^0(\{zeroes\}; \mathbb{R}) \to H^1(S, \{zeroes\}; \mathbb{R})$. Since all zeroes are labeled, holonomy of our flat connection acts on this subspace trivially. Thus, all $n-1$ Lyapunov exponents corresponding to this subspace are equal to zero.

The quotient over this vector bundle is a bundle with a fiber $H^1(S; \mathbb{R})$. It is also endowed with the Gauss—Manin flat connection. Note that the holonomy of this connection preserves the “intersection form”, so all matrices of holonomy are symplectic. Hence the corresponding Lyapunov exponents are symmetric: $\nu_j = -\nu_{2g-j+1}$ for $j = 1, \dots, 2g$.

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... and their Lyapunov exponents

We conclude that the spectrum of Lyapunov exponents of the vector bundle $H^1(S, \{zeroes\}; \mathbb{R})$ can be obtained by joining the above collection

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_g \geq -\nu_g \geq -\nu_2 \geq -\nu_1.$$

with the collection of $(n-1)$ Lyapunov exponents of the subbundle

$$image\big(H^0(S, \{zeroes\}; \mathbb{R}) \subset H^1(S, \{zeroes\}; \mathbb{R})\big)$$

which are trivial (i.e. zero).

We obtain the following spectrum of Lyapunov exponents for the vector bundle $H^1(S, \{zeroes\}; \mathbb{R})$:

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_g \geq \underbrace{0 = 0 = \cdots = 0}_{\text{number of conical points}-1} \geq -\nu_g \geq \cdots \geq -\nu_1$$

Symmetries of Lyapunov exponents of the Teichmüller flow

Recall that a neighborhood of $[\omega]$ in

$$H^1(S,\{zeroes\};\mathbb{C}) = H^1(S,\{zeroes\};\mathbb{R}) \oplus H^1(S,\{zeroes\}; i\mathbb{R})$$

serves as local coordinate chart for a neighborhood of (S,ω) in the stratum. The Teichmüller flow acts by uniform expansion by e^t on the first summand and by uniform contraction by e^t on the second summand.

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Thus, to obtain the spectrum of Lyapunov exponents of the Teichmüller flow we need to take two copies of spectra of $H^1(S,\{zeroes\};\mathbb{R})$ and shift all entries in one copy by $+1$ and in the other copy by -1:

$$1+\nu_1 \geq \cdots \geq 1+\nu_g \geq \underbrace{1 = 1 = \cdots = 1}_{\text{number of conical points}-1} \geq 1-\nu_g \geq \cdots \geq 1-\nu_1$$

$$-1+\nu_1 \geq \cdots \geq -1+\nu_g \geq \underbrace{-1 = \cdots = -1}_{\text{number of conical points}-1} \geq -1-\nu_g \geq \cdots \geq -1-\nu_1$$

One more invariant subbundle

Let $\omega = \omega_0 + i\,\omega_1$, where $\omega_0 = \mathrm{Re}(\omega)$ and $\omega_1 = \mathrm{Im}(\omega)$.
Let us compute the tangent vector to the Teichmüller geodesic flow in cohomological coordinates:

$$\frac{d}{dt}\Big|_{t=0}(e^t\omega_0 + e^{-t} i\,\omega_1) = \omega_0 - i\,\omega_1 = \bar\omega$$

This vector does not change the norm, so the corresponding Lyapunov exponent is trivial (i.e. equals to zero).

The vector ω itself can be considered as a vector "normal" to the hypersurface $\mathcal{H}_1(d_1, \ldots, d_n)$ so it does not expand or contract. Thus, both Lyapunov exponents corresponding to the invariant subspace $\mathcal{K}$ spanned by $[\omega_0]$ and $[\omega_1]$ are zero.

It is easy to show that $[\omega_0]$ is the most contracting vector in the vector bundle $H^1(S, \{zeroes\}; \mathbb{R})$ while $[\omega_1]$ is the most expanding vector. Hence **now** they are represented by Lyapunov exponents $1 - \nu_1$ and $-1 + \nu_1$. But we know that these numbers are zero! Hence $\nu_1 = 1$.

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Lyapunov exponents of the Teichmüller flow

Theorem *The spectrum of Lyapunov exponents of the Teichmüller flow acting on a hypersurface $\mathcal{H}_1(d_1,\dots,d_n)$ has the following form:*

$$2 \ge \dots \ge 1+\nu_g \ge \underbrace{1 = 1 = \dots = 1}_{\text{number of conical points}-1} \ge 1-\nu_g \ge \dots \ge 0 \ge$$

$$\ge \dots \ge -1+\nu_g \ge \underbrace{-1 = \dots = -1}_{\text{number of conical points}-1} \ge -1-\nu_g \ge \dots \ge -2$$

It is easy to show that the rightmost and leftmost inequalities are strict (W. Veech). It is true for the remaining inequalities, but it is already a difficult theorem...

Simplicity of the spectrum (after G. Forni, A. Avila and M. Viana)

Period matrix

Consider a *canonical basis of cycles* $\{a_1, b_1, \ldots, a_g, b_g\} \subset H_1(M, \mathbb{Z})$ on a Riemann surface S of genus $g \geq 2$ represented by paths intersecting as follows. For all i, $j \in \{1, \ldots, g\}$ we have

$$a_i \circ a_j = b_i \circ b_j = 0 \quad \text{and} \quad a_i \circ b_j = \delta_{ij} \,.$$

In other words a canonical homology basis is a symplectic basis with respect to the intersection form on the real homology $H_1(S, \mathbb{R})$.

Let $\{\theta_1, \ldots, \theta_g\}$ be the dual basis of the space of holomorphic (Abelian) differentials on M, characterized by the conditions $\theta_i(a_j) = \delta_{ij}$, for all i, $j \in \{1, \ldots, g\}$. The $g \times g$ complex matrix Π given by

$$\Pi_{ij}(S) := \int_{b_j} \theta_i \ , \quad i, j \in \{1, \ldots, g\} \,,$$

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Determinant locus

Let $q \in \mathcal{Q}_1$ be a holomorphic quadratic differential on the Riemann surface S. Let $(S_t, q_t) := g_t\big((S,q)\big)$, for $t \in \mathbb{R}$, be the Teichmüller orbit of (S,q) in the space Q_1 of flat surfaces of area one (= moduli space of quadratic differentials). The equation

$$\det\left[\frac{d}{dt}\Pi(S_t)|_{t=0}\right] = 0$$

defines a real analytic hypersurface $\mathcal{D} \subset \mathcal{Q}$ of real codimension 2 called *determinant locus*.

Theorem (G. Forni) *Let $\mathcal{M}$ be an $\mathrm{SL}(2,\mathbb{R})$-invariant subspace in the space $\mathcal{Q}_1$ of quadratic differentials or in the space $\mathcal{H}_1$ of holomorphic 1-forms. Let μ be an $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on $\mathcal{M}$. Let μ be ergodic with respect to the Teichmüller geodesic flow.*

If $\mu(\mathcal{M} - \mathcal{D}) > 0$ then the second Lyapunov exponent ν_2^μ (for the measure μ) is strictly positive, $\nu_2^\mu > 0$.

Simplicity of the spectrum

Recall that the top g Lyapunov exponents of the Teichmüller geodesic flow have the form $2 = 1 + \nu_1 > 1 + \nu_2 \geq \cdots \geq 1 + \nu_g$, where $1 = \nu_1 \geq \cdots \geq \nu_g \geq -\nu_g \geq \cdots \geq -\nu_1 = -1$ are the Lyapunov exponents of the vector bundle $H^1(S, \mathbb{R})$ transported by the Teichmüller flow using the Gauss—Manin connection.

Theorem (G. Forni, 2002) *For any connected component of any stratum $\mathcal{H}(d_1, \ldots, d_n)$ or $\mathcal{Q}(d_1, \ldots, d_n)$ one has $\nu_g > 0$.*

Theorem (A. Avila and M. Viana, 2007) *For any connected component of any stratum $\mathcal{H}(d_1, \ldots, d_n)$ of Abelian differentials the Lyapunov exponents ν_j have simple spectrum:*

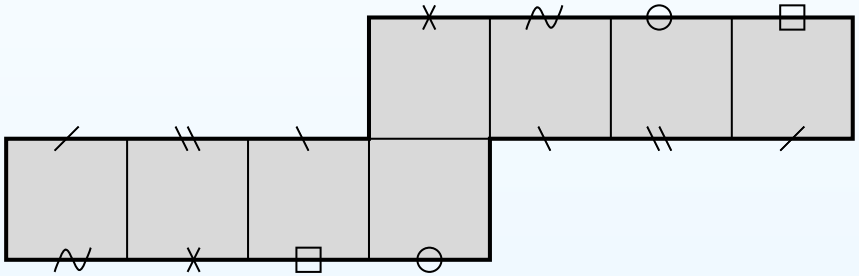
$$1 = \nu_1 > \nu_2 > \cdots > \nu_g \,.$$

Theorem (M. Bell, V. Delecroix, V. Gadre, R. Gutiérrez-Romo, S. Schleimer, 2021) *For any connected component of any stratum $\mathcal{Q}(d_1, \ldots, d_n)$ of quadratic differentials the Lyapunov exponents ν_j have simple spectrum:*

$$1 = \nu_1 > \nu_2 > \cdots > \nu_g \,.$$

Eierlegende Wollmilchsau

Consider a flat torus glued from a 2×2 square in a standard way. The following flat surface (S_0, ω_0) is a ramified double cover over the torus with four ramifications at integer points. It has genus three.



Its $\mathrm{SL}(2, \mathbb{R})$-orbit is closed (and, actually, is the same as the orbit of the torus). Consider a restriction of the vector bundle $H^1(S, \mathbb{R})$ to this orbit and transport it along the Teichmüller geodesic flow using the flat connections.

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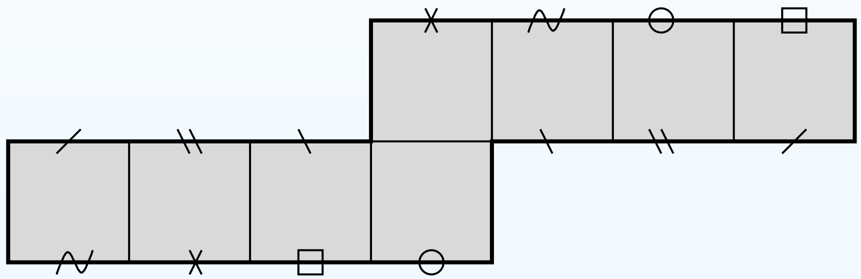

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Theorem (G. Forni) *The Lyapunov exponents ν_2 and ν_3 vanish for this orbit: $\nu_2 = \nu_3 = 0$.*

The entire $\mathrm{SL}(2, \mathbb{R})$-orbit belongs to the determinant locus!