Lecture notes for Tsingua Workshop

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Lecture 2: Hyperbolic dynamics

From here on, we will specialize to the case when f is a smooth diffeomorphism. To avoid doing any real geometry, we will always assume f acts on either the flat torus $M = \mathbb{T}^d$ or as a mapping $f : U \to \mathbb{R}^d$ where $U \subset \mathbb{R}^d$ is an open and bounded set. In either case, all tangent spaces can be identified with the same copy of $\mathbb{R}^d$ using the flat connection. Below and throughout, we write $D_x f : T_x M \to T_{fx} M$ for the total derivative of f.

Our goal for today is to describe an infinitesimal mechanism, *hyperbolicity*, which causes dynamical systems to 'randomize' and produce chaotic and seemingly random behavior.

Definition 1. Let f be a C^1 diffeomorphism. Let Λ be a compact, f-invariant set (i.e., $f(\Lambda) = \Lambda$). We say that $f|_\Lambda$ is *uniformly hyperbolic* if there exist constants $C \geq 1, \lambda \in (0, 1)$, and for each $x \in \Lambda$ there exists a splitting of tangent space $T_x M = E^u_x \oplus E^s_x$ into complementary subspaces $E^{u/s}_x$, such that for all $n \geq 0$,

$$\|D_x f^n|_{E^s_x}\|_{T_x M \to T_{f^n x} M} \leq C\lambda^n \,,$$
$$\|D_x f^{-n}|_{E^u_x}\|_{T_x M \to T_{f^{-n} x} M} \leq C\lambda^n \,.$$

If $\Lambda = M$ (that is, the entire domain is uniformly hyperoblic) then f is called an *Anosov diffeomorphism.*

Notice that the second line implies that $D_x f^n|_{E^u_x}$ strongly expands the vectors in E^u_x.

Example 2. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 diffeomorphism for which 0 is a saddle equilibrium (i.e., the spectrum of $D_0 f$ is disjoint from the unit circle and contains eigenvalues with absolute value both > 1 and < 1). It is easy to check that $f|_\Lambda$ with $\Lambda = \{0\}$ is uniformly hyperbolic, where E^u is the direct sum of the unstable ($|\lambda| > 0$) eigenspaces and E^s is the sum of the stable ($|\lambda| < 1$) eigenspaces.

Exercise 3. Show that if $f|_\Lambda$ is uniformly hyperbolic, then $D_x f(E^{u/s}_x) = E^{u/s}_{fx}$ holds for all $x \in \Lambda$.

Exercise 4. Let $f : \mathbb{T}^d \to \mathbb{T}^d$. Show that if $f|_\Lambda$ is uniformly hyperbolic with respect to the flat metric on $\mathbb{T}^d$, then it is uniformly hyperbolic with respect to any C^1 Riemannian metric on $\mathbb{T}^d$.

We call E^u_x, E^s_x, respectively, the unstable and stable subspaces for f at x. The presence of hyperbolicity indicates that at the infinitesimal level near each orbit initiated in Λ, there is some stretching and contracting in different directions that accumulates over time. Notice that points in Λ need not be equilibria themselves, so this stretching / contracting is occurring in the *moving frame* along each orbit $\{f^n x\}_{n\in\mathbb{Z}}, x \in \Lambda$.

Exercise 5. Show that the CAT map as defined in Example **??** is Anosov, i.e., uniformly hyperbolic on $\Lambda = \mathbb{T}^2$. What are E^u_x, E^s_x?

0.1 Continuity properties of $x \mapsto E_x^{u/s}$

Below, d_H denotes the Hausdorff distance on subspaces of $\mathbb{R}^d$, defined by

$$d_H(E, E') = \|P_E - P_{E'}\|\,,$$

where $P_E, P_{E'}$ denote the orthogonal projections onto E, E', respectively.

Lemma 6. *The mapping $x \mapsto E_x^{u/s}$ various Holder continuously in d_H as x varies in Λ.*

Proof. Let us prove continuity of $x \mapsto E_x^s$; that for E_x^u is a time-reversal of the argument we given. Holder continuity is more involved– see e.g. Brin and Stuck.

Let $x_n \to x$ be a convergent sequence, and assume for the sake of contradiction that there is a sequence of unit vectors $v_n \in E_{x_n}^s$ for which $\limsup_{n\to\infty} \operatorname{dist}(v_n, E_x^s) > 0$. Refining the sequence and using compactness, assume that the vectors v_n converge to some limiting unit vector $v \notin E_x^s$.

With v defined, let N be sufficiently large so that $\|D_x f^m v\| \geq 2$ for all $m \geq N$ (this step is justified in Exercise 8 below). Now,

$$\|D_x f^m v\| \leq \|D_x f^m - D_{x_n} f^m\| + \|D_{x_n} f^m\| \cdot \|v - v_n\| + \|D_{x_n} f^m(v_n)\|\,.$$

Fix $m \geq N$ large enough so that the third term is $\leq C\lambda^m \leq \frac{1}{2}$. With m fixed, take n large enough so that the first and second terms are each $\leq \frac{1}{4}$. Since $m \geq N$, the LHS is ≥ 2 but the RHS is ≤ 1. This is a contradiction. □

Remark 7. In fact, it can be shown that $x \mapsto E_x^{s/u}$ varies Holder continuously in x. This is the most regularity one can hope for in general uniformly hyperbolic systems, even when f is C^∞ or analytic: see Brin & Stuck for for a proof of Holder continuity and see Hasselblatt & Katok for more information on optimality of Holder continuity. This fact is a source of significant technical difficulty in hyperbolic dynamics.

Exercise 8. Let $f|_\Lambda$ be uniformly hyperbolic and $x \in \Lambda$. Let $v \in \mathbb{R}^d \setminus E_x^s$. Show that there is a constant $c(v) > 0$ for which $\|D_x f^n v\| \geq c(v)\lambda^{-n}$ for all n. What is $c(v)$ in terms of v, E_x^s, and the other parameters in the system? What happens to $c(v)$ as v gets closer to E_x^s?

Corollary 9. *Assume $f|_\Lambda$ is topologically transitive, i.e., for any open $U, V \subset \Lambda$ (in the subspace topology), we have that there exists an $n \geq 0$ for which $f^n(U) \cap V \neq \emptyset$. Then, $\dim E_x^{u/s}$ is constant in x.*

Exercise 10. Prove Corollary 9. *Hint: A useful step is to show that if $E, E' \subset \mathbb{R}^d$ are subspaces with $\dim E < \dim E'$, then $d_H(E, E') = 1$.*

0.2 Equivalent formulation: cones conditions

It is often easier to work with the following *cones formulation* of hyperbolicity. Below, given a subspace $E \subset T_x M$ and $\alpha \in (0, \pi/2)$, define

$$\mathcal{C}(E, \alpha) = \{0\} \cup \{v \in T_x M \setminus \{0\} : \angle(v, E) \leq \alpha\}\,.$$

Here, $\angle(v, E)$ is the minimal angle between v and E, and satisfies the formula

$$\sin \angle(v, E) = \frac{\|(I - P_E)v\|}{\|v\|}\,,$$

where P_E is the orthogonal projection onto E.

Proposition 11. *We have that $f|_\Lambda$ is uniformly hyperbolic if and only if the following holds. There are (1) constants $\alpha \in (0, \pi/2), \tilde{C} \geq 1, \tilde{\lambda} \in (0,1)$ and (2) continuously-varying assignments $x \mapsto \tilde{E}^s_x, \tilde{E}^u_x$ of subspaces of $T_x M$ (not necessarily Df-invariant) such that*

- $D_x f(\mathcal{C}(\tilde{E}^u_x, \alpha)) \subset \mathcal{C}(\tilde{E}^u_{fx}, \alpha)$,
- $D_x f^{-1}(\mathcal{C}(\tilde{E}^s_x, \alpha)) \subset \mathcal{C}(\tilde{E}^s_{f^{-1}x}, \alpha)$, *and*
- *we have*

$$\|D_x f^n v\| \geq \tilde{C}^{-1} \tilde{\lambda}^{-n} \|v\|\,, \quad v \in \mathcal{C}(\tilde{E}^u_x, \alpha)\,,$$
$$\|D_x f^{-n} v\| \geq \tilde{C}^{-1} \tilde{\lambda}^{-n} \|v\|\,, \quad v \in \mathcal{C}(\tilde{E}^s_x, \alpha)\,.$$

Exercise 12. Prove that the Smale Solenoid is uniformly hyperbolic on its attractor $\mathcal{A}$ by checking that the cones condition in Proposition 11 holds. *Hint: take $\tilde{E}^u_x$ to be parallel to the S^1 factor, and take $\tilde{E}^s_x$ to be parallel to the D factor.*

0.3 Stable and unstable manifolds theory

As you probably learned in an ODE class, hyperbolic equilibria admit stable/unstable manifolds which are the local nonlinear representatives of the subspaces E^u, E^s along which stretching/ contracting occurs. Since a given orbit in a hyperbolic set can be thought of as as saddle in a moving frame, it should come as no surprise then that we can obtain 'moving frame' stable/unstable manifolds along such orbits.

Theorem 13. *Assume $f|_\Lambda$ is hyperbolic. Then, for all sufficiently small $\epsilon > 0$ and all $x \in \Lambda$, the set*

$$W^s_{x,\epsilon} := \{y \in B_\epsilon(x) : d(f^n x, f^n y) \leq \epsilon \text{ for all } n\}$$

is a $\dim E^s_x$-dimensional embedded disk passing through x and tangent to E^s_x at x. Moreover, for any $p, q \in W^s_{x,\epsilon}$, we have $\operatorname{dist}(f^n p, f^n q) \leq \hat{C}\lambda^n$, where $\hat{C} \geq 1$ is a constant.

Analogously, we have

$$W^u_{x,\epsilon} := \{y \in B_\epsilon(x) : d(f^{-n} x, f^{-n} y) \leq \epsilon \text{ for all } n\}$$

is a $\dim E^u_x$-dimensional embedded disk passing through x and tangent to E^u_x at x, as well as the analogous contraction estimate $\operatorname{dist}(f^{-n} p, f^{-n} q) \leq \hat{C}\lambda^n$ for $p, q \in W^u_{x,\epsilon}$.

Lastly, if f is C^r-smooth, then $W^{u/s}_{x,\epsilon}$ are each C^r smooth.

Definition 14. The global stable/unstable manifolds at $x \in \Lambda$ are defined by

$$W^s_x = \bigcup_{n \geq 0} f^{-n}(W^s_{f^n x, \epsilon})$$
$$W^u_x = \bigcup_{n \geq 0} f^{n}(W^u_{f^{-n} x, \epsilon})$$

Remark 15. After seeing the proof of (part of) Theorem 13, you will be able to check that the above definitions do not depend on the precise value of ϵ. See Exercise 27 below.

Exercise 16. Show that

$$W^s_x = \{y \in M : d(f^n x, f^n y) \to 0 \text{ as } n \to \infty\}\,.$$

Conclude that if $W^s_x \cap W^s_y \neq \emptyset$ for some $x, y \in \Lambda$, then $W^s_x = W^s_y$.

That is, distinct (un)stable leaves may not cross each other. On the other hand, stable and unstable leaves may cross each other (obviously $W^s_x \cap W^u_x \supset \{x\}$). See Remark 18 for more information.

Since $x \mapsto E^{u/s}_x$ various continuously, it stands to reason that the nonlinear analogues $W^{u/s}_{x,\epsilon}$ should vary continuously as well. Below, given two compact sets A, B, we define the symmetric Hausdorff distance d_H between them by

$$d_H(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\}\,.$$

Corollary 17. *Let $\epsilon > 0$ be sufficiently small. Then, $x \mapsto W^{u/s}_{x,\epsilon}$ varies continuously in the Hausdorff metric d_H.*

The proof is left as an exercise. It proceeds along similar lines to the proof of Lemma 6.

Remark 18. Local stable manifolds are nice, embedded disks of the correct dimension. The corresponding global stable manifolds are not nice, and generally speaking are immersed submanifolds (e.g., they may accumulate on themselves). This phenomenon occurs in under very natural conditions (e.g. homoclinic transversal intersections) and is a mechanism for deterministic chaos (discovered first by Poincaré). See Chapter V of Brin and Stuck for more details.

0.4 Adapted charts

The goal for the rest of this lecture is to sketch a proof of the (un)stable manifold theorem using adapted charts and so-called *graph transforms*, which will be defined below. This is sometimes called the 'Hadamard' method, in contrast with the 'Perron' method which is more analytical in nature.

To describe the local nonlinear behavior along hyperbolic trajectories, it makes sense to pass into a moving frame and consider a chart 'adapted' to the nature of the hyperoblicity, i.e., for which hyperoblicity occurs in a single timestep. These constructions are very useful in the proof of the (un)stable manifold theorem (Theorem 13) which we will sketch. Notation: let $\langle \cdot, \cdot \rangle_x$ denote the Riemannian metric on T_xM at $x \in M$.

Let $\delta_0 > 0$ be fixed, $\delta_0 \ll |\log \lambda|$. Given $x \in \Lambda$, $u, u_1, u_2 \in E^u_x, v, v_1, v_2 \in E^s_x$, define

$$\begin{aligned}
\langle\langle u_1, u_2 \rangle\rangle_x &= \sum_{n=0}^{\infty} \frac{\langle D_x f^n u_1, D_x f^n u_2 \rangle_{f^n x}}{(e^{\delta_0}\lambda)^{2n}} \\
\langle\langle v_1, v_2 \rangle\rangle_x &= \sum_{n=0}^{\infty} \frac{\langle D_x f^n v_1, D_x f^n v_2 \rangle_{f^n x}}{(e^{\delta_0}\lambda)^{2n}} \\
\langle\langle u, v \rangle\rangle_x &= 0
\end{aligned}$$

Exercise 19. Define $\|\cdot\|'_x$ to be the norm on T_xM corresponding to $\langle\langle , \rangle\rangle_x$. Using continuity of $x \mapsto E^{u/s}_x$, show that $\|\cdot\|'_x$ is equivalent, up to a constant independent of x, to the norm $\|\cdot\|_x$ induced by the Riemannian metric $\langle , \rangle_x$ on T_xM.

Exercise 20. Assume $M = \mathbb{T}^d$ (for simplicity). Show that the inner products $\langle\langle , \rangle\rangle_x$ vary continuously in x, in the sense that

$$x \mapsto \langle\langle v_1, v_2 \rangle\rangle_x, \qquad v_1, v_2 \in \mathbb{R}^d \text{ fixed and arbitrary}$$

varies continuously in x.

Define $\lambda_0 = e^{\delta_0}\lambda \in (0,1)$. As one easily checks, for $u \in E^u_x, v \in E^s_x$, we have

$$\|D_x f u\|'_{fx} \geq \lambda_0^{-1} \|u\|'_x \,, \quad \|D_x f v\|'_{fx} \leq \lambda_0 \|v\|'_x \,.$$

We now turn to defining *connecting maps* and our desired *adapted charts system*. The purpose of doing this is to study the dynamics *in the moving frame* along a given orbit $\{f^n x\}_{n\in\mathbb{Z}}$ by mapping that moving frame to the fixed model space $\mathbb{R}^u \oplus \mathbb{R}^s$, where $\mathbb{R}^u = \mathbb{R}^{\dim E^u}$ and $\mathbb{R}^s = \mathbb{R}^{\dim E^s}$, equipped with the standard inner product $(\cdot,\cdot)$ with norm $|\cdot|$.

For this, we fix for each $x \in \Lambda$ linear maps $L_x : \mathbb{R}^u \oplus \mathbb{R}^s \to T_x M$ which send $(\cdot,\cdot)$ to $\langle\!\langle \cdot,\cdot \rangle\!\rangle_x$. We now define the *connecting mapss*

$$f_x = (\Phi_{fx})^{-1} \circ f \circ \Phi_x \,,$$

and iterates

$$f^n_x = \begin{cases} f_{f^{n-1}x} \circ \cdots \circ f_x & n \geq 0 \\ (f_{f^{-1}x} \circ \cdots \circ f_{f^n x})^{-1} & n < 0 \end{cases}$$

Here, we have set

$$\Phi_x = \exp_x \circ L_x \,,$$

where $\exp_x : T_x M \to M$ is the (geodesic) exponential map (on $M = \mathbb{R}^d$ or $\mathbb{T}^d$, this is given by $\exp_x(v) = x + v$; if $M = \mathbb{T}^d$ this equation is meant to be taken mod 1 in each coordinate).

We will consider these connecting maps as acting $f_x : B(r) \to \mathbb{R}^u \oplus \mathbb{R}^s$, where

$$B(r) = B^u(r) + B^s(r)$$

and $B^{u/s}(r)$ denotes the r-ball in $\mathbb{R}^{u/s}$, respectively. Here the parameter $r > 0$ is chosen sufficiently small so that the (nonlinear) f_x is sufficiently close to its linearization $D_0 f_x = L^{-1}_{fx} \circ D_x f \circ L_x$. The following summarizes the properties of this construction.

Lemma 21. *Let r be sufficiently small. Then, the chart map $\Phi_x = \exp_x \circ L_x : B(r) \to M$ satisfies*

$$C^{-1}\operatorname{dist}(\Phi_x(v), \Phi_x(w)) \leq |v - w| \leq C \operatorname{dist}(\Phi_x(v), \Phi_x(w))$$

for $v, w \in B(r)$, where $C > 0$ is a constant.

Moreover, let $f_x : B(r) \to \mathbb{R}^u \oplus \mathbb{R}^s$ denote the connecting map as above. Write $f_x = D_0 f_x + F_x$. Then, the following holds.

- *We have $D_0 f_x \mathbb{R}^{u/s} = \mathbb{R}^{u/s}$, and for $u \in \mathbb{R}^u, v \in \mathbb{R}^s$ we have that $|D_0 f_x u| \geq \lambda_0^{-1}|u|$ and $|D_0 f_x v| \leq \lambda_0 |v|$.*
- *Let $\delta > 0$ be arbitrary. By taking r small enough, we can ensure the nonlinearity F_x satisfies:*
  - $F_x(0) = 0$
  - $\mathrm{Lip}(F_x) \leq \delta$, *and*
  - $\mathrm{Lip}(DF_x) \leq C$ *where $C > 0$ is a constant.*

Exercise 22. Assume $M = \mathbb{T}^d$ for simplicity, and regard each L_x as above as a mapping $\mathbb{R}^u \oplus \mathbb{R}^s \to \mathbb{R}^d$. Show that $x \mapsto L_x$ can be made to vary continuously in x (c.f. Exercise 20). Conclude that the chart maps Φ_x can be constructed so as to vary continuously in x.

0.5 Graph transforms; the 'Hadamard' proof of the (un)stable manifold theorem

In this section we will summarize a proof of part of the (un)stable manifold theorem:

Proposition 23. *Let $\epsilon > 0$ be sufficiently small, $x \in \Lambda$. Then, the set*

$$W^u_{x,\epsilon} = \{y \in B_\epsilon(x) : d(f^{-n}x, f^{-n}y) \to 0 \text{ as } n \to \infty\}$$

is contained in a set of the form

$$\Phi_x \,\mathrm{graph}\, g_x$$

where $g_x : B^u(r) \to \mathbb{R}^s, g_x(0) = 0$ is a Lipshcitz-continuous mapping with $\mathrm{Lip}(g_x) \le 1/10$ (in the $|\cdot|$ norm on mappings $B^u(r) \to \mathbb{R}^s$).

A corresponding statement for $W^s_{x,\epsilon}$ can be proved by a time-reversal of the following arguments. Details are left as an exercise.

Proving that $W^u_{x,\epsilon}$ is in fact as smooth as the mapping f requires more work and falls outside the scope of these notes. Unfortunately this does not follow a naive argument, e.g., by showing that the graph transform is a contraction in a norm compatible with the C^r topology on graph mappings. Alternative arguments are needed using, e.g., implicit function theorems on appropriate spaces of mappings (following the Perron approach) or the Fiber Contraction Principle. See, e.g., the book of Pugh and Shub for details.

Proof of Proposition 23

We begin with the following highly useful *geometric cones lemma* describing how cones invariance for the linearization $D_0 f_x$ translates to invariance of the cone for the (nonlinear) connecting map f_x itself.

Some terminology: Let $z_i = (u_i, v_i) \in B(r), i = 1, 2$. Let us say that z_1, z_2 are *unstably related* if $|v_1 - v_2| \le \frac{1}{10}|u_1 - u_2|$, equivalently, if $z_1 - z_2$ belongs to the cone $\mathcal{C}^u = \{(u, v) : |v| \le \frac{1}{10}|u|\}$.

Lemma 24. *Let $r > 0$ be sufficiently small. Let $z_i \in B(r), i = 1, 2$ be unstably related. Then,*

- *$f_x(z_1), f_x(z_2)$ are also unstably related; and*
- *$|f_x(z_1) - f_x(z_2)| \ge (\lambda_0 - \delta)|z_1 - z_2|$.*

The graph transform

For $g : B^u(r) \to \mathbb{R}^s$, we define $\mathrm{graph}\, g = \{(u, g(u)) : u \in B^u(r)\}$. The *graph transform* of g is defined to be the mapping $\Gamma_x g : B^u(r) \to \mathbb{R}^s$ with the property that

$$f_x(\mathrm{graph}\, g) \cap B(r) = \mathrm{graph}\, \Gamma_x g$$

Define

$$\mathcal{W} = \{g : B^u(r) \to B(r) \text{ Lipschitz continuous } : \mathrm{graph}\, g \subset B(r) \text{ and } \mathrm{Lip}(g) \le 1/10\}\,.$$

Lemma 25.

(a) If $g \in \mathcal{W}$, then the graph transform $\Gamma_x g$ exists and belongs to $\mathcal{W}$.

(b) *The mapping* $\Gamma_x : \mathcal{W} \to \mathcal{W}$ *is a contraction mapping in the uniform norm on* $C(B^u(r), \mathbb{R}^s)$ *with a uniform (x-independent) contraction constant.*

Exercise 26. In this exercise, we will sketch a proof of Lemma 25. Below, $\mathrm{Pr}_{\mathbb{R}^{u/s}}$ denotes the orthogonal projection $\mathbb{R}^u \oplus \mathbb{R}^s \to \mathbb{R}^{u/s}$, respectively.

(1) Given $g \in \mathcal{W}$, define $\phi^u : B^u(r) \to \mathbb{R}^u$ by $\phi^u(u) = \mathrm{Pr}_{\mathbb{R}^u}(f_x(u, g(u)))$. Clearly ϕ^u is continuous. Show that $(\phi^u)^{-1}$ can be uniquely defined $B^u(r)$ by checking (a) and (b) below:

   (a) For all $u' \in \phi^u(\partial B^u(r))$, we have $|u| > r$.

   (b) For all $u_1, u_2 \in B^u(r)$, we have $|\phi^u(u_1) - \phi^u(u_2)| \geq (\lambda_0 - \delta)|u_1 - u_2|$.

   Conclude from (a) that the image of $B^u(r)$ under ϕ^u contains $B^u(r)$, and from (b) that $\phi^u : B^u(r) \to \mathbb{R}^u$ is injective. Now apply the following criterion: *Let* $\phi : X \to Y$ *be a continuous and injective mapping of topological spaces* X, Y*, where* X *and* Y *are metric spaces and* X *is compact. Then,* ϕ *is a homeomorphism from* X *onto its image* $\phi(X)$. (If this is new to you, try to prove it yourself.)

(2) The graph transform $\Gamma_x g : B^u(r) \to \mathbb{R}^s$ is now defined by

$$\Gamma_x g(u') = \mathrm{Pr}_{R^s}(f_x(u, g(u)))$$

   where $u' = \phi^u(u) \in B^u(r)$. Check that $f_x(\mathrm{graph}\, g) \supset \mathrm{graph}\, \Gamma_x g$.

(3) Carry out the contraction estimate on $|\Gamma_x g_1(u') - \Gamma_x g_2(u')|$ for $g_1, g_2 \in \mathcal{W}$ for $u' \in B^u(r)$.

Proposition 23 now follows by defining $g_x, x \in \Lambda$ so that

$$\{g_x\} = \bigcap_{n \geq 0} \Gamma_{f^{-1}x} \circ \cdots \circ \Gamma_{f^{-n}x}(\mathcal{W}),$$

noting that the limit exists and is unique since

$$\mathrm{diam}\left(\Gamma_{f^{-1}x} \circ \cdots \circ \Gamma_{f^{-n}x}(\mathcal{W})\right) \to 0 \quad \text{as } n \to \infty$$

by Lemma 25(b).

Slight re-definition of local unstable manifolds

Moving forward, it is technically extremely convenient to work with the following slightly re-defined local (un)stable manifolds:

$$\hat{W}^u_{x,r} = \Phi_x(\mathrm{graph}\, g_x)$$

where $r > 0$ is as above.

Exercise 27.

(a) Show that with this convention, we have that

$$f(\hat{W}^u_{x,r}) \supset \hat{W}^u_{fx,r}, \quad f(\hat{W}^s_{x,r}) \subset \hat{W}^s_{fx,r}.$$

(b) Show that the global (un)stable manifolds $W^{u/s}_x$ are given by

$$W^u_x = \cup_{n \geq 0} f^n(\hat{W}^u_{f^{-n}x,r}), \quad W^s_x = \cup_{n \geq 0} f^{-n}(\hat{W}^s_{f^n x,r})$$

   and that these are both increasing unions. In particular, show that these are independent of the parameter $r > 0$.