Lecture No 13 April 19, 2022 (Tue)

④ Invariance under transformations: Let B be a Brownian motion. Then, $\tilde{B}$ defined from B as follows are also Brownian motions.

(1) $\tilde{B}_t := B_{t+t_1} - B_{t_1}, \quad t \geq 0$

(2) $\tilde{B}_t := -B_t, \quad t \geq 0$

(3) $\tilde{B}_t := \gamma B(t/\gamma^2), \quad t \geq 0$

where $t_1 > 0$ and $\gamma > 0$ are fixed. □

⚇ The properties of $\tilde{B}_0 = 0$, continuity in t and independent increments are fine for $\tilde{B}_t$ in (1)–(3).

(1) follows by $\tilde{B}_t - \tilde{B}_s = B_{t+t_1} - B_{s+t_1} \overset{\text{law}}{=} N(0, t-s)$.

(2) follows by $\tilde{B}_t - \tilde{B}_s = -(B_t - B_s) \overset{\text{law}}{=} N(0, t-s)$.

(3) follows by

$$\tilde{B}_t - \tilde{B}_s = \gamma(B(t/\gamma^2) - B(s/\gamma^2)) \overset{\text{law}}{=} \gamma \cdot N(0, \tfrac{t-s}{\gamma^2}) \overset{\text{law}}{=} N(0, t-s).$$

Here, $\overset{\text{law}}{=}$ means that the laws (distributions) of both sides are the same. □

⑤ The total variation of B in the interval $[0, t]$ is ∞ for a.s. ω.

□

∵ (Similar argument was given in §10.5) Total variation of B in $[0, t]$ is defined by

$$V_t(B) := \sup_{\Delta} \sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right|,$$

where sup is taken over all divisions of $[0, t]$:
$\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. First, note

$$\sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}} \right)^2 \leq \max_{1 \leq i \leq n} |B_{t_i} - B_{t_{i-1}}| \times \sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right|.$$

Here, (LHS)$\to t$ in $L^2(\Omega)$ so that $\to t$ a.s. along a certain subsequence of $\{n\}$. On the other hand, the first term in (RHS) $\to 0$ a.s. by the continuity of B_t and the second term $\leq V_t(B)$. Therefore, if $V_t(B) < \infty$, we have a contradiction. Thus, $V_t(B) = \infty$ a.s. is proved. □

The following properties ⑥∼⑧ are known, but we omit the proof; see textbooks [2] or [3].

⑥ For a.s. ω, $B_t(\omega)$ is not differentiable at all $t \geq 0$. □

⑦ Law of iterated logarithm:

$$\overline{\lim_{t\downarrow 0}} \frac{B_t}{\sqrt{2t \log\log \frac{1}{t}}} = 1 \quad \text{a.s.}$$ □

⑧ Modulus of continuity: Kolmogorov's regularization theorem implies Hölder continuity of B_t with exponent α for all $\alpha < \frac{1}{2}$. The following gives more detailed modulus of continuity of B_t.

$$\overline{\lim_{\substack{t_2 - t_1 = \varepsilon \downarrow 0 \\ 0 \leq t_1 < t_2 \leq 1}}} \frac{|B_{t_2} - B_{t_1}|}{\sqrt{2\varepsilon \log \frac{1}{\varepsilon}}} = 1 \quad \text{a.s.}$$ □

Finally, we consider higher dimensional Brownian motion.

⑨ Rotation invariance: Let $B = (B_t)_{t\geq 0}$ be a d-dimensional Brownian motion starting from the origin $0 \in \mathbb{R}^d$. Let O be a $d \times d$ orthogonal matrix. Then, OB_t is also a d-dimensional Brownian motion.

In particular, the distribution of B_σ is a uniform probability measure on S. Here, $\sigma := \inf\{t > 0; B_t \in S\}$ is the first hitting time to the sphere $S := \{x \in \mathbb{R}^d; |x| = r\}$ and $B_{\sigma(\omega)}(\omega)$ (written simply by B_σ) denotes the hitting position.

⚇ • For $^\forall n \in \mathbb{N}$, $0 = t_0 \leq^\forall t_1 < \cdots <^\forall t_n$ and $^\forall \xi_k \in \mathbb{R}^d (1 \leq k \leq n)$, compute the characteristic function of the joint distribution of increments of (OB_t):

$$E\left[\exp\left\{i\sum_{k=1}^n \xi_k \cdot (OB_{t_k} - OB_{t_{k-1}})\right\}\right]$$

$$\underset{\perp\!\!\!\perp}{=} \prod_{k=1}^n E\left[\exp\left\{i\,{}^tO\xi_k \cdot (B_{t_k} - B_{t_{k-1}})\right\}\right],$$

where ${}^tO$ is the transposed matrix of O.

Recalling the characteristic function of Gaussian distribution,

$$(\text{Above}) = \prod_{k=1}^{n} \exp\{-|{}^tO\xi_k|^2(t_k - t_{k-1})/2\}$$
$$= \prod_{k=1}^{n} \exp\{-|\xi_k|^2(t_k - t_{k-1})/2\}$$

The last equality follows, since O is an orthogonal matrix. The last formula does not depend on O, and in particular, we see $OB \overset{\text{law}}{=} B$ (for any finite-dimensional distribution). Thus, we see that OB_t is a d-dimensional Brownian motion.

- Finally, by rotation invariance, for ${}^{\forall}E \in \mathcal{B}(\mathcal{S})$,

$$P(B_{\sigma_S} \in E) = P(OB_{\sigma_S} \in E) = P(B_{\sigma_S} \in O^{-1}E).$$

Note that $\sigma_S < \infty$ a.s. is shown from

$$P(\sigma_S = \infty) = \lim_{t\to\infty} P(\sigma_S \geq t) \leq \lim_{t\to\infty} \int_{\{|x|\leq r\}} p(t,x)dx = 0.$$

Therefore, the distribution $\mu(dy) := P(B_{\sigma_S} \in dy)$ of B_{σ_S} is a rotational invariant probability measure on S. □

§12 Stochastic integrals

- We want to define integrals $\int_0^t f_s \, dB_s$ by a Brownian motion.
- If $B_t(\omega)$ is of bounded variation in t, it is definable as a Stieltjes integral. However, since $B_t(\omega)$ is not of bounded variation (a.s.), this does not work.

[Example 12.1] Let us see that $\int_0^t B_s \, dB_s$ is not definable as a Riemann-Stieltjes integral. Consider the Riemann sum

$$S^{(\Delta)} := \sum_{i=1}^{n} B_{s_i}(B_{t_i} - B_{t_{i-1}}), \quad s_i \in [t_{i-1}, t_i].$$

for a division $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$.

(Recall that, if Stieltjes integral is definable, $\lim_{|\Delta| \to 0} S^{(\Delta)}$ does not depend on the choice of s_i.)

Set $\underline{S}^{(\Delta)}$ the sum taking $s_i = t_{i-1}$ (left edge of $[t_{i-1}, t_i]$) and $\overline{S}^{(\Delta)}$ taking $s_i = t_i$ (right edge). Then, as we showed before,

$$\overline{S}^{(\Delta)} - \underline{S}^{(\Delta)} = \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \underset{|\Delta| \to 0}{\longrightarrow} t \quad \text{in } L^2(\Omega)$$

In particular, two limits $\lim_{|\Delta| \to 0} \underline{S}^{(\Delta)}$ and $\lim_{|\Delta| \to 0} \overline{S}^{(\Delta)}$ are not the same even if they exist. □

- It will be important for Itô's stochastic integrals to take $s_i = t_{i-1}$ (left edge of small intervals $[t_{i-1}, t_i]$), though the integrand f_s may not be continuous in s.
- Roughly saying, we will define

$$\int_0^t f_s dB_s \left(\equiv \int_0^t f_s(\omega) dB_s(\omega) \right) := \lim_{|\Delta| \to 0} \sum_{i=1}^{n} f_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})$$

- However, this is not definable for arbitrary f_s, but for functions adapted to the Brownian motion, in the sense of $(\mathcal{F}_t^B)$-adapted functions. Indeed, this class of f_s would be enough later, for example, to apply for SDEs.

• Let the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and the right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ be given.

[Definition] $B = (B_t)_{t\geq 0}$ is $(\mathcal{F}_t)$-Brownian motion $\underset{\text{def}}{\Longleftrightarrow}$

(1) $B = (B_t)_{t\geq 0}$ is $(\mathcal{F}_t)$-adapted Brownian motion

(2) $0 \leq s \leq t \Longrightarrow$ increment $B_t - B_s \perp\!\!\!\perp \mathcal{F}_s$

[Remark 12.1] (1) When a Brownian motion is already given, we may take its natural filtration $(\mathcal{F}_t^B)_{t\geq 0}$. Then, B is $(\mathcal{F}_t^B)_{t\geq 0}$-Brownian motion. Recall that, by property ②, $(\mathcal{F}_t^B)_{t\geq 0}$ is a filtration; in particular, it is right-continuous.

(2) For example, when we consider higher dimensional Brownian motion, it is useful to generalize $(\mathcal{F}_t)_{t\geq 0}$ as in the above definition. □

• The class of integrands $f(t,\omega)$: Fix $T > 0$ and set

$$\begin{aligned}
\mathbb{L}^2_T :=& L^2([0, T] \times \Omega, dtdP) \\
\equiv& \{f; f = f(t,\omega) \text{ is measurable and } ||f||_{\mathbb{L}^2_T} < \infty\} \\
\mathcal{L}^2_T \equiv& \mathcal{L}^2_T(\mathcal{F}_t) := \{f \in \mathbb{L}^2_T; f \text{ is } (\mathcal{F}_t)\text{-adapted}\},
\end{aligned}$$

where $||f||^2_{\mathbb{L}^2_T} := E[\int_0^T f_t^2\, dt]$ and measurability of f means that as a map

$$(t,\omega) \in ([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}) \longmapsto f(t,\omega) \in (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

We identify elements of $\mathbb{L}^2_T$ when they coincide for a.e. (t,ω).

[Remark] More precisely, we assume $f \in \mathcal{L}^2_T$ is progressively measurable, that is, for $^\forall t \in [0, T]$, the map

$$(s,\omega) \in ([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t) \longmapsto f(s,\omega) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable. In fact, under the above assumption for f (i.e. $(\mathcal{F}_t)$-adapted and measurable), f has a progressively measurable modification; see Karatzas-Shreve [3], p.5, Proposition 1.12. □

- The class of functions appearing as stochastic integrals:

$$\mathcal{M}_{c,T}^2 := \{M = (M_t)_{t\in[0,T]}; \text{square integrable, continuous } (\mathcal{F}_t)\text{-martingale and } M_0 = 0 \text{ a.s.}\}$$

By Proposition 10.8, $\mathcal{M}_{c,T}^2$ is a real Hilbert space equipped with the norm $\|M\|_{\mathcal{M}_{c,T}^2} := E[M_T^2]^{\frac{1}{2}}$.

Step 1 (Step processes) $f = (f(t,\omega))_{t\in[0,T]}$ is called a step process, if it has the form

$$f(t,\omega) = \sum_{j=1}^{n} f_j(\omega) 1_{[t_{j-1},t_j)}(t), \quad t \in [0, T]$$

with $n \in \mathbb{N}$, $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ taken independently of ω and $\mathcal{F}_{t_{j-1}}$-measurable and bounded f_j.

Set $\mathcal{S}_T = \{\text{all step processes}\}$. Note that $\mathcal{S}_T \subset \mathcal{L}_T^2$.

[Definition 12.1] For $f \in \mathcal{S}_T$, define the stochastic integral by

$$I_t(f) \equiv \int_0^t f_s \, dB_s := \sum_{j=1}^{n} f_j \left(B_{t\wedge t_j} - B_{t\wedge t_{j-1}}\right), \quad t \in [0, T]$$

$$= \sum_{j=1}^{k-1} f_j \left(B_{t_j} - B_{t_{j-1}}\right) + f_k \left(B_t - B_{t_{k-1}}\right), \quad \text{if } t \in [t_{k-1}, t_k].$$

($\because$ For such t, $B_{t\wedge t_j} - B_{t\wedge t_{j-1}} = 0$ for $j \geq k+1$.) □

[Lemma 12.1] For $f, g \in \mathcal{S}_T$, $I(f) = (I_t(f)) \in \mathcal{M}^2_{c,T}$ and

$$\langle I(f), I(g)\rangle_t = \int_0^t f_s\, g_s\, ds, \quad t \in [0, T].$$

In particular, $E[I_t(f)] = 0$ and Itô isometry:

$$\|I(f)\|_{\mathcal{M}^2_{c,T}} = \|f\|_{\mathbb{L}^2_T}$$

and therefore we have

$$E[I_t(f)^2] = E\Big[\int_0^t f_s^2\, ds\Big]. \qquad \square$$

[Proof] • Since $\mathcal{M}^2_{c,T}$ is a linear space, $I(f) \in \mathcal{M}^2_{c,T}$ follows once we show it when $f(t,\omega) = f(\omega)1_{[a,b)}(t)$, $0 \le a < b \le T$ with f : $\mathcal{F}_a$-measurable and bounded. For such f,

$$I_t \equiv I_t(f) = f(B_{t\wedge b} - B_{t\wedge a}).$$

• "I_t: continuous, square integrable, $I_0 = 0$" are obvious.

- I_t: $(\mathcal{F}_t)$-martingale

⚇ We may show $E[I_t|\mathcal{F}_s] = I_s$ for $0 \le s \le t$. Write

$$E[I_t - I_s|\mathcal{F}_s] = E[f\{(B_{t\wedge b} - B_{t\wedge a}) - (B_{s\wedge b} - B_{s\wedge a})\}|\mathcal{F}_s].$$

When $0 \le s \le a$, RHS is (by tower property)

$$E[f\, E[B_{t\wedge b} - B_{t\wedge a}|\mathcal{F}_a]|\mathcal{F}_s] = 0$$

When $s > a$, (since f is $\mathcal{F}_a$-m'ble so that $\mathcal{F}_s$-m'ble), RHS is

$$f\, E[B_{t\wedge b} - B_{s\wedge b}|\mathcal{F}_s] = 0$$

These imply the martingale property of I_t. □

- Next, we show the identity for the cross variation.

It is enough to show

$$E\left[I_t(f)I_t(g) - I_s(f)I_s(g) - \int_s^t f_r g_r\, dr \middle| \mathcal{F}_s\right] = 0 \qquad (\star)$$

for $0 \le s \le t \le T$. (Actually $(\star)$ for $f = g$ is enough, but we write in the above form.)

∵ By the definition of cross variation, $\langle I(f), I(g)\rangle_t := \frac{1}{4}\{\langle I(f+g)\rangle_t - \langle I(f-g)\rangle_t\}$. But, $(\star)$ for $f=g$ implies $\langle I(f)\rangle_t = \int_0^t f_s^2\,ds$, from which we obtain the conclusion. □

To prove $(\star)$, we may assume $f = f1_{[a,b)}(t), g = g1_{[c,d)}(t)$, $a \le c$. (∵ General $f, g \in \mathcal{S}_T$ are finite sum of these functions. Note also that the cross variation $\langle M, N\rangle$ is bilinear in M, N.)

When $0 \le s \le c$, recalling $I_t(f) = f(B_{t\wedge b} - B_{t\wedge a})$, LHS of $(\star)$ is rewritten as

$$E[f\,g\,E[(B_{t\wedge b} - B_{t\wedge a})(B_{t\wedge d} - B_{t\wedge c}) - (t\wedge b\wedge d - c)|\mathcal{F}_c]|\mathcal{F}_s]$$

and this is 0. When $s \ge c$, LHS of $(\star)$ is rewritten as

$$\begin{aligned} f\,g\,E[&(B_{t\wedge b} - B_a)(B_{t\wedge d} - B_c) \\ &- (B_{s\wedge b} - B_a)(B_{s\wedge d} - B_c) - (t\wedge b\wedge d - s)|\mathcal{F}_s] \end{aligned}$$

and this is also 0. Thus, $(\star)$ is shown.

• $E[I_t(f)] = 0$ and Itô isometry are immediate consequence. Therefore, Lemma 12.1 is proved. □

Step 2 We define the stochastic integral of $f \in \mathcal{L}^2_T$.

[Lemma 12.2] For ${}^\forall f \in \mathcal{L}^2_T$, ${}^\exists f^n \in \mathcal{S}_T$ s.t.
$$||f - f^n||_{\mathbb{L}^2_T} \to 0 \text{ as } n \to \infty. \qquad \square$$

[Proof] • (cut-off of f) For $f \in \mathcal{L}^2_T$, set
$$f^m_t(\omega) := f_t(\omega) \times 1_{[-m,m]}(f_t(\omega)).$$
Then, $f^m \in \mathcal{L}^2_T$ (note that f^m is $(\mathcal{F}_t)$-adapted) and $||f - f^m||_{\mathbb{L}^2_T} \underset{m\to\infty}{\to} 0$ holds (by Lebesgue's convergence theorem). Thus, we may assume that f is bounded.

• (continuous modification of f) Next, set

$$f^\varepsilon_t(\omega) := \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^t f_s(\omega)\, ds, \quad \varepsilon > 0.$$

Then, $f^\varepsilon \in \mathcal{L}^2_T$ (note that f^ε is $(\mathcal{F}_t)$-adapted) and $||f - f^\varepsilon||_{\mathbb{L}^2_T} \underset{\varepsilon\downarrow 0}{\to} 0$ ($\to$ see the next page). Thus, we may assume f is continuous in t (for ${}^\forall\omega$).

P: Show that $||f - f^{\varepsilon}||_{\mathbb{L}^2_T} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$

[Hint] First recall L^2-continuity of $g \in L^2([0,T])$, i.e.

$$\lim_{\varepsilon\to 0}\int_0^T \left(g_{t+\varepsilon}1_{\{t+\varepsilon\in[0,T]\}} - g_t\right)^2 dt = 0.$$

Then, noting $f_t \in L^2([0,T])$ a.s. ω and the boundedness of f, we may apply Lebesgue's convergence theorem. □

- Finally, take

$$f_t^n(\omega) := \sum_{j=0}^{n-1} f_{t_j}(\omega) 1_{[t_j, t_{j+1})}(t). \quad t_j = \frac{j}{n}T$$

Then, $f^n \in \mathcal{S}_T$ and converges to f in $\mathbb{L}^2_T$. This completes the proof of Lemma 12.2. □

Under these preparation, we can define the stochastic integral of $f \in \mathcal{L}_T^2$.

- For $f^n \in \mathcal{S}_T$ obtained in Lemma 12.2, stochastic integrals $I(f^n) \in \mathcal{M}_{c,T}^2$ are defined.
- However, $\{I(f^n)\}_n$ is a Cauchy sequence in $\mathcal{M}_{c,T}^2$.

  ∵ By Itô isometry (Lemma 12.1),

$$\|I(f^n) - I(f^m)\|_{\mathcal{M}_{c,T}^2} = \|f^n - f^m\|_{\mathbb{L}_T^2} \underset{n,m\to\infty}{\longrightarrow} 0.$$

- $\mathcal{M}_{c,T}^2$ is a real Hilbert space. Therefore, the limit $I(f) = (I_t(f)) \in \mathcal{M}_{c,T}^2$ of $\{I(f^n)\}_n$ is determined.

[Definition 12.2] We write $I_t(f)$ determined as above $\int_0^t f_s\, dB_s$, $t \in [0, T]$ and call stochastic integral of $f \in \mathcal{L}_T^2$ by the Brownian motion $B = (B_t)$. □

- $I(f)$ is uniquely determined as an element of $\mathcal{M}_{c,T}^2$ (under the identification in a.s. sense) independently of the choice of $\{f^n\}$.
- For $t > s \geq 0$, we write $\int_s^t f_r\, dB_r$ for $I_t(f) - I_s(f)$.