Lecture No 19   May 13, 2022 (Fri)

§20 Invariant measure, reversible measure

Let $\alpha(x), b(x)$ be given and Lipschitz continuous. For $x \in \mathbb{R}^d$, we consider the unique solution $X = (X_t)$ of the SDE

$$dX_t = \alpha(X_t)dB_t + b(X_t)dt \tag{1}$$

starting at x. We denote the distribution of X on the path space $W^d = C([0,\infty), \mathbb{R}^d)$ by P_x.

Let $\mathcal{P}(\mathbb{R}^d)$ be the family of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

[Definition 20.1] Let $\mu \in \mathcal{P}(\mathbb{R}^d)$.
(1) μ is called an invariant (probability) measure (or stationary measure, equilibrium measure) of X iff (= if and only if)

$$\mu(A) = \int_{\mathbb{R}^d} P_x(X_t \in A)\,\mu(dx)\ (=: P_\mu(X_t \in A))$$

hold for ${}^\forall t > 0$ and ${}^\forall A \in \mathcal{B}(\mathbb{R}^d)$.
(2) μ is called a reversible (probability) measure of X iff

$$\int_{A_0} P_x(X_t \in A_1)\,\mu(dx) = \int_{A_1} P_x(X_t \in A_0)\,\mu(dx)$$

hold for ${}^\forall t > 0$ and ${}^\forall A_0, A_1 \in \mathcal{B}(\mathbb{R}^d)$. □

We set
$\mathcal{I} := \{\text{all invariant measures}\}$,
$\mathcal{R} := \{\text{all reversible measures}\}$.

[Remark] (1) (Invariance) RHS in (1) is the distribution of X_t with initial distribution μ at time t. Therefore, (1) means that the distribution of X_t does not change after time passes:

$$P_\mu(X_0 \in A) = P_\mu(X_t \in A).$$

(2) (Reversibility) (2) is rephrased by means of P_μ as

$$P_\mu(X_0 \in A_0, X_t \in A_1) = P_\mu(X_0 \in A_1, X_t \in A_0).$$

In other words, if the initial distribution is μ, by interchanging the time 0 and t, i.e., by reversing the time, this distribution of the process X does not change. In fact, by using the Markov property of X, one can show that

$$P_\mu(X_{t_0} \in A_0, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)$$
$$= P_\mu(X_{t_n} \in A_0, X_{t_n - t_1} \in A_1, \ldots, X_{t_n - t_n} \in A_n)$$

for $0 = t_0 <^\forall t_1 < \cdots <^\forall t_n$, ${}^\forall A_0, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}^d)$; cf. e.g., Liggett, "Interacting Particle Systems" p.93, Springer, 1985.

Accordingly, we see that the distributions of $(X_t)_{t\in[0,T]}$ and $(X_{T-t})_{t\in[0,T]}$ on the path space W^d are the same under P_μ.

□

• If μ is reversible, it is invariant.

⚇ Take $A_1 = \mathbb{R}^d$ in (2). Then, (LHS) $= \mu(A_0)$ and (RHS) $= P_\mu(X_t \in A_0)$. Thus, we obtain (1). □

[Lemma 20.1] Let $\mathcal{L}$ be the generator of the SDE (1), i.e.,

$$\mathcal{L}\varphi(x) = \frac{1}{2} a^{ij}(x) \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x) + b^i(x) \frac{\partial \varphi}{\partial x^i}(x).$$

Then, if $\mu \in \mathcal{P}(\mathbb{R}^d)$ is an invariant measure of the SDE (1), then

$$\int_{\mathbb{R}^d} \mathcal{L}\varphi(x)\mu(dx) = 0$$

holds for ${}^\forall \varphi \in C_b^2(\mathbb{R}^d)$. □

[Proof] As we showed, P_x is a solution of the $\mathcal{L}$-martingale problem, i.e. $\varphi(X_t) - \int_0^t \mathcal{L}\varphi(X_s)ds$ is a martingale. In particular, taking the expectation under P_μ, we have

$$E^{P_\mu}[\varphi(X_t)] - \int_0^t E^{P_\mu}[\mathcal{L}\varphi(X_s)]ds = E^{P_\mu}[\text{martingale}].$$

However, by the invariance of μ, we have

$$E^{P_\mu}[\varphi(X_t)] = \int_{\mathbb{R}^d} \varphi(x)\mu(dx),$$
$$E^{P_\mu}[\mathcal{L}\varphi(X_s)] = \int_{\mathbb{R}^d} \mathcal{L}\varphi(x)\mu(dx).$$

Moreover, (RHS) = constant in t. Therefore, by differentiating both sides in t, we obtain the conclusion. □

We now discuss the converse of Lemma 20.1 for invariant measure and corresponding result for reversible measure; see Theorems 20.2 and 20.3 below.

- We took $C_b^2(\mathbb{R}^d)$ as a class of test functions for $\mathcal{L}$-martingale problem. This can be generalized as follows.

[Definition 20.2] In general, let $\Phi \subset C^2(\mathbb{R}^d)$ be a class of functions on $\mathbb{R}^d$. We call $(\mathcal{L}, \Phi)$-martingale problem is well-posed, if $(\mathcal{L}, \Phi)$-martingale problem defined on $(W^d, \mathcal{B}(W^d); (\mathcal{F}_t))$:

(i) $P(\omega(0) = x) = 1$

(ii) $\varphi(\omega(t)) - \int_0^t \mathcal{L}\varphi(\omega(s))ds$ is a $(P, (\mathcal{F}_t))$-martingale for $^\forall \varphi \in \Phi$,

has a unique solution P for each $x \in \mathbb{R}^d$. $\square$.

The following results are known by taking $\Phi \subset C^2(\mathbb{R}^d)$, which is a dense subspace of

$$C_\infty(\mathbb{R}^d) := \{\varphi \in C(\mathbb{R}^d); \lim_{|x| \to \infty} \varphi(x) = 0\}.$$

The proofs are omitted.

[Theorem 20.2] (Echeverria, Z. Wahrsch. Verw. Gebiete **61**, 1982) Assume that $(\mathcal{L}, \Phi)$-martingale problem is well-posed. If $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \mathcal{L}\varphi(x)\mu(dx) = 0$$

for ${}^\forall \varphi \in \Phi$, then $\mu \in \mathcal{I}$. □

[Theorem 20.3] (Fukushima-Stroock, Adv. Math. Suppl. Stud., **9**, 1986) Under the same assumption as Theorem 20.2, if $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \varphi(x)\mathcal{L}\psi(x)\mu(dx) = \int_{\mathbb{R}^d} \psi(x)\mathcal{L}\varphi(x)\mu(dx)$$

for ${}^\forall \varphi, \psi \in \Phi$, then $\mu \in \mathcal{R}$. □

Let us apply Theorem 20.3 to the Ornstein-Uhlenbeck process.

[Lemma 20.4] Under the assumption that $[\alpha, A] = 0$, A is symmetric and negative, $\mu = N(0, \alpha\alpha^*(-2A)^{-1})$ is a reversible measure of the Ornstein-Uhlenbeck process. In particular,

$$\begin{aligned} -\int_{\mathbb{R}^d} \varphi(x)\mathcal{L}\psi(x)\mu(dx) &= -\int_{\mathbb{R}^d} \psi(x)\mathcal{L}\varphi(x)\mu(dx) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} (\alpha\alpha^*)_{ij}\frac{\partial\varphi}{\partial x^i}(x)\frac{\partial\psi}{\partial x^j}(x)\mu(dx) \end{aligned} \tag{2}$$

for $^\forall\varphi, \psi \in C_b^2(\mathbb{R}^d)$. The last formula is called the Dirichlet form and denoted by $\mathcal{D}(\varphi, \psi)$. □

[Proof] μ is written as

$$\mu(dx) = Ce^{-\frac{1}{2}(x,(\alpha\alpha^*(-2A)^{-1})^{-1}x)}dx = Ce^{(x,A(\alpha\alpha^*)^{-1}x)}dx.$$

The generator of the OU process is given by

$$\mathcal{L}\varphi(x) = \frac{1}{2}(\alpha\alpha^*)_{ij}\frac{\partial^2\varphi}{\partial x^i\partial x^j}(x) + (Ax)_i(x)\frac{\partial\varphi}{\partial x^i}(x).$$

We may check the condition in Theorem 20.3.

To this end, by integration by parts formula, we first note

$$\int_{\mathbb{R}^d} \varphi(x) \frac{\partial \psi}{\partial x^i}(x) e^{(x, A(\alpha\alpha^*)^{-1}x)} dx = -\int_{\mathbb{R}^d} \psi(x) \frac{\partial}{\partial x^i} \left\{ \varphi e^{(x, A(\alpha\alpha^*)^{-1}x)} \right\} dx$$

Here, in the RHS,

$$\begin{aligned}
&\frac{\partial}{\partial x^i} \left\{ \varphi e^{(x, A(\alpha\alpha^*)^{-1}x)} \right\} \\
&\quad = \frac{\partial \varphi}{\partial x^i} e^{(x, A(\alpha\alpha^*)^{-1}x)} + \frac{\partial}{\partial x^i} (x, A(\alpha\alpha^*)^{-1}x) \cdot \varphi e^{(x, A(\alpha\alpha^*)^{-1}x)} \\
&\quad = \left\{ \frac{\partial \varphi}{\partial x^i} + 2(A(\alpha\alpha^*)^{-1}x)_i \cdot \varphi \right\} e^{(x, A(\alpha\alpha^*)^{-1}x)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^d} \varphi(x) (\alpha\alpha^*)_{ij} \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x) e^{(x, A(\alpha\alpha^*)^{-1}x)} dx \\
&\quad = -\frac{1}{2} \int_{\mathbb{R}^d} (\alpha\alpha^*)_{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial \psi}{\partial x^j} e^{(x, A(\alpha\alpha^*)^{-1}x)} dx \\
&\qquad - \frac{1}{2} \int_{\mathbb{R}^d} (\alpha\alpha^*)_{ij} 2(A(\alpha\alpha^*)^{-1}x)_i \cdot \varphi \frac{\partial \psi}{\partial x^j} e^{(x, A(\alpha\alpha^*)^{-1}x)} dx.
\end{aligned}$$

However, since $[A, \alpha\alpha^*] = 0$ and $\alpha\alpha^*$ is symmetric, we have

$\sum_i \frac{1}{2}(\alpha\alpha^*)_{ij} 2(A(\alpha\alpha^*)^{-1}x)_i = (Ax)_j$.

This shows (2) and we obtain the conclusion. □

- As a more general example, let a potential function $V \in C^1(\mathbb{R}^d)$ such that its derivative ∇V is Lipschitz continuous be given and consider the SDE on $\mathbb{R}^d$:

$$dX_t = dB_t - \frac{1}{2}\nabla V(X_t)dt. \tag{3}$$

- This SDE is sometimes called the Langevin equation and the solution X_t a distorted Brownian motion.
- For the OU process with $\alpha = I$, we may take $V = \frac{1}{2}(x, Ax)$ with a symmetric A. In particular, V is quadratic, which is called the harmonic potential.

[Lemma 20.5] Assume $Z := \int_{\mathbb{R}^d} e^{-V(x)} dx < \infty$. Then, $d\mu := \frac{1}{Z} e^{-V(x)} dx$ is reversible for X: $\mu \in \mathcal{R}$.
The corresponding Dirichlet form is given by

$$\mathcal{D}(\varphi, \psi) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla\psi d\mu. \quad \square$$

[Proof] Taking $\Phi = C_0^2(\mathbb{R}^d)$, the generator is given by $\mathcal{L} = \frac{1}{2}\Delta - \frac{1}{2}\nabla V \cdot \nabla$. Thus, again by integration by parts,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi \mathcal{L} \psi \, d\mu &= \frac{1}{2Z} \int_{\mathbb{R}^d} \varphi \big(\Delta\psi - \nabla V \cdot \nabla\psi \big) \, e^{-V} dx \\ &= -\frac{1}{2Z} \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla\psi \, e^{-V} dx \\ &= \int_{\mathbb{R}^d} \psi \mathcal{L} \varphi \, d\mu. \end{aligned}$$

The last identity follows by the symmetry of the above equation in φ and ψ. $\square$

- Taking $A = (\alpha_{ij})_{1 \le i,j \le d}$, we modify the Dirichlet form as
$$\tilde{\mathcal{D}}(\varphi, \psi) := \frac{1}{2} \int A\nabla\varphi \cdot A\nabla\psi \, e^{-V} dx$$
$$= - \int \psi \, \tilde{L}\varphi \, e^{-V} dx,$$
where
$$\tilde{L}\varphi = \frac{1}{2} A^* A \Delta\varphi - \frac{1}{2} A^* A \nabla\varphi \cdot \nabla V$$
$$= \frac{1}{2} A^* A \Delta\varphi - \frac{1}{2} A^* A \nabla V \cdot \nabla\varphi.$$
- (Fluctuation-dissipation relation) The corresponding SDE is changed as
$$dY_t = A dB_t - \frac{1}{2} A^* A \nabla V(Y_t) dt.$$
- Note that $A^* A$ is symmetric and nonnegative so that $\sqrt{A^* A}$ is well-defined. Then, $AB_t \overset{\text{law}}{=} \sqrt{A^* A} B_t$.

  P: Prove this by showing the covariance is the same: $E[(AB_t, \xi)^2] = t(\xi, AA^*\xi) = E[(\sqrt{A^* A} B_t, \xi)^2]$, $^\forall \xi \in \mathbb{R}^d$.

- Existence of invariant measure

We apply the argument in Liggett "Interacting Particle Systems" p.10.

Let X_t be a Markov process with (any) initial distribution $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Let $\mu_t := P_{\mu_0}(X_t \in \cdot) \in \mathcal{P}(\mathbb{R}^d)$ be the distribution of X_t at time t. Consider its Cesàro mean:

$$\bar{\mu}_T := \frac{1}{T} \int_0^T \mu_t dt.$$

[Proposition 20.6] If there exist a sequence $\{T_n \nearrow \infty\}$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that $\bar{\mu}_{T_n}$ converges weakly to ν as $n \to \infty$, then ν is invariant for X: $\nu \in \mathcal{I}$. □

[Proof] Setting $T_t\varphi(x) = E_x[\varphi(X_t)]$, the expectation under the process X starting at x and $\varphi \in C_\infty(\mathbb{R}^d)$, it is enough to prove $\int_{\mathbb{R}^d} T_t\varphi d\nu = \int_{\mathbb{R}^d} \varphi d\nu$. However,

$$\int_{\mathbb{R}^d} T_t\varphi(x) d\nu = \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} ds \int_{\mathbb{R}^d} T_t\varphi(x)\mu_s(dx).$$

Here, by the Markov property of X,

$$\begin{aligned}\int_{\mathbb{R}^d} T_t\varphi(x)\mu_s(dx) &= E_{\mu_0}[T_t\varphi(X_s)] = E_{\mu_0}[E_{X_s}[\varphi(X_t)]] \\ &= E_{\mu_0}[\varphi(X_{t+s})] = \int_{\mathbb{R}^d} \varphi(x)\mu_{t+s}(dx).\end{aligned}$$

Thus, the RHS in the above formula is (by shifting $t+s \to s$)

$$= \lim_{n\to\infty} \frac{1}{T_n} \int_t^{T_n+t} ds \int_{\mathbb{R}^d} \varphi(x)\mu_s(dx).$$

This is actually

$$= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} ds \int_{\mathbb{R}^d} \varphi(x)\mu_s(dx), \qquad (4)$$

since the difference of two limits is estimated as

$$\Big|\frac{1}{T_n}\Big\{\int_t^{T_n+t}-\int_0^{T_n}\Big\}ds\int_{\mathbb{R}^d}\varphi(x)\mu_s(dx)\Big|\le\frac{1}{T_n}2t\|\varphi\|_\infty\underset{n\to\infty}{\longrightarrow}0.$$

However, the limit in (4) $=\int_{\mathbb{R}^d}\varphi(x)d\nu$ by the condition of the proposition. Thus, we have shown $\int_{\mathbb{R}^d}T_t\varphi d\nu=\int_{\mathbb{R}^d}\varphi d\nu$. □

[Lemma 20.7] If the solution X_t of the SDE on $\mathbb{R}^d$ satisfies

$$\sup_{t\ge0}E[|X_t|^p]<\infty$$

for some $p>0$, then the assumption of Proposition 20.6 holds. □

[Proof] It is enough to show the tightness of $\{\bar{\mu}_T\}_{T>0}$. First note that

$$\int_{\mathbb{R}^d} |x|^p \bar{\mu}_T(dx) = \frac{1}{T}\int_0^T dt \int_{\mathbb{R}^d} |x|^p \mu_t(dx) = \frac{1}{T}\int_0^T E[|X_t|^p]dt \le C < \infty.$$

Thus,

$$\bar{\mu}_T(|x| \ge K) \le \frac{C}{K^p} \underset{K\to\infty}{\longrightarrow} 0 \quad \text{(uniformly in } T\text{)},$$

which implies that for $^\forall\varepsilon > 0$ there exists $K > 0$ such that $\bar{\mu}_T(|x| \ge K) \le \varepsilon$ for $^\forall T > 0$. However, since the set $\{|x| \ge K\}^c$ is relatively compact in $\mathbb{R}^d$, we see that $\{\bar{\mu}_T\}_{T>0}$ is tight by Prokhorov's theorem (recall Lect-5). □

[Remark] Note that the OU process satisfies the condition of Lemma 20.7, while the Brownian motion does not. □

§21 Other examples of SDE

(1) d-dimensional Bessel process: Let W_t be a d-dimensional Brownian motion starting at $W_0 = x \neq 0 (\in \mathbb{R}^d)$. Set $r_t := |W_t|$. Then, r_t satisfies the following SDE:

$$dr_t = \frac{d-1}{2r_t} dt + dB_t,$$

where

$$B_t := \sum_{i=1}^{d} \int_0^t \frac{W_s^i}{r_s} dW_s^i$$

is a 1-dimensional Brownian motion. We call r_t a d-dimensional Bessel process. It is known that its transition probability can be written by using the modified Bessel function.

[Proof] Set $f(x) := |x| = \sqrt{\sum_{i=1}^d x_i^2}$ for $x = (x_i)_{i=1}^d \in \mathbb{R}^d$. Then, since $\frac{\partial f}{\partial x_i} = \frac{x_i}{|x|}$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}$ and $\Delta f = \frac{d-1}{|x|}$, by Itô's formula, we have

$$dr_t = \sum_{i=1}^d \frac{\partial f}{\partial x_i} dW_t^i + \frac{1}{2} \Delta f dt$$
$$= \sum_{i=1}^d \frac{W_t^i}{r_t} dW_t^i + \frac{1}{2} \frac{d-1}{r_t} dt$$

This shows the SDE for r_t. We see that B_t is a Brownian motion, since its quadratic variation is given by

$$d\langle B \rangle_t = \sum_{i=1}^d \frac{(W_t^i)^2}{r_t^2} dt = dt,$$

and Lévy's characterization of the Brownian motion applies.

□

(2) Brownian bridge: Let W_t be a 1-dimensional Brownian motion such that $W_0 = 0$. Consider the SDE:

$$dX_t = dW_t + \frac{a - X_t}{1 - t} dt, \quad 0 \leq t < 1, \quad X_0 = 0$$

for $a \in \mathbb{R}$. Then, the solution is given by

$$X_t = at + (1 - t) \int_0^t \frac{1}{1 - s} dW_s, \quad 0 \leq t < 1.$$

⨀ By Itô's formula,

$$dX_t = adt - \left(\int_0^t \frac{1}{1 - s} dW_s \right) dt + dW_t.$$

However, since

$$\int_0^t \frac{1}{1 - s} dW_s = \frac{X_t - at}{1 - t} = \frac{X_t - a}{1 - t} + a,$$

we obtain the SDE. □

From this representation of X_t and Itô isometry, we have

$$E[(X_t - at)^2] = (1-t)^2 \int_0^t \frac{ds}{(1-s)^2}$$
$$= (1-t)^2 \Big(\frac{1}{1-t} - 1 \Big) = t(1-t) \underset{t \nearrow 1}{\longrightarrow} 0.$$

This we see that X_t converges to a as $t \nearrow 1$ in L^2-sense.
Moreover, we see that $X_t \stackrel{\text{law}}{=} N(at, t(1-t))$.

(3) Application to stochastic analysis (Itô, 1976): The stochastic integral $\int_0^1 B_1 dB_t$ in Itô's sense is undefinable, if we take $\mathcal{F}_t := \sigma\{B_s; s \le t\} \vee \mathcal{N}$ as usual, since the integrand is not $(\mathcal{F}_t)$-adapted.

Now we take $\tilde{\mathcal{F}}_t := \mathcal{F}_t \vee \sigma\{B_1\}$. Then, note that B_t is not an $(\tilde{\mathcal{F}}_t)$-martingale, since

$$E[B_t|\tilde{\mathcal{F}}_s] = \frac{1-t}{1-s}B_s + \frac{t-s}{1-s}B_1 \ (\neq B_s), \quad 0 \leq s \leq t < 1.$$

P: Show this equality.

However, under $(\tilde{\mathcal{F}}_t)$, one can decompose B_t as $B_t = W_t + A_t$ with $A_t := \int_0^t \frac{B_1 - B_s}{1-s} ds$ and $W_t := B_t - A_t$, which is indeed an $(\tilde{\mathcal{F}}_t)$-Brownian motion.

☺ In the SDE for Brownian bridge, one can take $a = B_1$ and obtain

$$dB_t = dW_t + \frac{B_1 - B_t}{1-t} dt. \qquad \square$$

Based on this decomposition, one can define and compute as

$$\int_0^1 B_1 dB_t := \int_0^1 B_1 dW_t + \int_0^1 B_1 dA_t = B_1(W_1 - W_0) + B_1(A_1 - A_0) = B_1^2.$$