Panorama of Dynamics and Geometry of Moduli Spaces and Applications

Lecture 3. Count of closed geodesics and of saddle connections on translation surfaces

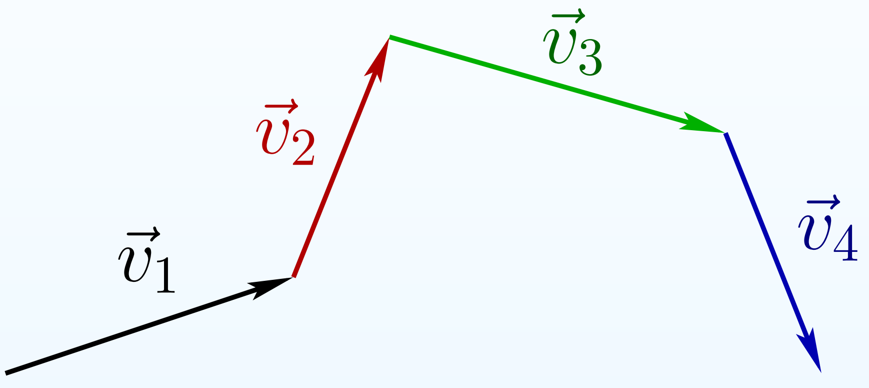
Anton Zorich
University Paris Cité

YMSC, Tsinghua University, April 12, 2022

Very flat surfaces

Very flat surfaces: construction from a polygon

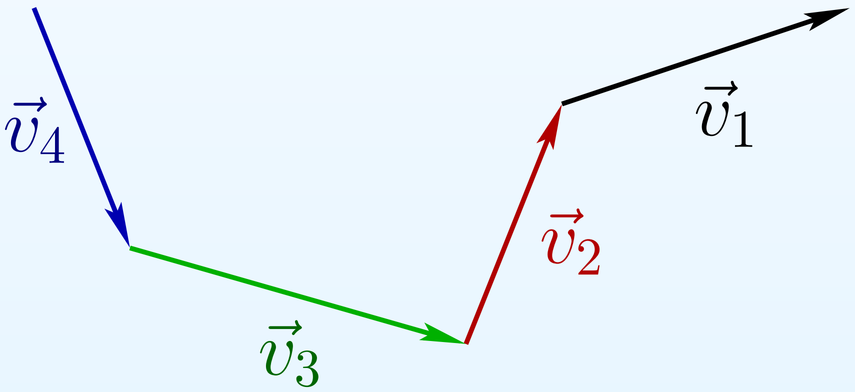
Consider a broken line constructed from vectors $\vec{v}_1, \ldots, \vec{v}_k$.



and another one constructed from the same vectors taken in another order.

Very flat surfaces: construction from a polygon

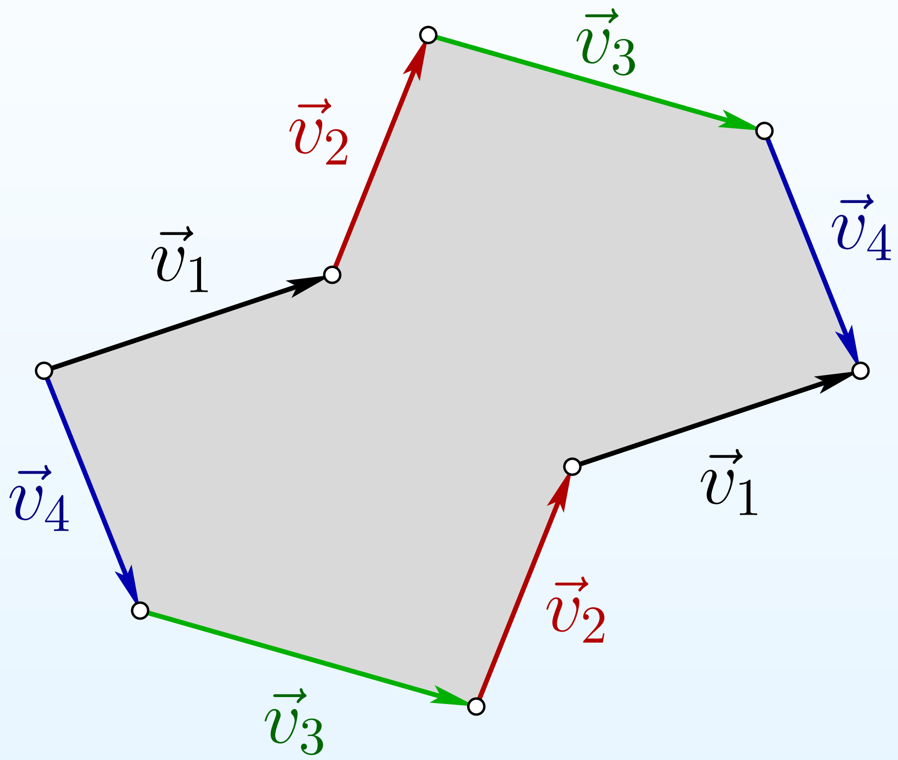
Consider a broken line constructed from vectors $\vec{v}_1, \dots, \vec{v}_k$.



and another one constructed from the same vectors taken in another order.

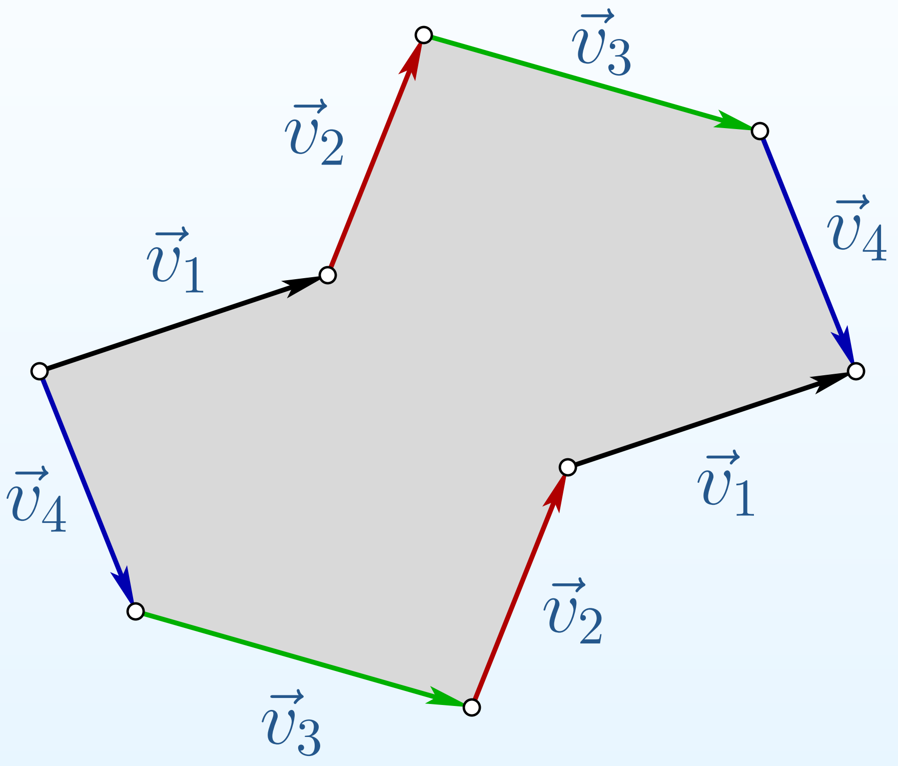
Very flat surfaces: construction from a polygon

Consider a broken line constructed from vectors $\vec{v}_1, \dots, \vec{v}_k$.



and another one constructed from the same vectors taken in another order. If we are lucky enough the two broken lines do not intersect and form a polygon.

Very flat surfaces: construction from a polygon



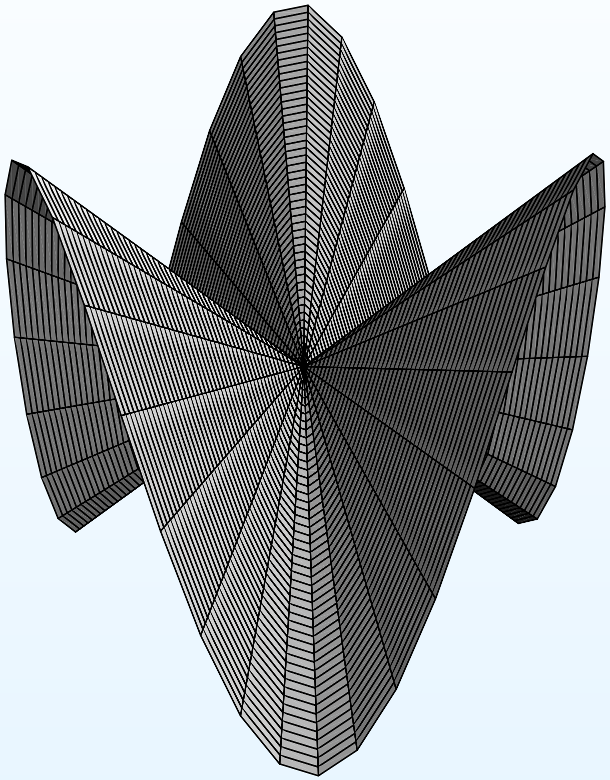
Identifying the corresponding pairs of sides by parallel translations we get a closed surface endowed with a flat metric.

Properties of very flat surfaces

- The flat metric is nonsingular outside of a finite number of conical singularities (inherited from the vertices of the polygon).
- The flat metric has trivial holonomy, i.e. parallel transport along any closed path brings a tangent vector to itself.
- In particular, all cone angles are integer multiples of 2π.
- By convention, the choice of the vertical direction (“direction to the North”) will be considered as a part of the “very flat structure”.
  For example, a surface obtained from a rotated polygon is considered as a different very flat surface.
- A conical singularity with the cone angle $2\pi \cdot N$ has N outgoing directions to the North.

Example: conical singularity with cone angle 6π

Locally a neighborhood of a conical point looks like a "*monkey saddle*".



A neighborhood of a conical point with a cone angle 6π can be glued from six metric half discs. At this conical point we have 3 distinct directions to the North.

Families of flat surfaces

The polygon in our construction depends continuously on the vectors $\vec{v}_j$. This means that the combinatorial geometry of the resulting flat surface (its genus g, the number n and types $2\pi(d_1+1), \ldots, 2\pi(d_n+1)$ of the resulting conical singularities) does not change under small deformations of the vectors $\vec{v}_j$. This allows to consider a flat surface as an element of a **family** of flat surfaces sharing common combinatorial geometry.

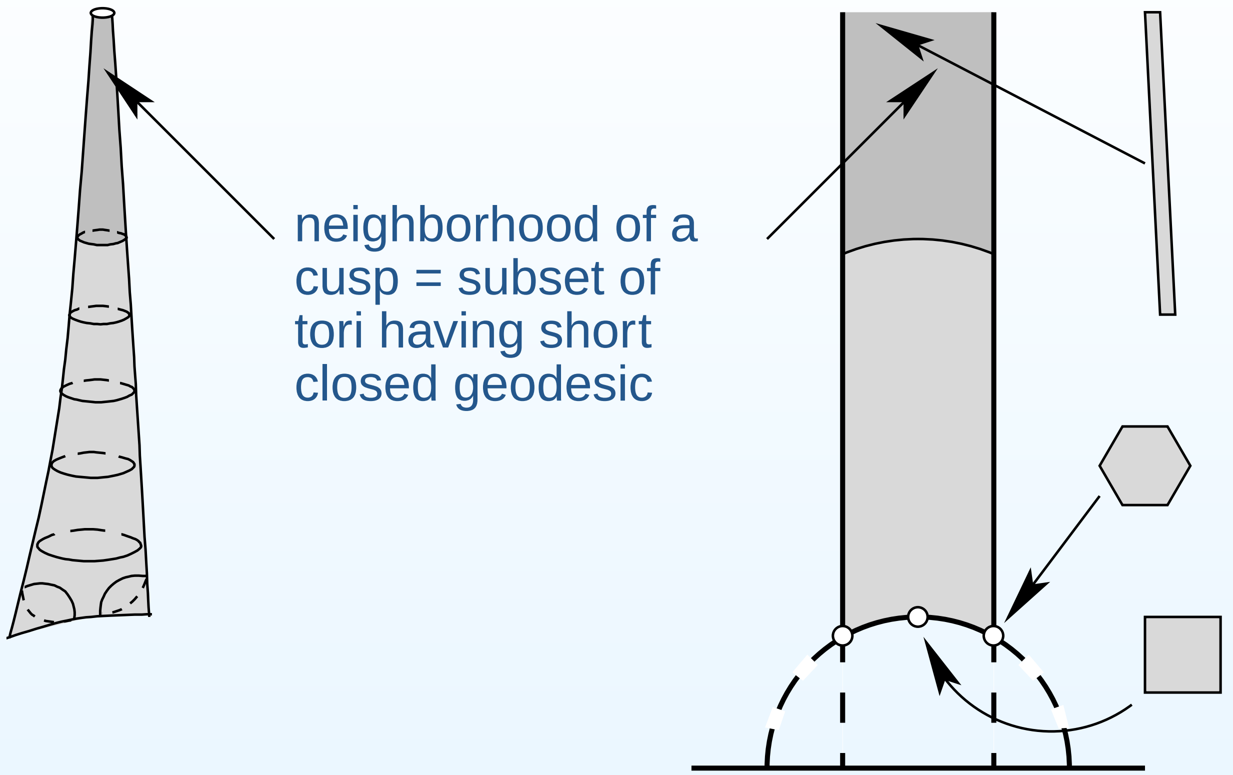
Families of flat surfaces

The polygon in our construction depends continuously on the vectors $\vec{v}_j$. This means that the combinatorial geometry of the resulting flat surface (its genus g, the number n and types $2\pi(d_1+1),\dots,2\pi(d_n+1)$ of the resulting conical singularities) does not change under small deformations of the vectors $\vec{v}_j$. This allows to consider a flat surface as an element of a **family** of flat surfaces sharing common combinatorial geometry.

As an example of such family one can consider a family of flat tori of area one, which can be identified with the space of lattices of area one:

$$\left.{}_{\mathrm{SO}(2,\mathbb{R})}\backslash\, \mathrm{SL}(2,\mathbb{R}) \middle/ {}_{\mathrm{SL}(2,\mathbb{Z})}\right. \quad = \quad \mathbb{H}^2 / \mathrm{SL}(2,\mathbb{Z})$$

Family of flat tori



The corresponding “modular surface” is not compact: flat tori representing points, which are close to the cusp, are almost degenerate: they have a very short closed geodesic.

Holomorphic 1-forms and quadratic differentials versus very flat surfaces

Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane, where lives the polygon. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points inherits the complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane, where lives the polygon. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points inherits the complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of a zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is a **degree** of the zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
- Any holomorphic 1-form corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly n zeroes of degrees $d_1, \dots, d_n$.

Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of a zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is a **degree** of the zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
- Any holomorphic 1-form corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly n zeroes of degrees $d_1, \dots, d_n$.

Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of a zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is a **degree** of the zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
- Any holomorphic 1-form corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly n zeroes of degrees $d_1, \dots, d_n$.

Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of a zero a holomorphic 1-form can be represented as $w^d \, dw$, where d is a **degree** of the zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
- Any holomorphic 1-form corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly n zeroes of degrees $d_1, \dots, d_n$.

Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of a zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is a **degree** of the zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \ldots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
- Any holomorphic 1-form corresponding to a fixed stratum $\mathcal{H}(d_1, \ldots, d_n)$ has exactly n zeroes of degrees $d_1, \ldots, d_n$.

Concise geometro-analytic dictionary

flat structure (including a choice of the vertical direction)	complex structure and a choice of a holomorphic 1-form ω
conical point with a cone angle $2\pi(d+1)$	zero of degree d of the holomorphic 1-form ω (in local coordinates $\omega = w^d\, dw$)
side $\vec{v}_j$ of a polygon	relative period $\int_{P_j}^{P_{j+1}} \omega = \int_{\vec{v}_j} dz$ of the 1-form ω
family of flat surfaces sharing the same cone angles $2\pi(d_1+1), \dots, 2\pi(d_n+1)$	stratum $\mathcal{H}(d_1, \dots, d_n)$ in the moduli space of holomorphic 1-forms
local coordinates in the family: vectors $\vec{v}_i$ defining the polygon	local coordinates in $\mathcal{H}(d_1, \dots, d_n)$: relative periods of ω in $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$

Homologous saddle connections and closed geodesics

Saddle connections

A *saddle connection* is a geodesic segment joining a pair of conical singularities or a conical singularity to itself without any singularities in its interior.

Similar to the torus case regular closed geodesics on flat surface always appear in families; any such family fills a maximal cylinder bounded on each side by a closed saddle connection or by a chain of parallel saddle connections.

Let $N_{sc}(S, L)$ be the number of saddle connections of length at most L on a flat surface S. Let $N_{cg}(S, L)$ be the number of maximal cylinders filled with closed regular geodesics of length at most L on S. It was proved by H. Masur that for any flat surface S both counting functions $N(S, L)$ grow quadratically in L:

$$const_1(S) \leq \frac{N(S, L)}{L^2} \leq const_2(S)$$

Saddle connections

A *saddle connection* is a geodesic segment joining a pair of conical singularities or a conical singularity to itself without any singularities in its interior.

Similar to the torus case regular closed geodesics on flat surface always appear in families; any such family fills a maximal cylinder bounded on each side by a closed saddle connection or by a chain of parallel saddle connections.

Let $N_{sc}(S, L)$ be the number of saddle connections of length at most L on a flat surface S. Let $N_{cg}(S, L)$ be the number of maximal cylinders filled with closed regular geodesics of length at most L on S. It was proved by H. Masur that for any flat surface S both counting functions $N(S, L)$ grow quadratically in L:

$$const_1(S) \leq \frac{N(S, L)}{L^2} \leq const_2(S)$$

Saddle connections

A *saddle connection* is a geodesic segment joining a pair of conical singularities or a conical singularity to itself without any singularities in its interior.

Similar to the torus case regular closed geodesics on flat surface always appear in families; any such family fills a maximal cylinder bounded on each side by a closed saddle connection or by a chain of parallel saddle connections.

Let $N_{sc}(S,L)$ be the number of saddle connections of length at most L on a flat surface S. Let $N_{cg}(S,L)$ be the number of maximal cylinders filled with closed regular geodesics of length at most L on S. It was proved by H. Masur that for any flat surface S both counting functions $N(S,L)$ grow quadratically in L:

$$const_1(S) \le \frac{N(S,L)}{L^2} \le const_2(S)$$

Exact quadratic asymptotics

Theorem (A. Eskin and H. Masur) *For almost all flat surfaces S of area 1 in any stratum $\mathcal{H}(d_1,\dots,d_n)$ the counting functions $N_{sc}(S,L)$ and $N_{cg}(S,L)$ have exact quadratic asymptotics*

$$\lim_{L\to\infty}\frac{N_{sc}(S,L)}{\pi L^2}=c_{sc}(S)\qquad \lim_{L\to\infty}\frac{N_{cg}(S,L)}{\pi L^2}=c_{cg}(S)$$

where the Siegel–Veech constants $c_{sc}(S)$ and $c_{cg}(S)$ depend only on the connected component of the stratum.

Analogous statement is valid for any closed $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold.

Consider some saddle connection $\gamma_1=[P_1P_2]$ with an endpoint at P_1. Memorize its direction, say, let it be the North-West direction. Let us launch a geodesic from the same starting point P_1 in one of the remaining $k-1$ North-West directions. Let us study how big is the chance to hit P_2 ones again, and how big is the chance to hit it after passing the same distance as before.

Exact quadratic asymptotics

Theorem (A. Eskin and H. Masur) *For almost all flat surfaces* S *of area* 1 *in any stratum* $\mathcal{H}(d_1,\dots,d_n)$ *the counting functions* $N_{sc}(S,L)$ *and* $N_{cg}(S,L)$ *have exact quadratic asymptotics*

$$\lim_{L\to\infty}\frac{N_{sc}(S,L)}{\pi L^2}=c_{sc}(S) \qquad \lim_{L\to\infty}\frac{N_{cg}(S,L)}{\pi L^2}=c_{cg}(S)$$

where the Siegel–Veech constants $c_{sc}(S)$ *and* $c_{cg}(S)$ *depend only on the connected component of the stratum.*

Analogous statement is valid for any closed $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold.

Consider some saddle connection $\gamma_1=[P_1P_2]$ with an endpoint at P_1. Memorize its direction, say, let it be the North-West direction. Let us launch a geodesic from the same starting point P_1 in one of the remaining $k-1$ North-West directions. Let us study how big is the chance to hit P_2 ones again, and how big is the chance to hit it after passing the same distance as before.

Phenomenon of multiple saddle connections

Theorem (A. Eskin, H. Masur, A. Zorich) *For almost any flat surface S in any stratum and for any pair P_1, P_2 of conical singularities on S the function $N_2(S,L)$ counting the number of pairs of parallel saddle connections of the same length joining P_1 to P_2 also has exact quadratic asymptotics*

$$\lim_{L\to\infty} \frac{N_2(S,L)}{\pi L^2} = c_2 > 0.$$

For almost all flat surfaces S in any stratum one cannot find neither a single pair of parallel saddle connections on S of different length, nor a single pair of parallel saddle connections joining different pairs of singularities.

Phenomenon of multiple saddle connections

Theorem (A. Eskin, H. Masur, A. Zorich) *For almost any flat surface S in any stratum and for any pair P_1, P_2 of conical singularities on S the function $N_2(S, L)$ counting the number of pairs of parallel saddle connections of the same length joining P_1 to P_2 also has exact quadratic asymptotics*

$$\lim_{L\to\infty} \frac{N_2(S,L)}{\pi L^2} = c_2 > 0.$$

For almost all flat surfaces S in any stratum one cannot find neither a single pair of parallel saddle connections on S of different length, nor a single pair of parallel saddle connections joining different pairs of singularities.

Rigid collections of saddle connections

Any saddle connection on a flat surface persists under small deformations of S inside the ambient stratum.

It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections.

We say that a collection $\{\gamma_1, \ldots, \gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \cdots : |\gamma_n|$ of the lengths of all saddle connections in the collection.

Theorem (Eskin, Masur, Zorich) *Let S be a flat surface corresponding to a holomorphic 1-form ω. A collection $\gamma_1, \ldots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \ldots, \gamma_n$ are homologous in $H^1(S, \{\text{zeroes of } \omega\}; \mathbb{C})$.*

Rigid collections of saddle connections

Any saddle connection on a flat surface persists under small deformations of S inside the ambient stratum.

It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections.

We say that a collection $\{\gamma_1, \dots, \gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_n|$ of the lengths of all saddle connections in the collection.

Theorem (Eskin, Masur, Zorich) *Let S be a flat surface corresponding to a holomorphic 1-form ω. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are homologous in $H^1(S, \{\text{zeroes of } \omega\}; \mathbb{C})$.*

Rigid collections of saddle connections

Any saddle connection on a flat surface persists under small deformations of S inside the ambient stratum.

It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections.

We say that a collection $\{\gamma_1, \dots, \gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_n|$ of the lengths of all saddle connections in the collection.

Theorem (Eskin, Masur, Zorich) *Let S be a flat surface corresponding to a holomorphic 1-form ω. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are homologous in $H^1(S, \{\text{zeroes of } \omega\}; \mathbb{C})$.*

Rigid collections of saddle connections

Any saddle connection on a flat surface persists under small deformations of S inside the ambient stratum.

It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections.

We say that a collection $\{\gamma_1, \dots, \gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_n|$ of the lengths of all saddle connections in the collection.

Theorem (Eskin, Masur, Zorich) *Let S be a flat surface corresponding to a holomorphic 1-form ω. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are homologous in $H^1(S, \{\text{zeroes of } \omega\}; \mathbb{C})$.*

Homologous saddle connections

Directions and lengths of saddle connections can be expressed in terms of integrals of the holomorphic 1-form ω along corresponding paths,

$$\overrightarrow{P_1 P_2} = \int_{[P_1 P_2]} \omega \in \mathbb{C} \simeq \mathbb{R}^2$$

Hence

- Homologous saddle connections $\gamma_1, \dots, \gamma_n$ are parallel and have equal length and
  - either all of them join the same pair of distinct singular points,
  - or all γ_i are closed loops.

Homologous saddle connections

Directions and lengths of saddle connections can be expressed in terms of integrals of the holomorphic 1-form ω along corresponding paths,

$$\overrightarrow{P_1 P_2} = \int_{[P_1 P_2]} \omega \in \mathbb{C} \simeq \mathbb{R}^2$$

Hence

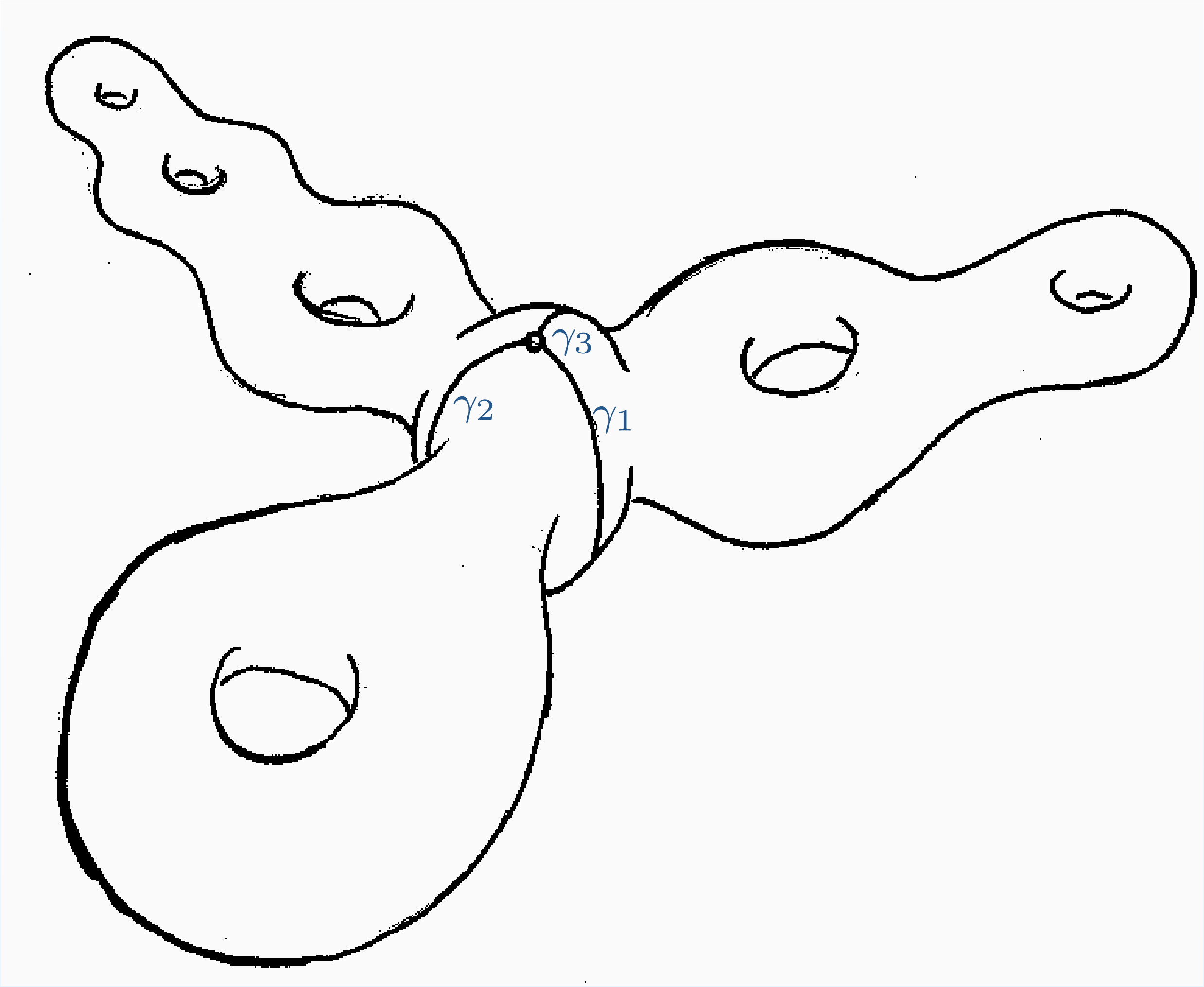
- Homologous saddle connections $\gamma_1, \ldots, \gamma_n$ are parallel and have equal length and
  - either all of them join the same pair of distinct singular points,
  - or all γ_i are closed loops.

Homologous saddle connections

Hence

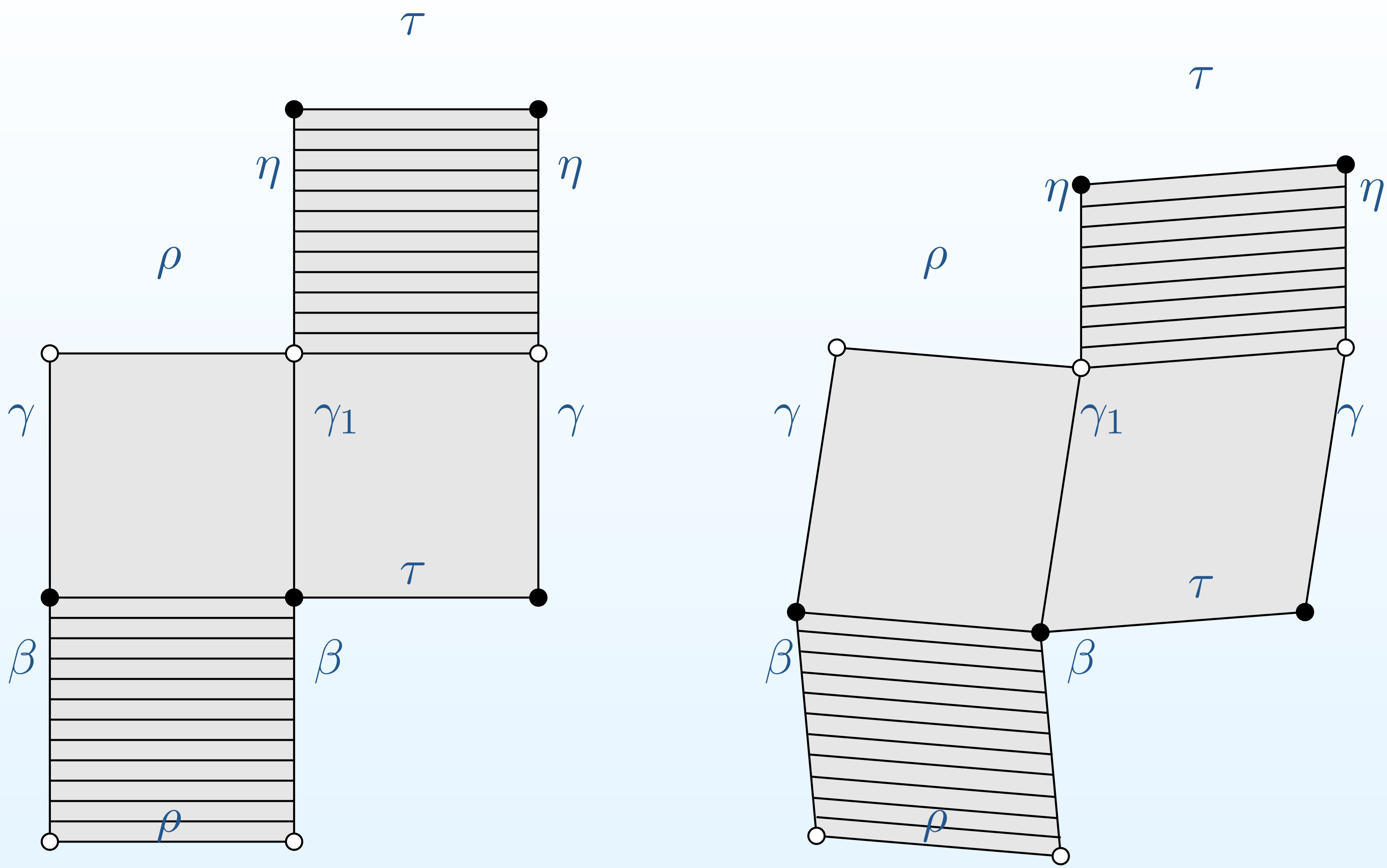
- Homologous saddle connections $\gamma_1, \ldots, \gamma_n$ are parallel and have equal length and
  - either all of them join the same pair of distinct singular points,
  - or all γ_i are closed loops.

Saddle connections joining distinct zeroes



Multiple homologous saddle connections, topological picture.

Saddle connections joining distinct zeroes



Saddle connections γ and γ_1 are homologous. They stay parallel and isometric, $|\gamma_1| = |\gamma|$, under any small deformation of the flat surface.

Why multiple saddle connections occur often

Note that our saddle connections persist and remain homologous for any small deformation of the surface. Hence we can find such configuration of saddle connections for all surfaces in an open domain in the ambient stratum.

On the other hand a linear action of $\mathrm{SL}(2,\mathbb{R})$ sends a configuration of homologous saddle connections to a configuration of homologous saddle connections.

Hence, by ergodicity of the linear action, if we managed to find some configuration of homologous saddle connections on a single flat surface, we shall find a configuration of homologous saddle connections of the same combinatorial type on almost every surface in the same connected component of the ambient stratum.

The number of combinatorial types of configurations of homologous saddle connections for any given stratum is finite.

Why multiple saddle connections occur often

Note that our saddle connections persist and remain homologous for any small deformation of the surface. Hence we can find such configuration of saddle connections for all surfaces in an open domain in the ambient stratum.

On the other hand a linear action of $\mathrm{SL}(2,\mathbb{R})$ sends a configuration of homologous saddle connections to a configuration of homologous saddle connections.

Hence, by ergodicity of the linear action, if we managed to find some configuration of homologous saddle connections on a single flat surface, we shall find a configuration of homologous saddle connections of the same combinatorial type on almost every surface in the same connected component of the ambient stratum.

The number of combinatorial types of configurations of homologous saddle connections for any given stratum is finite.

Why multiple saddle connections occur often

Note that our saddle connections persist and remain homologous for any small deformation of the surface. Hence we can find such configuration of saddle connections for all surfaces in an open domain in the ambient stratum.

On the other hand a linear action of $\mathrm{SL}(2,\mathbb{R})$ sends a configuration of homologous saddle connections to a configuration of homologous saddle connections.

Hence, by ergodicity of the linear action, if we managed to find some configuration of homologous saddle connections on a single flat surface, we shall find a configuration of homologous saddle connections of the same combinatorial type on almost every surface in the same connected component of the ambient stratum.

The number of combinatorial types of configurations of homologous saddle connections for any given stratum is finite.

Why multiple saddle connections occur often

Note that our saddle connections persist and remain homologous for any small deformation of the surface. Hence we can find such configuration of saddle connections for all surfaces in an open domain in the ambient stratum.

On the other hand a linear action of $\mathrm{SL}(2,\mathbb{R})$ sends a configuration of homologous saddle connections to a configuration of homologous saddle connections.

Hence, by ergodicity of the linear action, if we managed to find some configuration of homologous saddle connections on a single flat surface, we shall find a configuration of homologous saddle connections of the same combinatorial type on almost every surface in the same connected component of the ambient stratum.

The number of combinatorial types of configurations of homologous saddle connections for any given stratum is finite.

However, not too often

Siegel–Veech constants for k cylinders of homologous closed geodesics for the principal strata $\mathcal{H}(1,\dots,1)$ in $g=2,3,4$

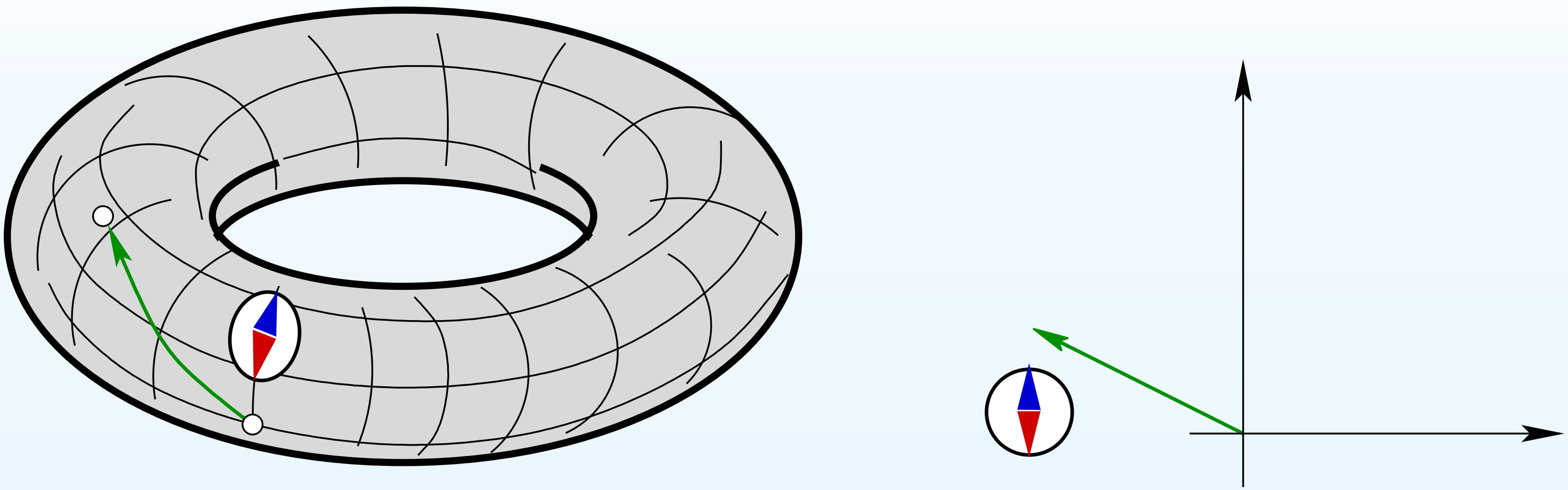
k	$g=1$	$g=2$	$g=3$	$g=4$
1	$\frac{1}{2\zeta(2)} \approx 0.3$	$\frac{5}{2}\cdot\frac{1}{\zeta(2)} \approx 1.5$	$\frac{36}{7}\cdot\frac{1}{\zeta(2)} \approx 3.13$	$\frac{3150}{377}\cdot\frac{1}{\zeta(2)} \approx 5.08$
2	–	–	$\frac{3}{14}\cdot\frac{1}{\zeta(2)} \approx 0.13$	$\frac{90}{377}\cdot\frac{1}{\zeta(2)} \approx 0.145$
3	–	–	–	$\frac{5}{754}\cdot\frac{1}{\zeta(2)} \approx 0.004$

In genus $g=4$ a closed regular geodesic belongs to a one-cylinder family with “probability” 97.1%, to a two-cylinder family with “probability” 2.8% and to a three-cylinder family with “probability” only 0.1%.

Siegel—Veech constants and cusps of the moduli space

Holonomy vector of a saddle connection

To every saddle connection γ on a flat surface S assign a vector $\vec{v}(\gamma)$ in $\mathbb{R}^2$ having the length and the direction of γ. In other words, $\vec{v} = \int_\gamma \omega$, where we consider a complex number as a vector in $\mathbb{R}^2 \simeq \mathbb{C}$.

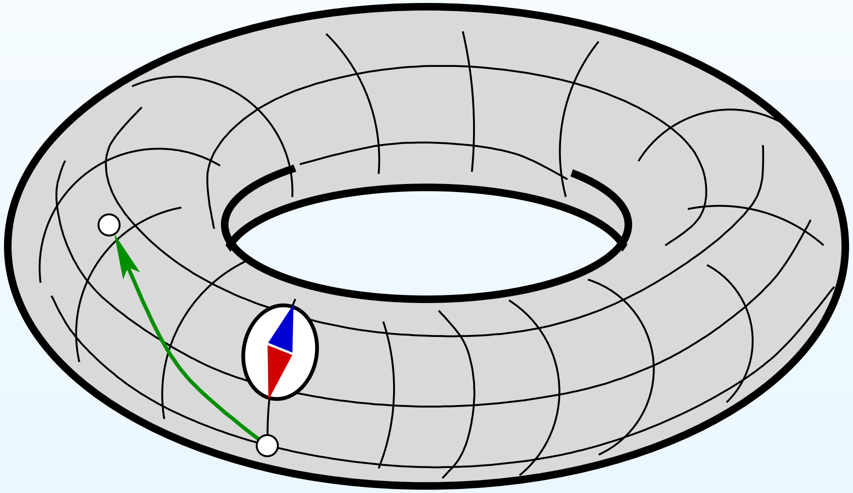


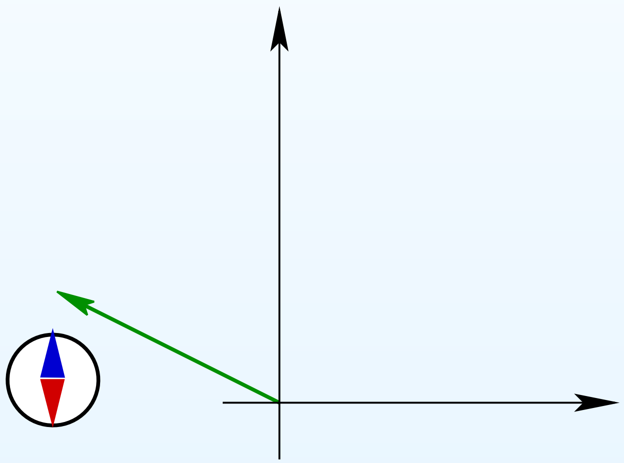
When we have a configuration of several *homologous* closed geodesics or saddle connections, the corresponding holonomy vectors coincide.

Fix a combinatorial type of a configuration. For each configuration of this type construct the corresponding holonomy vector. We obtain a discrete set V in $\mathbb{R}^2$.

Holonomy vector of a saddle connection

To every saddle connection γ on a flat surface S assign a vector $\vec{v}(\gamma)$ in $\mathbb{R}^2$ having the length and the direction of γ. In other words, $\vec{v} = \int_\gamma \omega$, where we consider a complex number as a vector in $\mathbb{R}^2 \simeq \mathbb{C}$.



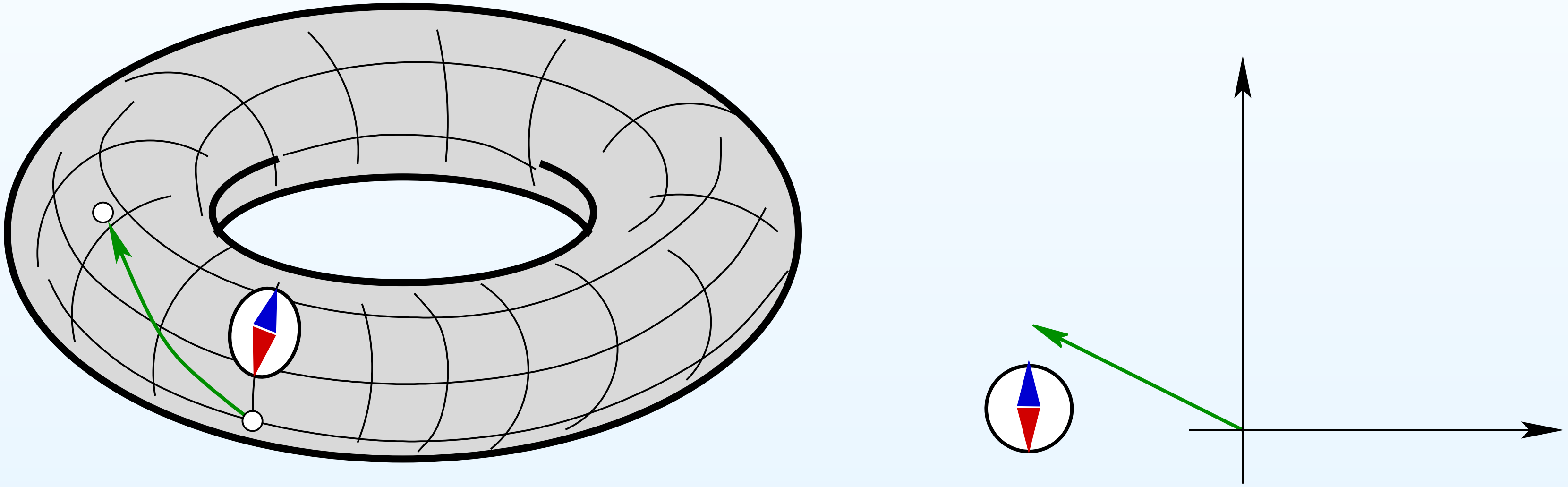


When we have a configuration of several *homologous* closed geodesics or saddle connections, the corresponding holonomy vectors coincide.

Fix a combinatorial type of a configuration. For each configuration of this type construct the corresponding holonomy vector. We obtain a discrete set V in $\mathbb{R}^2$.

Holonomy vector of a saddle connection

To every saddle connection γ on a flat surface S assign a vector $\vec{v}(\gamma)$ in $\mathbb{R}^2$ having the length and the direction of γ. In other words, $\vec{v} = \int_\gamma \omega$, where we consider a complex number as a vector in $\mathbb{R}^2 \simeq \mathbb{C}$.



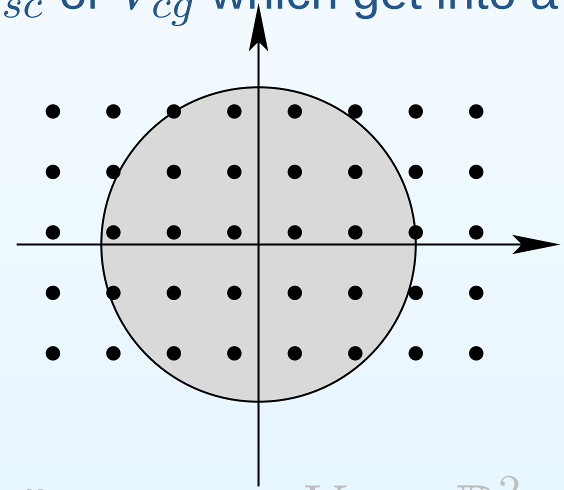
When we have a configuration of several *homologous* closed geodesics or saddle connections, the corresponding holonomy vectors coincide.

Fix a combinatorial type of a configuration. For each configuration of this type construct the corresponding holonomy vector. We obtain a discrete set V in $\mathbb{R}^2$.

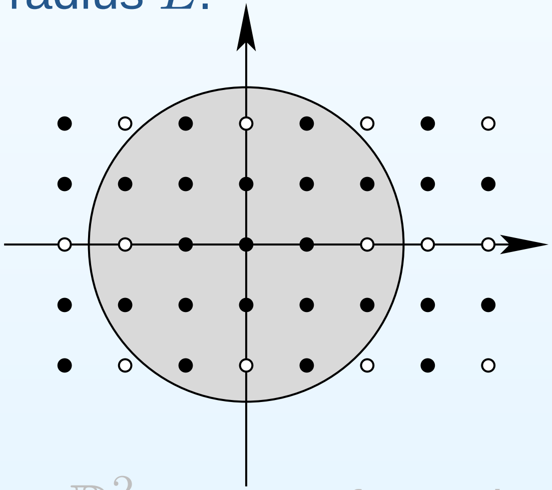
Holonomy sets for saddle connections and for closed geodesics on a torus

Mark two points on a torus and consider all geodesic segments joining these two points. They mimic saddle connections. We associate to them a set V_{sc} of holonomy vectors. Consider also all closed geodesics; we associate to them the set V_{cg} of holonomy vectors. To count the number of saddle connections or closed geodesics of length bounded by L is the same as to count the number of points of V_{sc} or V_{cg} which get into a disc of radius L.

V_{sc}



V_{cg}

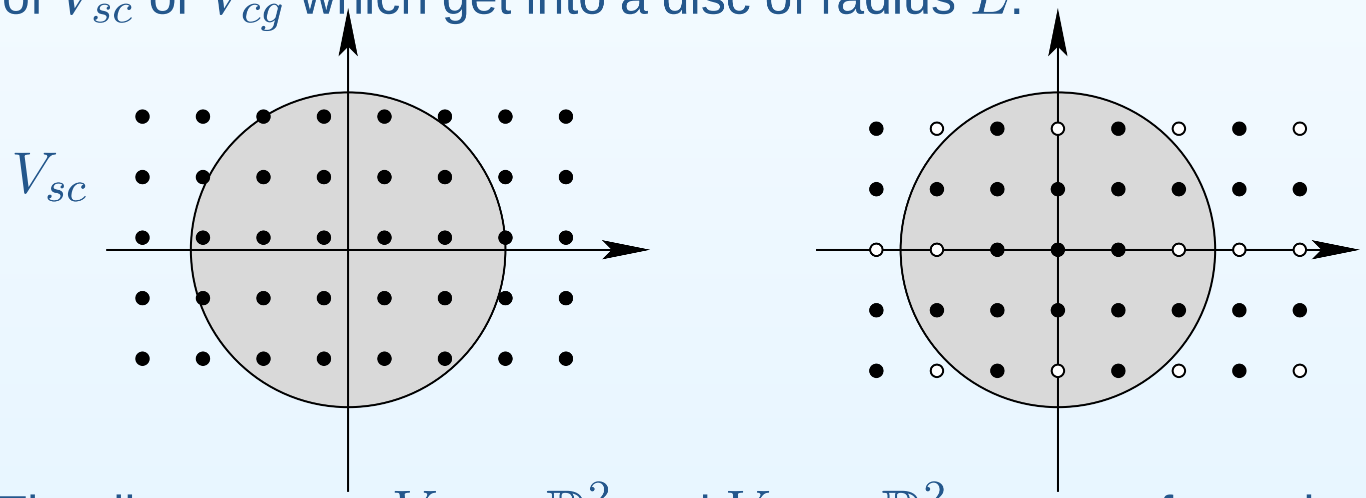


Remark The discrete sets $V_{sc} \subset \mathbb{R}^2$ and $V_{cg} \subset \mathbb{R}^2$ are transformed equivariantly with respect to the group action:

$$V(gS) = gV(S) \quad \text{for any } g \in \mathrm{GL}(2, \mathbb{R}) \, .$$

Holonomy sets for saddle connections and for closed geodesics on a torus

Mark two points on a torus and consider all geodesic segments joining these two points. They mimic saddle connections. We associate to them a set V_{sc} of holonomy vectors. Consider also all closed geodesics; we associate to them the set V_{cg} of holonomy vectors. To count the number of saddle connections or closed geodesics of length bounded by L is the same as to count the number of points of V_{sc} or V_{cg} which get into a disc of radius L.



Remark The discrete sets $V_{sc} \subset \mathbb{R}^2$ and $V_{cg} \subset \mathbb{R}^2$ are transformed equivariantly with respect to the group action:

$$V(gS) = gV(S) \quad \text{for any } g \in \mathrm{GL}(2, \mathbb{R})\,.$$

Siegel–Veech formula

Consider the following operator $f \mapsto \hat{f}$ from functions with compact support on $\mathbb{R}^2$ to functions on the stratum $\mathcal{H}_1(d_1, \ldots, d_n)$:

$$\hat{f}(S) := \sum_{\vec{v} \in V(S)} f(\vec{v})$$

Function $\hat{f}(S)$ generalizes the counting function $N(S, L)$: when $f(x, y)$ is the characteristic function χ_L of the disc of radius L with the center at the origin, $\hat{\chi}_L(S) = N(S, L)$ counts the number of chosen configurations of homologous saddle connections of length at most L on a flat surface S.

Lemma (W. Veech) *The functional*

$$f \mapsto \int_{\mathcal{H}_1^{comp}(d_1,\ldots,d_n)} \hat{f}(S)\, d\nu_1$$

is $\mathrm{SL}(2, \mathbb{R})$*-invariant.*

Siegel–Veech formula

Consider the following operator $f \mapsto \hat{f}$ from functions with compact support on $\mathbb{R}^2$ to functions on the stratum $\mathcal{H}_1(d_1,\dots,d_n)$:

$$\hat{f}(S) := \sum_{\vec{v}\in V(S)} f(\vec{v})$$

Function $\hat{f}(S)$ generalizes the counting function $N(S,L)$: when $f(x,y)$ is the characteristic function χ_L of the disc of radius L with the center at the origin, $\hat{\chi}_L(S) = N(S,L)$ counts the number of chosen configurations of homologous saddle connections of length at most L on a flat surface S.

Lemma (W. Veech) *The functional*

$$f \mapsto \int_{\mathcal{H}_1^{comp}(d_1,\dots,d_n)} \hat{f}(S)\, d\nu_1$$

is $\mathrm{SL}(2,\mathbb{R})$*-invariant.*

Siegel–Veech formula

Consider the following operator $f \mapsto \hat{f}$ from functions with compact support on $\mathbb{R}^2$ to functions on the stratum $\mathcal{H}_1(d_1, \ldots, d_n)$:

$$\hat{f}(S) := \sum_{\vec{v} \in V(S)} f(\vec{v})$$

Function $\hat{f}(S)$ generalizes the counting function $N(S, L)$: when $f(x, y)$ is the characteristic function χ_L of the disc of radius L with the center at the origin, $\hat{\chi}_L(S) = N(S, L)$ counts the number of chosen configurations of homologous saddle connections of length at most L on a flat surface S.

Lemma (W. Veech) *The functional*

$$f \mapsto \int_{\mathcal{H}_1^{comp}(d_1,\ldots,d_n)} \hat{f}(S)\, d\nu_1$$

is $\mathrm{SL}(2, \mathbb{R})$*-invariant.*

Siegel–Veech formula

Theorem (W. Veech) *For any function $f:\mathbb{R}^2\to\mathbb{R}$ with compact support the following equality is valid:*

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}\int_{\mathcal{H}_1(d_1,\dots,d_n)}\hat{f}(S)\,d\nu_1 = C\int_{\mathbb{R}^2} f(x,y)\,dx\,dy\,,$$

where the constant C does not depend on the function f.

Proof: The only $\mathrm{SL}(2,\mathbb{R})$-invariant functionals are the integral over $\mathbb{R}^2$, the value in the origin, and their linear combinations.

Theorem (A. Eskin, H. Masur) *For almost all flat surfaces S (in a connected component of a stratum) the Siegel–Veech constant $c(S)$ in quadratic asymptotics $N(S,L) = c(S)\cdot\pi L^2$ coincides with the constant C in the Theorem of Veech.*

Siegel–Veech formula

Theorem (W. Veech) *For any function $f : \mathbb{R}^2 \to \mathbb{R}$ with compact support the following equality is valid:*

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}\int_{\mathcal{H}_1(d_1,\dots,d_n)} \hat{f}(S)\,d\nu_1 = C\int_{\mathbb{R}^2} f(x,y)\,dx\,dy\;,$$

where the constant C does not depend on the function f.

Proof: The only $\mathrm{SL}(2,\mathbb{R})$-invariant functionals are the integral over $\mathbb{R}^2$, the value in the origin, and their linear combinations.

Theorem (A. Eskin, H. Masur) *For almost all flat surfaces S (in a connected component of a stratum) the Siegel–Veech constant $c(S)$ in quadratic asymptotics $N(S,L) = c(S)\cdot\pi L^2$ coincides with the constant C in the Theorem of Veech.*

Siegel–Veech formula

Theorem (W. Veech) *For any function $f:\mathbb{R}^2\to\mathbb{R}$ with compact support the following equality is valid:*

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}\int_{\mathcal{H}_1(d_1,\dots,d_n)}\hat{f}(S)\,d\nu_1 = C\int_{\mathbb{R}^2} f(x,y)\,dx\,dy\,,$$

where the constant C does not depend on the function f.

Proof: The only $\mathrm{SL}(2,\mathbb{R})$-invariant functionals are the integral over $\mathbb{R}^2$, the value in the origin, and their linear combinations.

Theorem (A. Eskin, H. Masur) *For almost all flat surfaces S (in a connected component of a stratum) the Siegel–Veech constant $c(S)$ in quadratic asymptotics $N(S,L)=c(S)\cdot\pi L^2$ coincides with the constant C in the Theorem of Veech.*

Calculation of Siegel–Veech constants: key idea

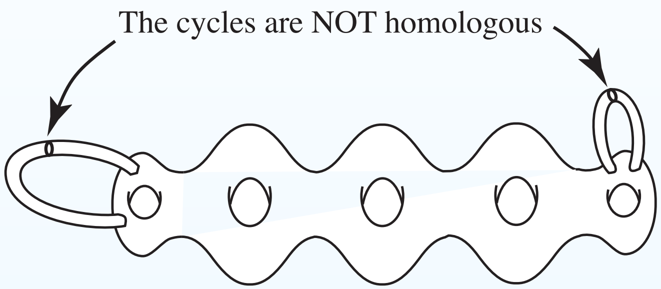
To compute C it is sufficient to evaluate $\int_{\mathcal{H}_1} \hat{f}(S)\, d\nu_1$ for a single function f. Consider a characteristic function $\chi_\varepsilon(x,y)$ of a disc of a very small radius ε in $\mathbb{R}^2$. Then $\hat{\chi}_\varepsilon(S)$ counts how many ε-short saddle connections (closed geodesics) we can find on a flat surface S. We have

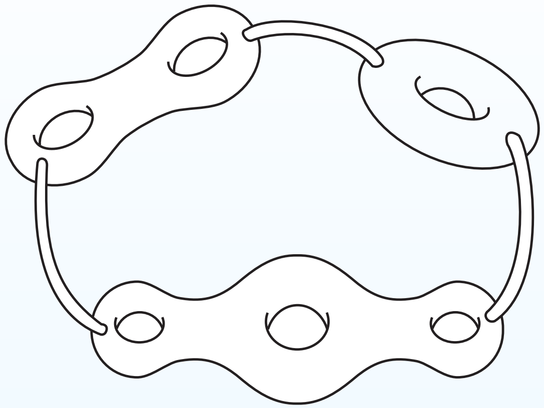
$$\hat{\chi}_\varepsilon(S) = \begin{cases} 0 & \text{for most of the surfaces } S \\ 1 & \text{for } S \in \mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) \\ > 1 & \text{for } S \in \mathcal{H}_1^{\varepsilon,thin}(d_1,\dots,d_n) \end{cases}$$

where $\mathcal{H}_1^{\varepsilon,thin}(d_1,\dots,d_n)$ is the subset of surfaces containing at least two nonhomologous saddle connections of length at most ε. We get

$$\int_{\mathcal{H}_1} \hat{\chi}_\varepsilon(S)\, d\nu_1 = \operatorname{Vol} \mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) \; + \int_{\mathcal{H}_1^{\varepsilon,thin}} \hat{\chi}_\varepsilon(S)\, d\nu_1 .$$

Typical and nontypical degenerations



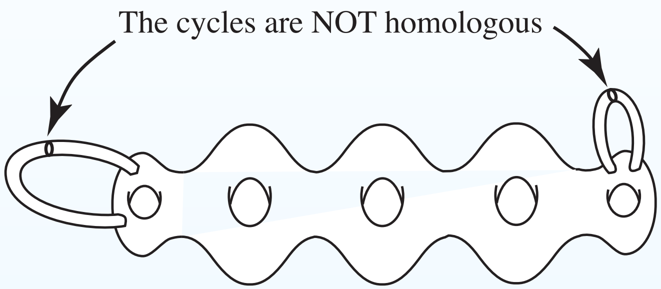


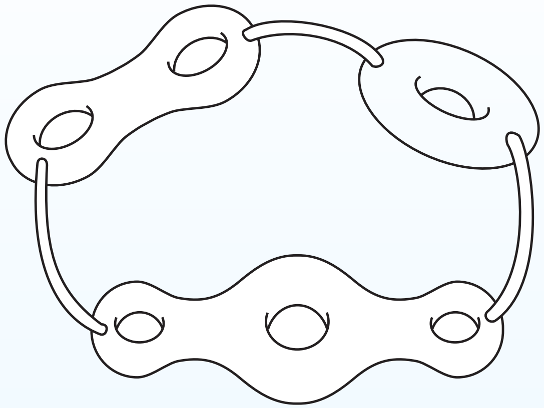
Theorem *(H. Masur, J. Smillie) The set of surfaces as on the right such that the waist curve of the cylinder is shorter than ε has measure $O(\varepsilon^2)$ in $\mathcal{H}_1(d_1,\dots,d_n)$ no matter what is the number of components.*
The set of surfaces as on the left such that the waist curve of the cylinder is shorter than ε has measure $O(\varepsilon^4)$.

A similar statement is true for short saddle connections. In our language:

$$\operatorname{Vol}\mathcal{H}_1^{\varepsilon,thin}(d_1,\dots,d_n) = O(\varepsilon^4)\,.$$

Typical and nontypical degenerations





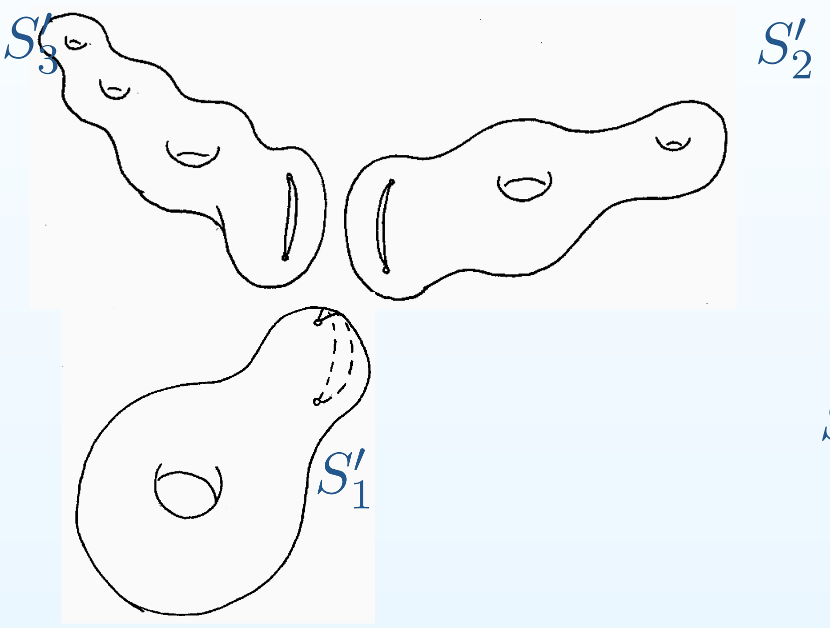
Theorem *(H. Masur, J. Smillie) The set of surfaces as on the right such that the waist curve of the cylinder is shorter than ε has measure $O(\varepsilon^2)$ in $\mathcal{H}_1(d_1,\dots,d_n)$ no matter what is the number of components.*
The set of surfaces as on the left such that the waist curve of the cylinder is shorter than ε has measure $O(\varepsilon^4)$.

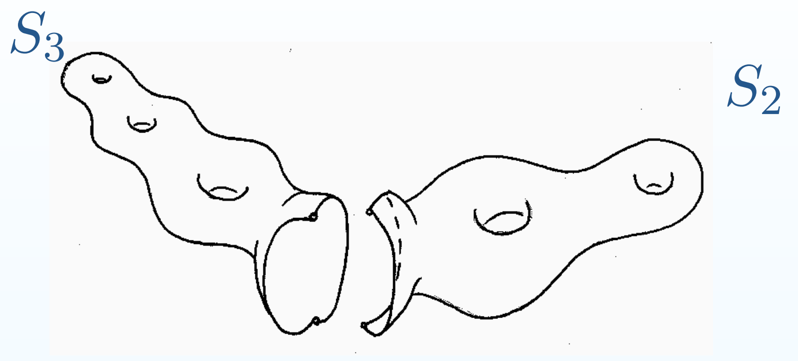
A similar statement is true for short saddle connections. In our language:

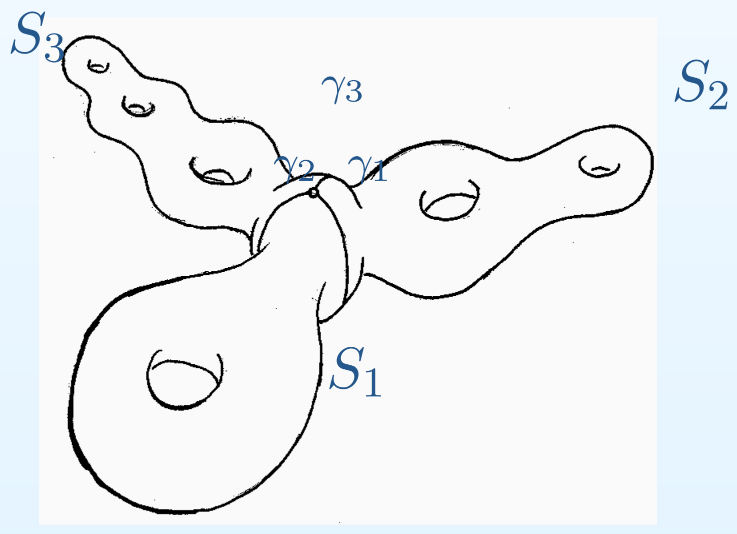
$$\operatorname{Vol}\mathcal{H}_1^{\varepsilon,thin}(d_1,\dots,d_n) = O(\varepsilon^4)\ .$$

More artistic picture of a generic degeneration

Another example of a generic degeneration

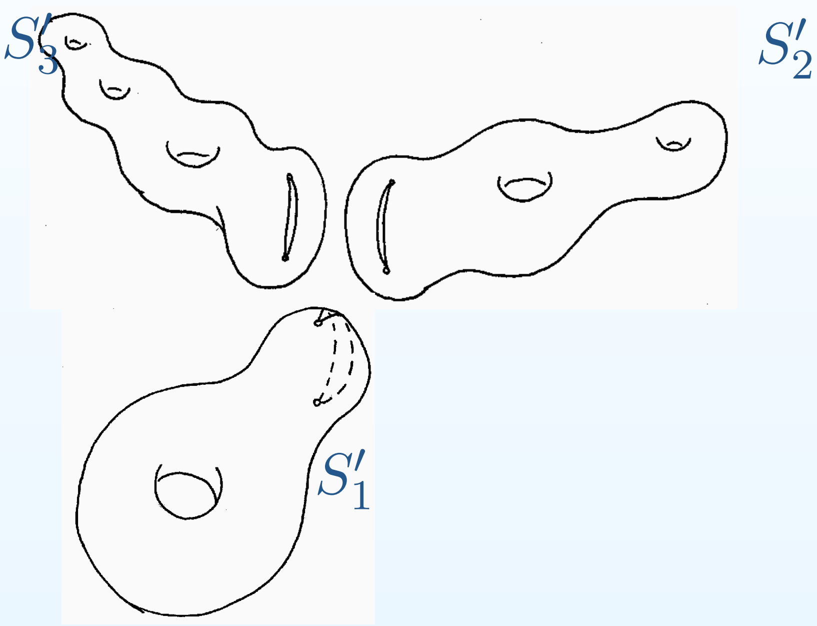


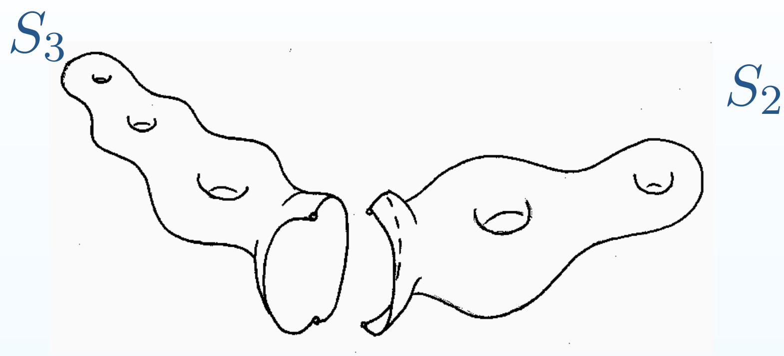


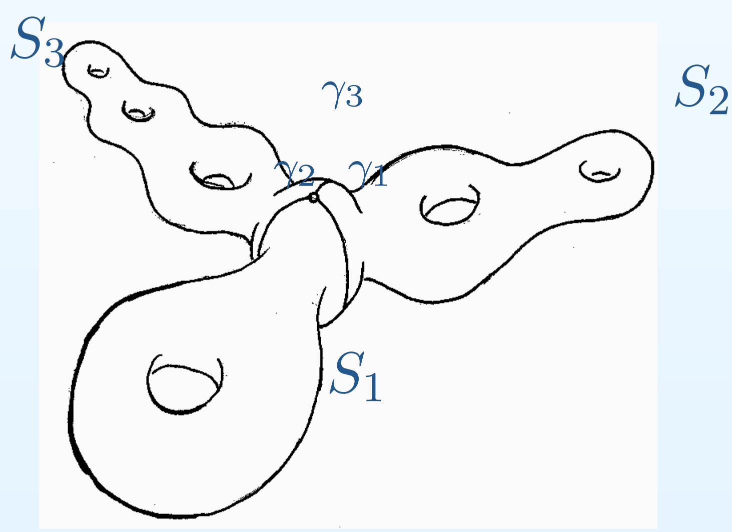


Remark Contracting the slits we get a “forbidden stable curve” from the point of view of the Deligne—Mamford compactification.

Another example of a generic degeneration







Remark Contracting the slits we get a “forbidden stable curve” from the point of view of the Deligne—Mamford compactification.

Calculation of Siegel–Veech constants (continued)

For a characteristic function $\chi_\varepsilon(x,y)$ of a disc of radius ε the Siegel–Veech formula gives us:

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}\int_{\mathcal{H}_1}\hat\chi_\varepsilon(S)\,d\nu_1 = C\int_{\mathbb{R}^2}\chi_\varepsilon(x,y)\,dx\,dy = C\cdot\pi\varepsilon^2$$

On the other hand, by definition of $\hat\chi_\varepsilon$, of the thick and the thin parts:

$$\int_{\mathcal{H}_1}\hat\chi_\varepsilon(S)\,d\nu_1 = \operatorname{Vol}\mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) + \int_{\mathcal{H}_1^{\varepsilon,thin}}\hat\chi_\varepsilon(S)\,d\nu_1 .$$

Theorem (A. Eskin, H. Masur)

$$\int_{\mathcal{H}_1^{\varepsilon,thin}}\hat\chi_\varepsilon(S)\,d\nu_1 = o(\varepsilon^2)$$

Calculation of Siegel–Veech constants: key idea

Corollary

$$\int_{\mathcal{H}_1} \hat{\chi}_\varepsilon(S)\, d\nu_1 = \operatorname{Vol} \mathcal{H}_1^{\varepsilon, thick}(d_1, \dots, d_n) \; + \; o(\varepsilon^2).$$

Applying Siegel–Veech formula we obtain

$$\frac{\operatorname{Vol} \mathcal{H}_1^{\varepsilon}(d_1, \dots, d_n)}{\operatorname{Vol} \mathcal{H}_1(d_1, \dots, d_n)} + o(\varepsilon^2) = C \cdot \pi \varepsilon^2$$

In order to compute the constant C it is sufficient to compute the asymtotics of the volume of the subset $\mathcal{H}_1^{\varepsilon}(d_1, \dots, d_n)$ of surfaces containing a saddle connection of length at most ε, i.e. the volume of a "ε-neighborhood of the cusp". Then

$$C = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \frac{\operatorname{Vol}(\text{“}\varepsilon\text{-neighborhood of the cusp”})}{\operatorname{Vol} \mathcal{H}_1(d_1, \dots, d_n)}$$

Calculation of Siegel–Veech constants: key idea

Corollary

$$\int_{\mathcal{H}_1} \hat{\chi}_\varepsilon(S)\, d\nu_1 = \operatorname{Vol}\mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) \,+\, o(\varepsilon^2).$$

Applying Siegel–Veech formula we obtain

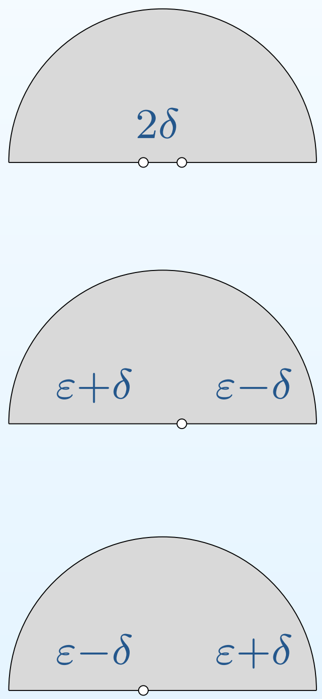
$$\frac{\operatorname{Vol}\mathcal{H}_1^{\varepsilon}(d_1,\dots,d_n)}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)} + o(\varepsilon^2) = C\cdot\pi\varepsilon^2$$

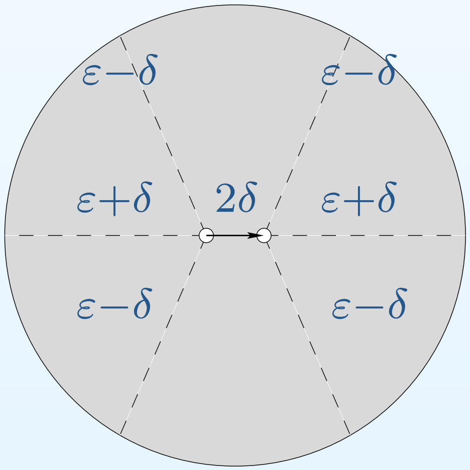
In order to compute the constant C it is sufficient to compute the asymtotics of the volume of the subset $\mathcal{H}_1^{\varepsilon}(d_1,\dots,d_n)$ of surfaces containing a saddle connection of length at most ε, i.e. the volume of a "ε-neighborhood of the cusp". Then

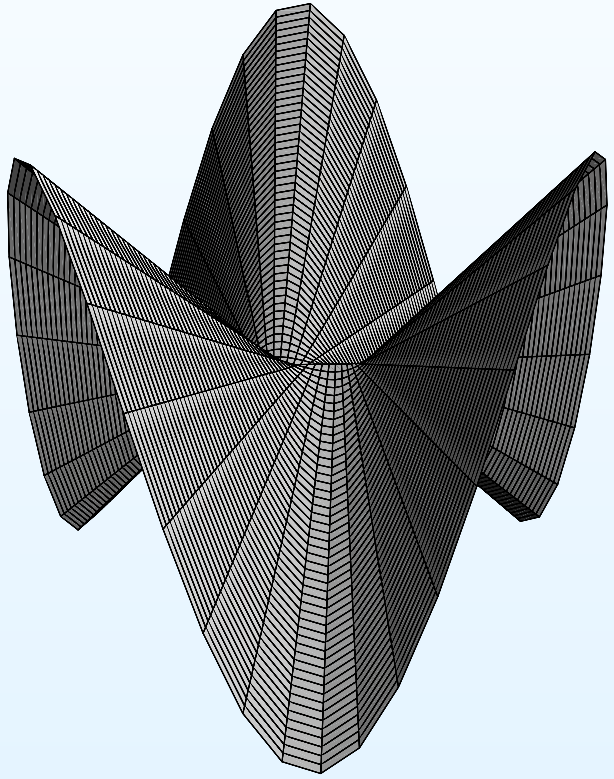
$$C = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\frac{\operatorname{Vol}(\text{“}\varepsilon\text{-neighborhood of the cusp”})}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}$$

Breaking up a double zero into two simple ones

Cut an ε-neighborhood of the double zero out of the surface. Decompose it into six metric half-disks of radius ε. Now change identifications of diameters of these half-discs as indicated and paste the result into the surface.







Simplest cusp: short simple saddle connection

We want to compute the measure of the subset of surfaces having a single short saddle connection joining two simple zeroes. There is a canonical way to shrink the saddle connection on $S \in \mathcal{H}_1^{\varepsilon,thick}(1,1)$ coalescing two zeroes into one. This provides us with an (almost) fiber bundle

$$\begin{array}{c} \mathcal{H}_1^{\varepsilon,thick}(1,1) \\ \Big\downarrow {\scriptstyle \tilde{D}^2_\varepsilon} \\ \mathcal{H}_1(2) \end{array}$$

where $\tilde{D}^2_\varepsilon$ is a ramified cover of order 3 over a standard metric disc of radius ε. Moreover, the measure on $\mathcal{H}_1^{\varepsilon,thick}(1,1)$ disintegrates into a product of the standard measure on $\tilde{D}^2_\varepsilon$ and the natural measure on $\mathcal{H}_1(2)$ which implies:

$$\operatorname{Vol}(\text{“}\varepsilon\text{-neighborhood of the cusp”}) \approx 3 \cdot \pi\varepsilon^2 \cdot \operatorname{Vol}\mathcal{H}_1(2)$$

General formula for Siegel–Veech constants

Consider a configuration of homologous saddle connections on a flat surface $S \in \mathcal{H}(\alpha)$. Contracting the saddle connections we get a (possibly degenerate) limiting flat surface $S' = S'_1 \sqcup \cdots \sqcup S'_k$, where components $S'_j \in \mathcal{H}(\alpha'_k)$ are already regular flat surfaces of smaller genera. Let C be the Siegel–Veech constant responsible for counting configurations of homologous saddle connections of the above type on almost any flat surface in $\mathcal{H}(\alpha)$.

$$C = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \frac{\operatorname{Vol}(\text{“}\varepsilon\text{-neighborhood of the cusp”})}{\operatorname{Vol}\mathcal{H}_1(\alpha)} =$$

$$= const \cdot \frac{\prod_{j=1}^{k} \operatorname{Vol}\mathcal{H}_1(\alpha'_j)}{\operatorname{Vol}\mathcal{H}_1(\alpha)},$$

where $const$ is an explicit combinatorial constant.

Example: configurations of geodesic saddle connections for the stratum $\mathcal{H}(3,1,1,1)$

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$(1+1,3,1)\succ$	2	1	9	$\frac{729}{62}$	11.7581
$(1+3,1,1)\succ$	1	1	15	$\frac{185625}{7936}$	23.3902
$(0+0)\succ(0+0,3,1)\succ$	2	1	3	$\frac{15}{62}$	0.241935
$(0+0)\succ(0+2,1,1)\succ$	1	1	9	$\frac{2025}{3968}$	0.510333
$(0+2)\succ(0+0,1,1)\succ$	1	1	9	$\frac{405}{7936}$	0.0510333
$(0+1,1)\succ(0+1,1)\succ$	1	2	12	$\frac{3}{62}$	0.0483871

Some recent results

Counting ignoring multiplicities

Working with translation surfaces (holomorphic forms) in $\mathcal{H}(m_1, \dots, m_n)$ one usually labels all conical singularities $P_1, \dots, P_n$. Fix any two of them, P_i and P_j. Let us count saddle connections joining P_i to P_j neglecting multiplicities (i.e., let us count saddle connections looking only at their holonomy vectors in $\mathbb{R}^2$). The corresponding Siegel–Veech constant $c_{i,j}^{hom}$ is the sum of all Siegel–Veech constants corresponding to all possible configurations of homologous saddle connections joining P_i to P_j.

Theorem (D. Chen, M. Möller, A. Sauvaget, D. Zagier, 2020). *For any nonhyperelliptic component of any stratum $\mathcal{H}(m_1, \dots, m_n)$ of Abelian differentials one has* $c_{i,j}^{hom} = (m_i + 1)(m_j + 1)$.

The formula has the following (somehow misleading) heuristic interpretation: the cone angle $2\pi(m_i + 1)$ at the conical point P_i is $(m_i + 1)$ times larger than at a regular point. So there are $(m_i + 1)$ times more saddle connections getting out of P_i than from a regular point. Multiplying, $(m_i + 1)$ by $(m_j + 1)$ we get the answer.

There are yet no analogous formulae valid for quadratic differentials!

Area Siegel—Veech constant

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills a *maximal cylinder* having conical points on each of the boundary components. We have seen that sometimes we might get a *configuration* $\mathcal{C}$ of several cylinders, with homologous waste curves (sharing the same length and direction).

Denote by $N_{area}(S,L)$ the sum of areas of all cylinders spanned by geodesics of length at most L on a translation surface S of area 1.

Theorem [W. Veech; Ya. Vorobets] *For every* $\mathrm{SL}(2,\mathbb{R})$*-invariant finite ergodic measure the following ratio is constant (i.e. does not depend on the value of a positive parameter* L*):*

$$\frac{1}{\pi L^2}\int N_{area}(S,L)\,d\nu_1 = c_{area}(d\nu_1)$$

The constant c_{area} is called the *area Siegel–Veech constant*.

Area Siegel—Veech constant

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills a *maximal cylinder* having conical points on each of the boundary components. We have seen that sometimes we might get a *configuration* $\mathcal{C}$ of several cylinders, with homologous waste curves (sharing the same length and direction).

Denote by $N_{area}(S, L)$ the sum of areas of all cylinders spanned by geodesics of length at most L on a translation surface S of area 1.

Theorem [W. Veech; Ya. Vorobets] *For every* $\mathrm{SL}(2, \mathbb{R})$*-invariant finite ergodic measure the following ratio is constant (i.e. does not depend on the value of a positive parameter* L*):*

$$\frac{1}{\pi L^2} \int N_{area}(S, L)\, d\nu_1 = c_{area}(d\nu_1)$$

The constant c_{area} is called the *area Siegel–Veech constant*.

Large genus asymptotics

The result below (in a slightly weaker form) was conjectured by A. Eskin and A. Zorich about 2003. The conjecture was proved in 2020 by D. Chen, M. Möller, A. Sauvaget, D. Zagier, and independently in 2019 by A. Aggarwal (in a slightly weaker form by different methods).

Theorem. *For any nonhyperelliptic component of any stratum $\mathcal{H}(m_1,\dots,m_n)$ of Abelian differentials one has*

$$c_{area} = \frac{1}{2} - \frac{1}{2\sum_{i=1}^{n}(m_i+1)} + O(1/g^2) \text{ as } g \to +\infty\,,$$

where the implied constants are independent of the partition $m_1+\dots+m_n = 2g-2$ and of g.

Combining the theorem with further results of D. Chen, M. Möller, A. Sauvaget, D. Zagier, and of A. Aggarwal on large genus asymptotics of Masur–Veech volumes (confirming another conjecture of A. Eskin and A. Zorich) one gets

Theorem (A. Zorich, 2020). *The relative contribution of all configurations of saddle connections of multiplicity 2 and more to c_{area} and to $c^{hom}_{i,j}$ tends to 0 uniformly in partitions $m_1+\dots+m_n=2g-2$ and in g as $g\to+\infty$.*