2019 Summer School on Modern Dynamics

Introduction to Dynamics on Moduli Spaces

1. Dani Correspondence

1. Let μ be an invariant Borel probability measure such that $\mu(U) > 0$ for every nonempty open subset U. Show that the forward orbit of any point that is generic with respect to μ is dense.
2. Show that any bounded orbit of a map $T : \mathbf{Z} \to \mathbf{Z}$ is periodic.
3. Show that any smooth vector field on a closed manifold is complete.
4. Show that a periodic orbit of an action by $\mathbf{R}$ on a Hausdorff space is closed.
5. Show that the map $T(x, y) = (x + y, x + 2y)$ on $\mathbf{R}^2/\mathbf{Z}^2$
   (i) has a dense set of periodic orbits
   (ii) has orbits with finite ω-limit sets
   (iii) has a full measure set of points with dense orbits.
6. Show that the g_t-orbit of $\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{Z}^2$ is closed. (Here, $g_t = \mathrm{diag}(e^t, e^{-t})$ and $\phi = \tan^{-1}(\sqrt{5} - 1)/2$.)
7. Show that a discrete subgroup of $\mathbf{R}^n$ is finitely generated.
8. Show that the volume of a fundamental parallelopiped for a lattice $\Lambda \subset \mathbf{R}^n$ is independent of the representation $\Lambda = \mathbf{Z}u_1 + \cdots + \mathbf{Z}u_n$. (Such a set of generators is called an *integral basis.*)
9. Show that if $|\mathrm{Re}\ \tau| > 1/2$ then $\mathbf{Z} + \mathbf{Z}\tau$ contains an element that is shorter and not real.
10. Show that the Möbius action on the upper half plane leaves $\frac{|dz|}{y}$ invariant, but not $\frac{dz}{y}$.
11. Show that $e^t i$ and $x_0 + R\tanh(t - t_0) + iR\,\mathrm{sech}(t - t_0)$ are unit speed geodesics with respect to the hyperbolic metric $g = \frac{dx^2+dy^2}{y^2}$.
12. Compute the hyperbolic area of $\Delta = \{\tau \in \mathbb{H} : |\tau| \geq 1, |\mathrm{Re}\ \tau| \leq 1/2\}$ by two methods: (i) directly by multiple integration, and (ii) using Gauss-Bonnet theorem.
13. Show that the euclidean systole is given by $\mathrm{sys}_2(\tau) = \frac{1}{\sqrt{\mathrm{Im}\ \tau}}$.
14. Verify that the systole defines a proper function on $\mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z})$.