PART II. Deligne-Lusztig theory for finite reductive groups

Schedule.

- Next lecture on Wednesday, Dec. 14
- NO LECTURE on Friday, Dec. 16
- Two lectures:
  - → Wednesday, Dec. 21
  - → Friday, Dec. 23

Chapter 7. Harish-Chandra theory.

- G connected reductive algebraic group
- $F : G \longrightarrow G$ Frobenius endomorphism / $\mathbb{F}_q$
- $G^F = \{ g \in G \mid F(g) = g \}$ (finite reductive group)
- B_0 F-stable Borel subgroup
  $\cup$
  T_0 F-stable maximal torus
- $W = N_G(T_0)/T_0$: Weyl group
- $S = \{ s \in W \mid \dim B_0 s B_0 - \dim B_0 = 1 \}$

Example. $G = GL_n(\mathbb{F})$

7.A. Harish-Chandra induction and restriction

- $P = L \ltimes V$: F-stable parabolic subgroup

  L : F-stable Levi complement

  $V = R_u(P)$.

(ind) $\mathcal{R}^G_{L\subset P}(M) = K[G^F/V^F] \otimes_{KL^F} M$

$\simeq \mathrm{Ind}_{P^F}^{G^F} \tilde{M}$ by V^F acts trivially

(res) ${}^*\mathcal{R}^G_{L\subset P}(N) = K[V^F\backslash G^F] \otimes_{KG^F} N$

$\simeq N^{V^F}$.

$\rightsquigarrow$

$$\begin{tikzcd} \mathrm{Class}(L^F) \arrow[r, bend left, "R^G_{L\subset P}"] & \mathrm{Class}(G^F) \arrow[l, bend left, "{}^*R^G_{L\subset P}"] \end{tikzcd}$$

Transitivity 7.1.

$$\begin{tikzcd} KL'^F\text{-mod} \arrow[r, bend left, "\mathcal{R}^L_{L'\subset P'}"] \arrow[rr, bend left=60, "\mathcal{R}^G_{L'\subset P'V}"] & KL^F\text{-mod} \arrow[l, bend left, "{}^*\mathcal{R}^L_{L'\subset P'}"] \arrow[r, bend left, "\mathcal{R}^G_{L\subset P}"] & KG^F\text{-mod} \arrow[l, bend left, "{}^*\mathcal{R}^G_{L\subset P}"] \arrow[ll, bend left=60, "{}^*\mathcal{R}^G_{L'\subset P'V}"] \end{tikzcd}$$

7.B. Cuspidality.

Definition 7.2. An irreducible E of G^F is called cuspidal if ${}^*\mathcal{R}^G_{L\subset P}\, E = 0$ for all F-stable parabolic subgroup $P \subsetneq G$.

By adjunction:

Proposition 7.3. E is cuspidal if and only if E does not occur in any $\mathcal{R}^G_{L\subset P}(KL^F)$ where P runs over the set of F-stable parabolic subgroups $\neq G$.

$$\mathrm{Irr}_{cus}(G^F) = \{\text{cusp. irr. rep. of } G^F\}$$
$$\cap$$
$$\mathrm{Irr}(G^F)$$

$\mathrm{CusPairs}(G) = \{ (L, E) \mid L$ is an F-stable Levi complement of an F-stable parab. subgroup and $E \in \mathrm{Irr}_{cus}(L^F) \}$

If $(L, E) \in \mathrm{CusPairs}(G)$, we set

$$HC(G^F, L, E) = \{\text{irr. comp. of } R^G_{L\subset P} E\}$$
$$\cap$$
$$\mathrm{Irr}\, G^F$$

(Harish-Chandra series)

Theorem 7.4 (Harish-Chandra theory)

(a) $$\mathrm{Irr}\, G^F = \dot{\bigcup_{(L,E) \in \mathrm{CusPairs}(G)/\sim_{G^F}}} HC(G^F, L, E)$$

(b) $$HC(G^F, L, E) \xleftrightarrow{\sim} \mathrm{Irr}\, W_{G^F}(L, E)$$

where $W_{G^F}(L, E) = N_{G^F}(L, E)/L^F$.

Remarks 7.5. (1) $R^G_{G\subset G} = \mathrm{Id}$

So if $E \in \mathrm{Irr}_{cus}(G^F)$, then

$$HC(G^F, G, E) = \{E\}$$

"discrete series"

(2) If G is a torus, then $G = B_0 = T_0$ so $\mathrm{Irr}_{cus}(G^F) = \mathrm{Irr}(G^F)$.

(3) $G = SL_2(\bar{\mathbb{F}})$; $G^F = SL_2(\mathbb{F}_q)$

$$\mathrm{Irr}\, G^F = \underbrace{1_{G^F},\ \mathrm{St}}_{HC(G^F, T_0, 1_{T_0^F})} \qquad \underbrace{R(\alpha_0)^+ \quad R(\alpha_0)^-}_{HC(G^F, T_0, \alpha_0)}$$

$$R(\alpha) \longrightarrow HC(G^F, T_0, \alpha)$$

$$\underbrace{R'(\theta_0)^+,\ R'(\theta_0)^-,\ R'(\theta)}_{\text{cuspidal}}$$

$$W_{G^F}(T_0, 1_{T_0^F}) = W_{G^F}(T_0, \alpha_0) = \langle \dot{s} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

$$W_{G^F}(T_0, \alpha) = 1 \quad \text{if } \alpha \neq \alpha^{-1}.$$

7.C. The Mackey formula

Let $P = L \ltimes V$ and $P' = L' \ltimes V'$ be F-stable Levi decompositions of F-stable parabolic subgroups ($V = R_u(P)$ and $V' = R_u(P')$)

We set $S(L,L') = \{ g \in G \mid L \cap {}^gL' \text{ contains a maximal torus of } G \}$

$$L^F \backslash S(L,L')^F / L'^F \xleftrightarrow{\sim} P^F \backslash G^F / P'^F$$

$g L'^F = g L'^F g^{-1} g$

Theorem 7.6 (Mackey formula)

$$ {}^*\mathcal{R}^G_{L\subset P} \circ \mathcal{R}^G_{L'\subset P'}(M) \simeq \bigoplus_{g \in L^F \backslash S(L,L')^F/L'^F} \mathcal{R}^L_{L\cap {}^gL' \subset L \cap {}^gP'} \circ {}^*\mathcal{R}^{{}^gL'}_{L\cap {}^gL' \subset P\cap {}^gL'}({}^gM)$$

Sketch of the proof.

$$K[V^F\backslash G^F] \otimes_{KG^F} K[G^F/V'^F] \xrightarrow{\sim} K[V^F\backslash G^F/V'^F] = \bigoplus_{g\in L^F\backslash S(L,L')^F/L'^F} K[V^F\backslash P^F g P'^F/V'^F]$$

$$K[L^F/(L^F\cap {}^gV'^F)] \otimes_{K(L\cap {}^gL')^F} K[(V^F\cap {}^gL'^F)\backslash gL'^F] \xrightarrow{\sim} K[V^F\backslash P^F g P'^F/V'^F]$$

$$\ell(L^F\cap {}^gV'^F) \otimes (V^F\cap {}^gL'^F) g\ell' \longmapsto V^F \ell g \ell' V'^F \quad \blacksquare$$

not trivial (see Digne-Michel)

Corollary 7.7. $\langle R^G_{L\subset P}\lambda, R^G_{L'\subset P'}\lambda'\rangle_{G^F} = \sum_{g\in L^F\backslash S(L,L')^F/L'^F} \langle {}^*R^L_{L\cap {}^gL' \subset L\cap {}^gP'}\lambda, {}^*R^{{}^gL'}_{L\cap {}^gL' \subset P\cap {}^gL'} {}^g\lambda'\rangle_{L^F\cap {}^gL'^F}$

Corollary 7.8. Up to isomorphism, $\mathcal{R}^G_{L\subset P}(M)$ does not depend on P.

Proof. By induction on $\dim G$ and using 7.7, we get

$$\langle R^G_{L\subset P}\lambda - R^G_{L\subset P'}\lambda', R^G_{L\subset P}\lambda - R^G_{L\subset P'}\lambda'\rangle_{G^F} = 0. \ \blacksquare$$

Example. $G = GL_3(\overline{\mathbb{F}})$, $G^F = GL_3(\mathbb{F}_q)$

$$P = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\} \qquad P' = \left\{\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}\right\}$$

$$L = \left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}$$

- $R^G_{L\subset P}(K_{L^F}) = \mathrm{Ind}_{P^F}^{G^F} K_{P^F} = K[G^F/P^F]$
- $R^G_{L\subset P'}(K_{L^F}) = K[G^F/P'^F]$ $\cong$ (coro. 7.8)

But P^F and P'^F are not conjugate in G^F (exercise).

So the G^F-sets G^F/P^F and G^F/P'^F are not isomorphic.

But their associated permutation modules are isomorphic!! ■

7.D. Sketch of the proof of Theorem 7.4.

Theorem 7.4 (Harish-Chandra theory)

(a) $\mathrm{Irr}\, G^F = \dot{\bigcup}_{(L,E)\in \mathrm{CusPairs}(G)/\sim_{G^F}} HC(G^F, L, E)$

(b) $HC(G^F, L, E) \overset{\sim}{\longleftrightarrow} \mathrm{Irr}\, W_{G^F}(L,E)$

where $W_{G^F}(L,E) = N_{G^F}(L,E)/L^F$.

(a) Let (L,E) and $(L',E') \in \mathrm{CusPairs}(G)$ be such that $HC(G^F, L, E) \cap HC(G^F, L', E') \neq \emptyset$.

Then
$$\langle R^G_{L\subset P}\, \chi_E,\ R^G_{L'\subset P'}\, \chi_{E'}\rangle_{G^F} \neq 0.$$
By the Mackey formula 7.7, $\exists\, g \in S(L,L')^F$ such that
$$\langle {}^*R^L_{L\cap {}^gL'}(\underbrace{\chi_E}_{\text{cuspidal}}),\ {}^*R^{{}^gL'}_{L\cap {}^gL'}(\underbrace{{}^g\chi_{E'}}_{\text{cuspidal}})\rangle_{L^F\cap {}^gL'^F} \neq 0$$

So $L = L\cap {}^gL' = {}^gL'$ and $\langle \chi_E, {}^g\chi_{E'}\rangle \neq 0$.

So $(L,E) \sim_{G^F} (L',E')$.

Endomorphism algebra. Let Γ be a finite group and let M be a $K\Gamma$-module. Write

$$M \simeq \bigoplus_{V \in \mathrm{Irr}\,\Gamma} V^{\oplus n_V}$$

Then (Schur's lemma)

$$(7.9) \quad \mathrm{End}_{K\Gamma} M \simeq \prod_{\substack{V \in \mathrm{Irr}\,\Gamma \\ n_V \neq 0}} \mathrm{Mat}_{n_V}(K)$$

Corollary 7.10.

$$\{\text{irr. comp. of } M\} \xleftrightarrow{\sim} \mathrm{Irr}\,\mathrm{End}_{K\Gamma}(M)$$
$$V_{\mathcal{E}} \longleftarrow\!\shortmid \mathcal{E}$$

and $$M \simeq \bigoplus_{\mathcal{E} \in \mathrm{Irr}\,\mathrm{End}_{K\Gamma}(M)} V_{\mathcal{E}}^{\oplus \dim \mathcal{E}}$$

(b) First (by 7.7)

$$\langle R_{LCP}^{G}\,\chi_E,\ R_{LCP}^{G}\,\chi_E \rangle = |W_{G^F}(L,E)|$$

$$\dim \mathrm{End}_{KG^F}\left(\mathcal{R}_{LCP}^{G} E\right) = |W_{G^F}(L,E)|$$

More precisely:

Theorem 7.11. If $(L,E) \in \mathrm{CusPairs}(G)$, then

$$\mathrm{End}_{KG^F}\left(\mathcal{R}_{LCP}^{G} E\right) \simeq K W_{G^F}(L,E)$$

Corollary 7.12.

$$\mathrm{HC}(G^F, L, E) \xleftrightarrow{\sim} \mathrm{Irr}\, W_{G^F}(L,E)$$
$$V_{\mathcal{E}} \longleftarrow\!\shortmid \mathcal{E}$$

and

$$\mathcal{R}_{LCP}^{G} E = \bigoplus_{\mathcal{E} \in \mathrm{Irr}\, W_{G^F}(L,E)} V_{\mathcal{E}}^{\oplus \dim \mathcal{E}}$$

Theorem 7.11 (Tits, Howlett-Lehrer, Lusztig, Geck). If (L,E) is a cuspidal pair, then

$$\mathrm{End}_{KG^F}\left(\mathcal{R}^G_{L\subset P} E\right) \simeq K W_{G^F}(L,E)$$

Sketch of the proof. If $w \in N_{G^F}(L,E)$, there exists $\rho_w : E \xrightarrow{\sim} E$ such that

$$\rho_w \cdot \ell \cdot \rho_w^{-1} = {}^w\ell \quad \text{on } E$$

We set
$$T_w : K[G^F/V^F] \otimes_{KL^F} E \longrightarrow K[G^F/V^F] \otimes_{KL^F} E$$
$$gV^F \otimes_{KL^F} e \longmapsto \frac{1}{|V^F|} \sum_{v \in V^F} g v w^{-1} V^F \otimes_{KL^F} \rho_w(e)$$

Check: $T_w \in \mathrm{End}_{KG^F}(\mathcal{R}^G_{L\subset P} E)$

Schur's Lemma $\Rightarrow \exists\, \alpha(w,w') \in K^\times$ s.t. $\rho_w \rho_{w'} = \underbrace{\alpha(w,w')}_{\text{2-cocycle}} \rho_{ww'}$

Fact (Howlett-Lehrer). $\mathrm{End}\, \mathcal{R}^G_{L\subset P} E = \bigoplus_{w \in W_{G^F}(L,E)} K T_w$

is a deformation of the twisted group algebra $(K W_{G^F}(L,E))_\alpha$.

Tits' deformation Theorem. $\simeq (K W_{G^F}(L,E))_\alpha$

$\left(SL_2 : \quad T_s^2 = (q-1)T_s + q \right)$

$\left(\mathcal{R}^G_{T\subset B} K_{T^F}\right)$

Lusztig. If $Z(G)$ is connected, then $\alpha \sim 1$.

Lusztig + Geck. $\alpha \sim 1$ in general.

Example 7.13. $G^F = GL_n(\mathbb{F}_q)$

$$P = B_0 \,,\quad L = T_0 \,,\quad E = K_{T_0^F}.$$

$$W_{G^F}(T_0, K_{T_0^F}) = S_n$$

$$\mathrm{Irr}\, S_n \xleftrightarrow{\sim} \{\text{partitions of } n\}$$

$$\chi_\lambda \longleftarrow\!\shortmid \lambda$$

$$\mathcal{R}^G_{T_0 \subset B_0}(K_{T_0^F}) = K[G^F/B_0^F]$$

$$\simeq \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus \chi_\lambda(1)}$$

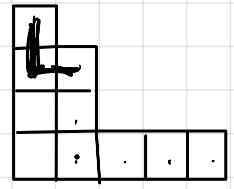
$$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n) \,,\quad \lambda_1 + \cdots + \lambda_n = n$$

Young diagram:

$$Y(\lambda) = \{(i,j) \in \mathbb{Z}_{\geqslant 1} \mid 1 \leqslant i \leqslant \lambda_j\}$$

$$a_\lambda = \sum_{i=1}^{n} (i-1)\lambda_i \in \mathbb{Z}_{\geqslant 0}$$

$h\ell_\lambda(1,3) = 3$



$\lambda = (5,2,2,1) \vdash 10$

$h\ell_\lambda(2,1) = 6$

$$f_\lambda(t) = t^{a_\lambda} \frac{\prod_{i=1}^{n} (t^i - 1)}{\prod_{y \in Y(\lambda)} (t^{h\ell_\lambda(y)} - 1)}$$

Fact. $$\begin{cases} \chi_\lambda(1) = f_\lambda(1) \\ \dim V_\lambda = f_\lambda(q) \end{cases}$$

$$\mathbb{C}[X_1,\dots,X_n] / \langle \mathbb{C}[X_1,\dots,X_n]^{\mathfrak{S}_n}_+ \rangle = A$$

$$f_\lambda(t) = \sum_{i \geqslant 0} \langle \chi_{\underbrace{A_i}_{\text{hom. comp. of degree } i \text{ in } A.}}, \chi_\lambda \rangle t^i$$

(fake degree!)

There are many Harish-Chandra theories:

- Finite reductive groups
- Langlands program
- Generalized Springer correspondence (Lusztig, '80s)
- Character sheaves (Lusztig, '80s)
- Representations of Cherednik algebras "at $t=1$" (Bezrukavnikov-Etingof, Shan)
- Symplectic leaves in generalized Calogero-Moser spaces (Bellamy, Losev, 2010)
- Blocks of finite reductive groups (d-Harish-Chandra theory of Broué-Malle-Michel 1993)