Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 1. Arnold’s Problem on Interval Exchange Permutations (after a joint work with V. Delecroix, E. Goujard, P. Zograf)

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Arnold's problem

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2002-8. The (C, B, A)-permutation of the set $\{1, 2, \ldots, n\}$ transports to the last place the subset $A = \{1, 2, \ldots, a\}$ preceded by the transported set $B = \{a+1, \ldots, a+b\}$ while the starting position is occupied by $C = \{a+b+1, \ldots, n\}$.

Some of these $(n-1)(n-2)/2$ permutations permute *cyclically* (like the addition of a constant to the residues mod n), and some of these cyclic permutations are *transitive* (like the addition of the constant 1).

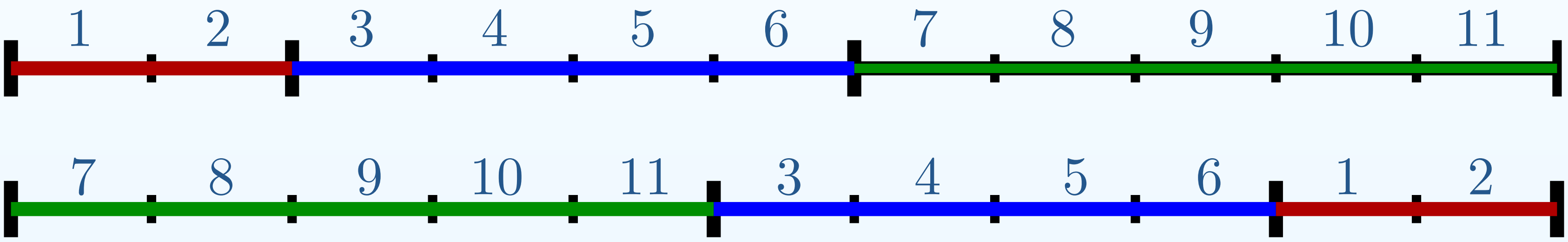
Find the proportion of both the cyclic and the transitive cyclic permutations among the (C, B, A) -permutations for large n.

More generally, starting from a permutation of k elements, one defines a permutation of the set $\{1, \ldots, n\}$ from its decomposition into k segments $\{a_i + 1, a_{i+1} - 1\}$. The problem is to study the statistics of the Young diagrams formed by the cycle lengths of the resulting permutations, for the case of large n and random decompositions of n into k parts.

Example of a non-cyclic (C,B,A)-permutation

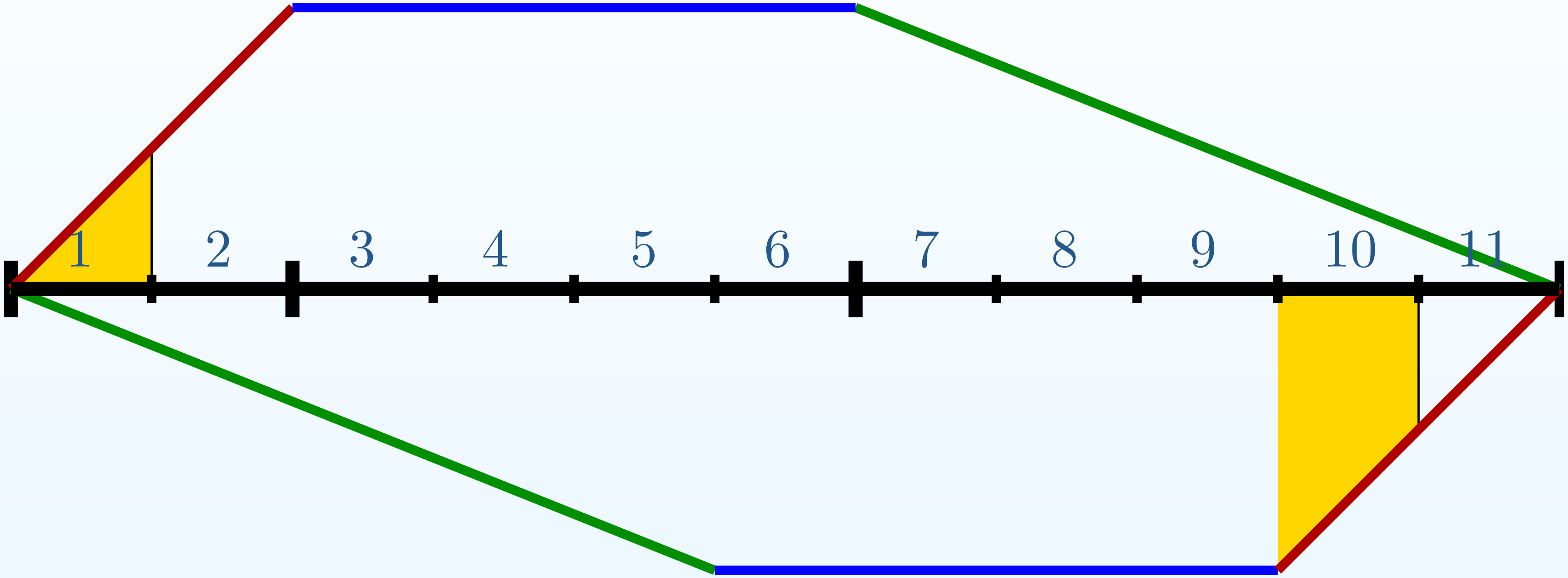
Let us chop the interval $X = [0, n[$ into three subintervals $X_A = [0, a[$, $X_B = [a+1, a+b[$, and $X_C = [a+b+1, n[$.
In our example $A = \{1, 2\}, B = \{3, 4, 5, 6\}, C = \{7, 8, 9, 10, 11\}$.

Example of a non-cyclic (C,B,A)-permutation



The decomposition a (C, B, A)-permutation into disjoint cycles can be studied through the interval exchange transformation placing the subintervals in the order X_C, X_B, X_A and mapping the resulting interval to the original interval X by isometry.

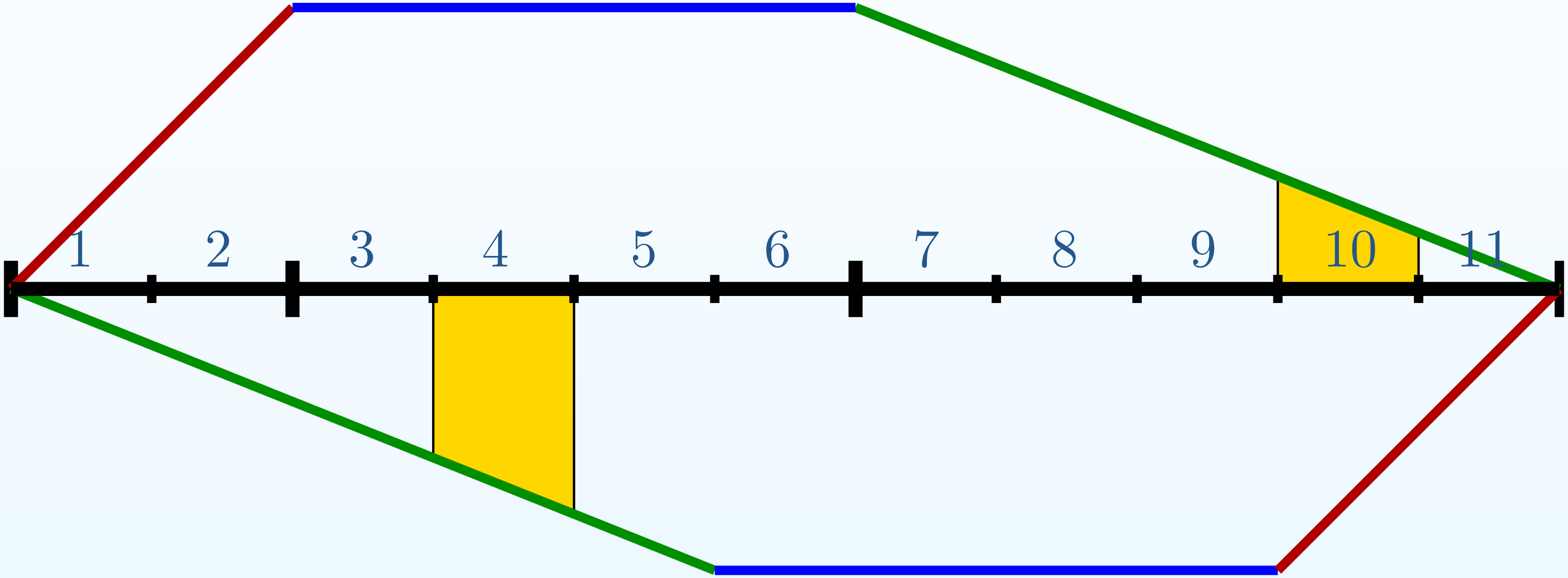
Example of a non-cyclic (C,B,A)-permutation



It is convenient to use a suspension over the interval exchange transformation to study its orbits. We get

$1 \mapsto 10$

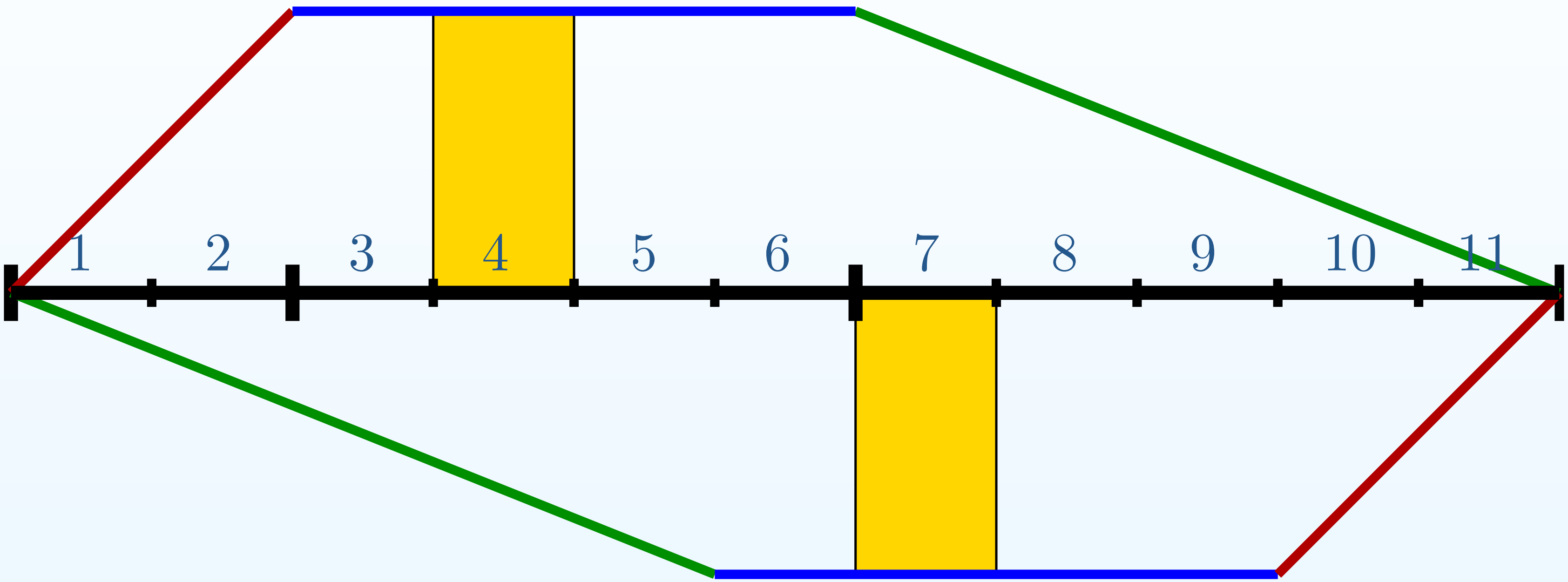
Example of a non-cyclic (C,B,A)-permutation



It is convenient to use a suspension over the interval exchange transformation to study its orbits. We get

$1 \mapsto 10 \mapsto 4$

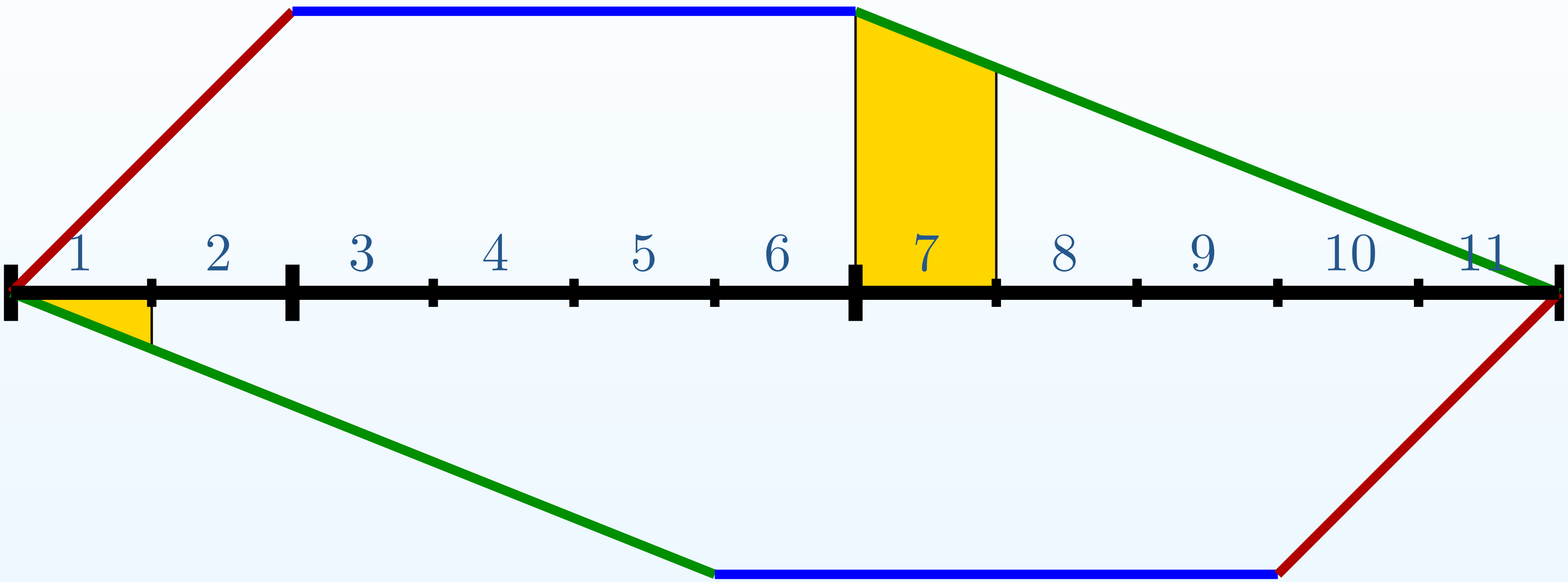
Example of a non-cyclic (C,B,A)-permutation



It is convenient to use a suspension over the interval exchange transformation to study its orbits. We get

$1 \mapsto 10 \mapsto 4 \mapsto 7$

Example of a non-cyclic (C,B,A)-permutation



It is convenient to use a suspension over the interval exchange transformation to study its orbits. We get

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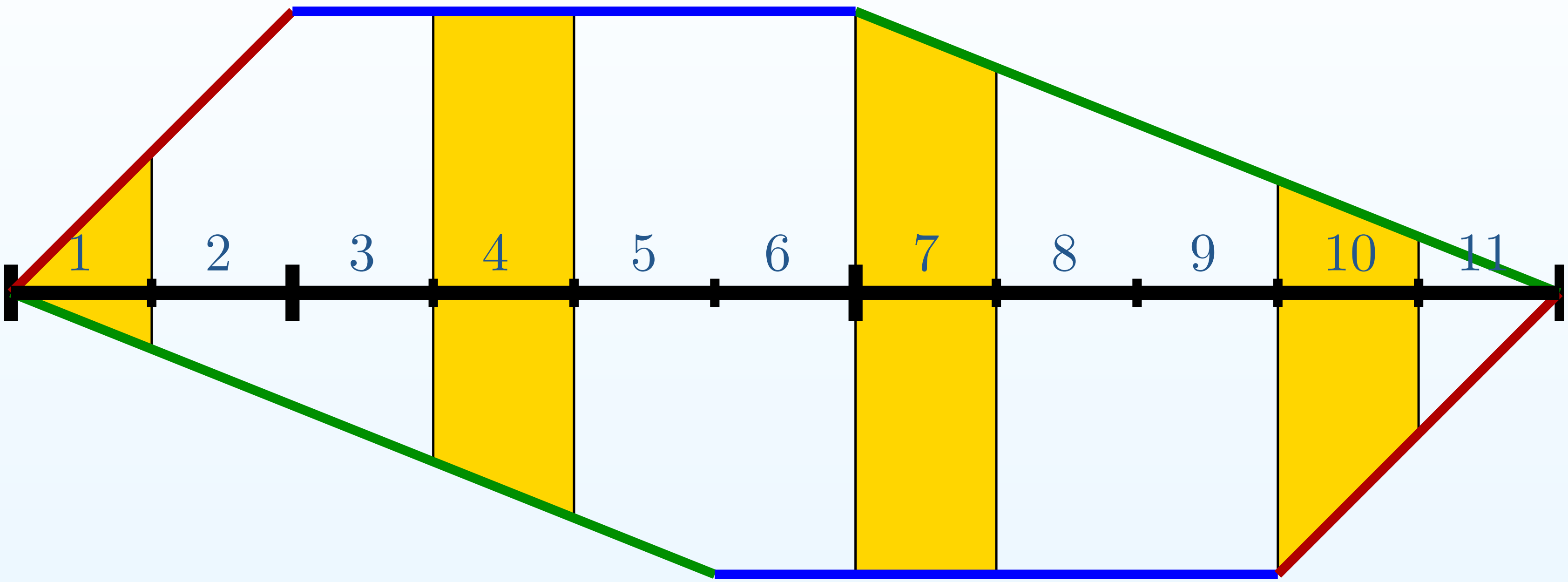
and the first cycle closes up.

Example of a non-cyclic (C,B,A)-permutation



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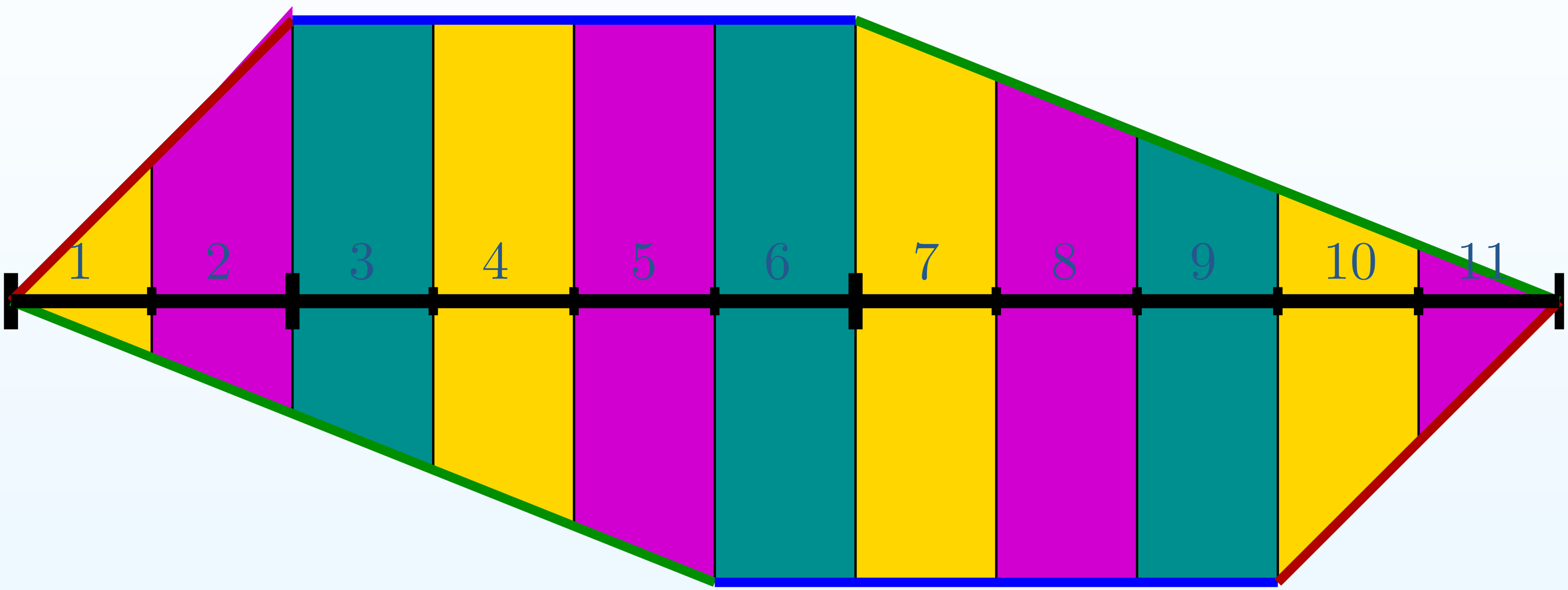
Example of a non-cyclic (C,B,A)-permutation

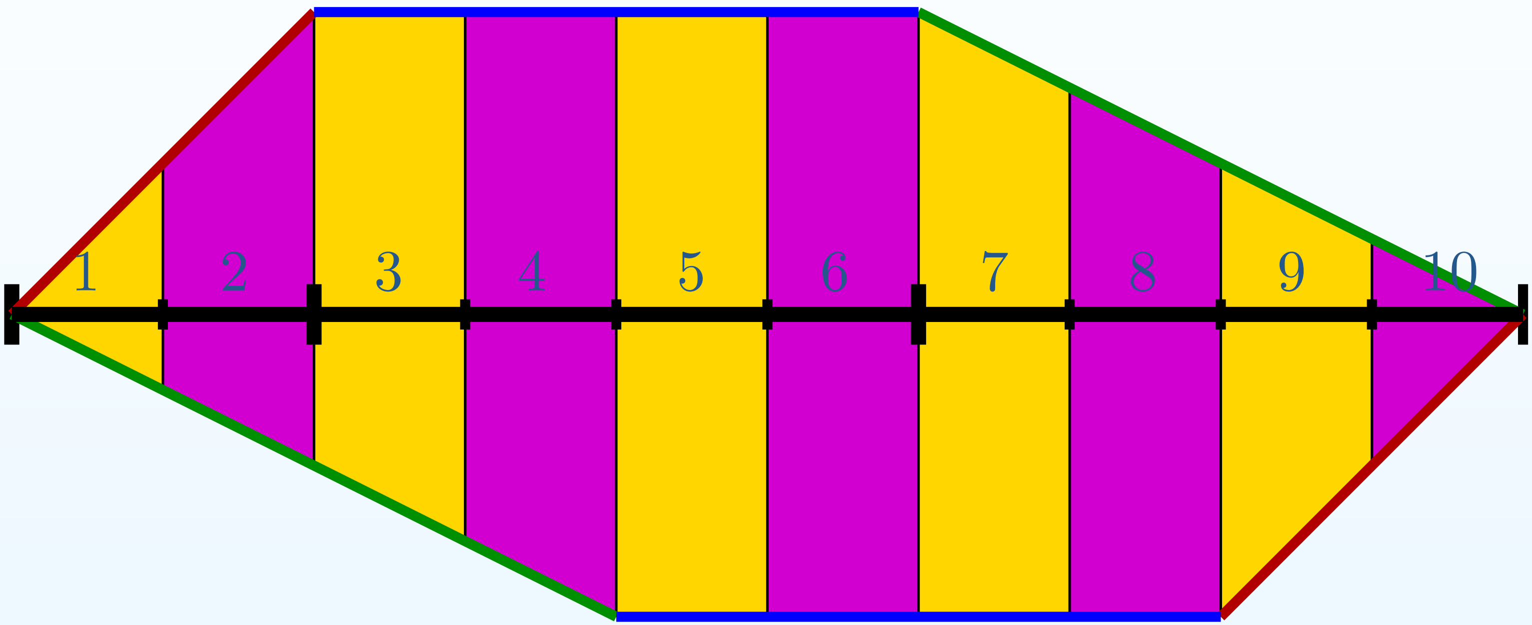


In a similar way we get the complete cyclic decomposition of our permutation:

$$(1, 10, 4, 7)(2, 11, 5, 8)(3, 6, 9)\,.$$

We considered the example, where $|A| = 2, |B| = 4, |C| = 5$, and observed that the resulting (C, B, A)-permutation is not “cyclic” in the sense of Arnold (and, hence, also non transitive).

Example of a cyclic but nontransitive (C,B,A)-permutation



Choosing $|A| = 2, |B| = 4, |C| = 4$ we get the following cyclic decomposition of the resulting (C, B, A)-permutation:

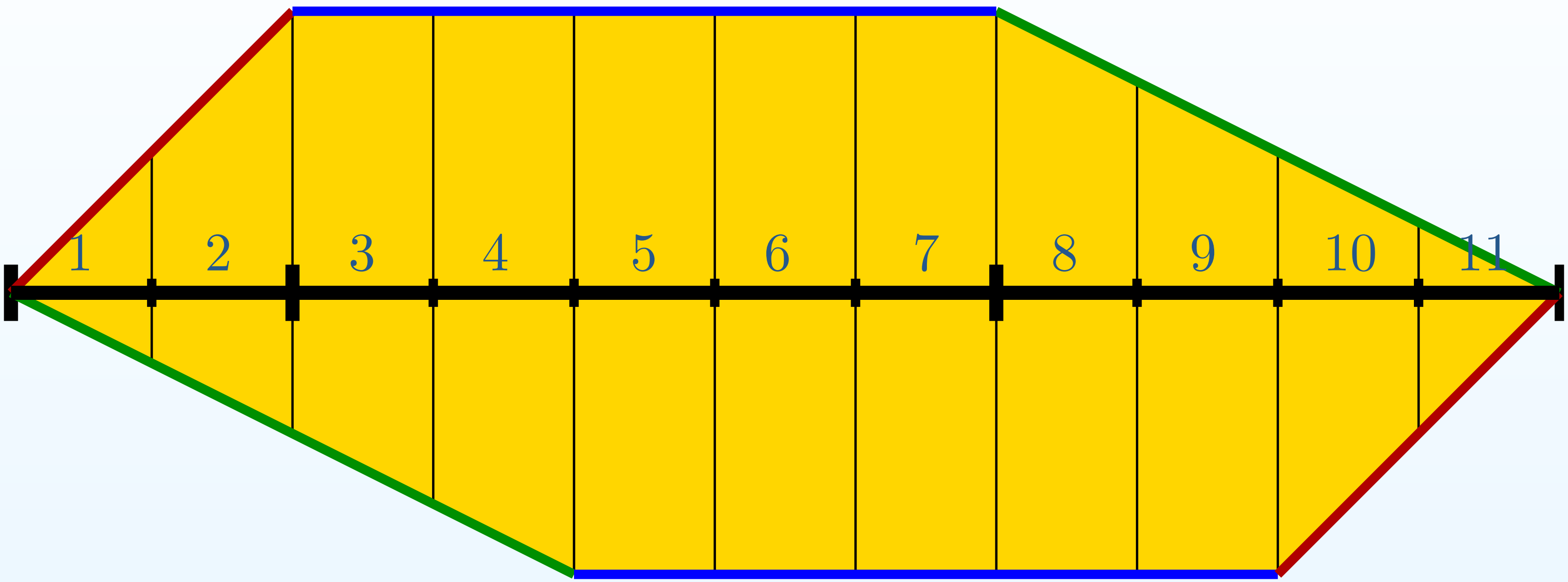
$$(1, 9, 3, 5, 7)(2, 10, 4, 6, 8)$$

Forgetting the ordering in the two cycles, we get two unordered sets

$$\{1, 3, 5, 7, 9\} \sqcup \{2, 4, 6, 8, 10\} ,$$

which mimic orbits of a cyclic permutation (as when adding the constant 2). However, since there are two distinct orbits and not a single orbit, this permutation is “cyclic” but not “transitive” in the sense of Arnold.

Example of a transitive (C,B,A)-permutation

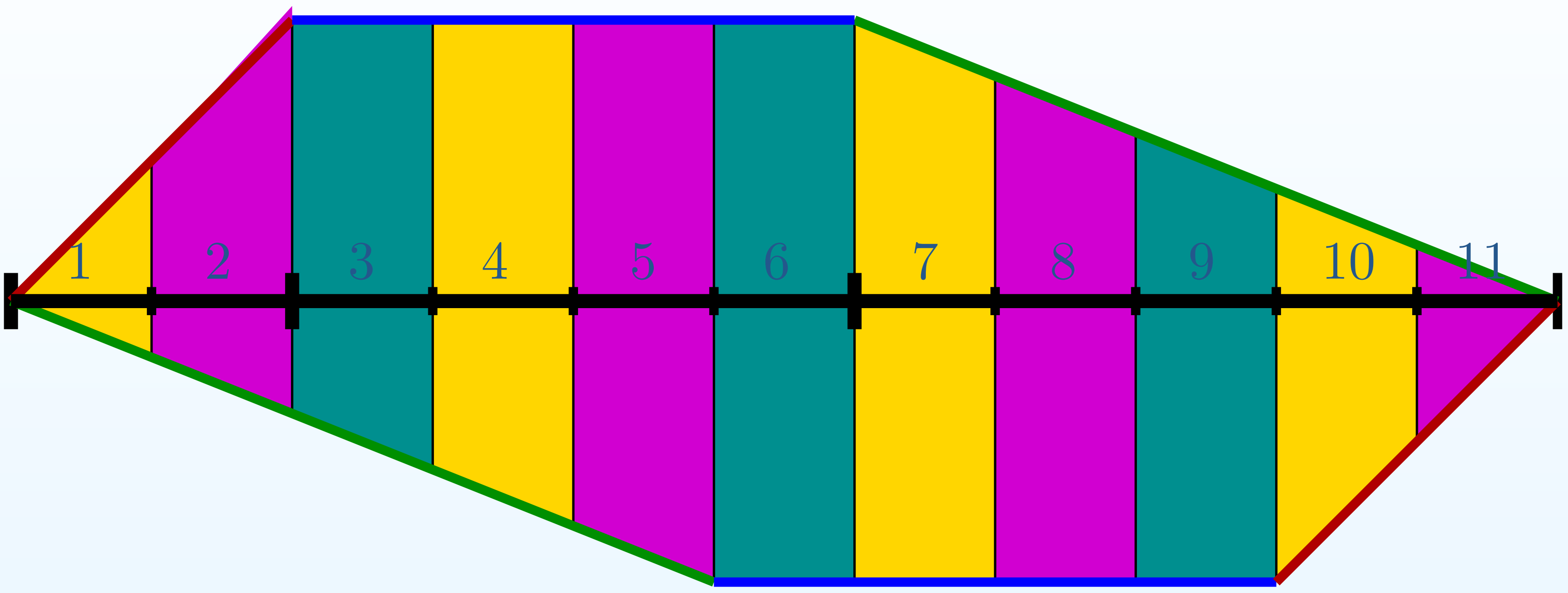


Choosing $|A| = 2, |B| = 5, |C| = 4$ we get the following cyclic decomposition of the resulting (C, B, A)-permutation:

$$(1, 10, 3, 5, 7, 9, 2, 11, 4, 6, 8)$$

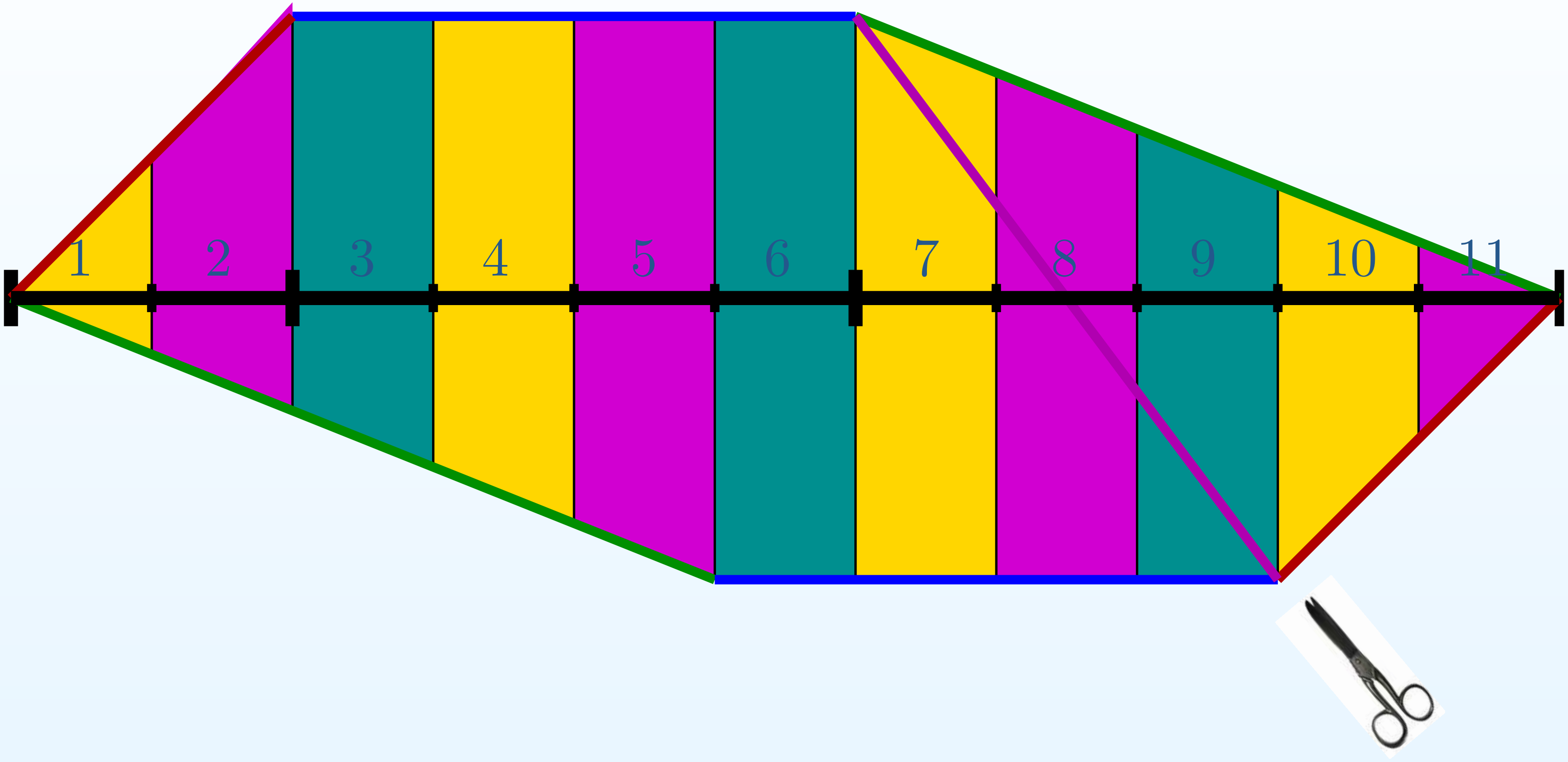
Our permutation acts transitively on the set $\{1, \ldots, 11\}$. This permutation is "transitive" in the sense of Arnold.

Suspension flow on the flat torus



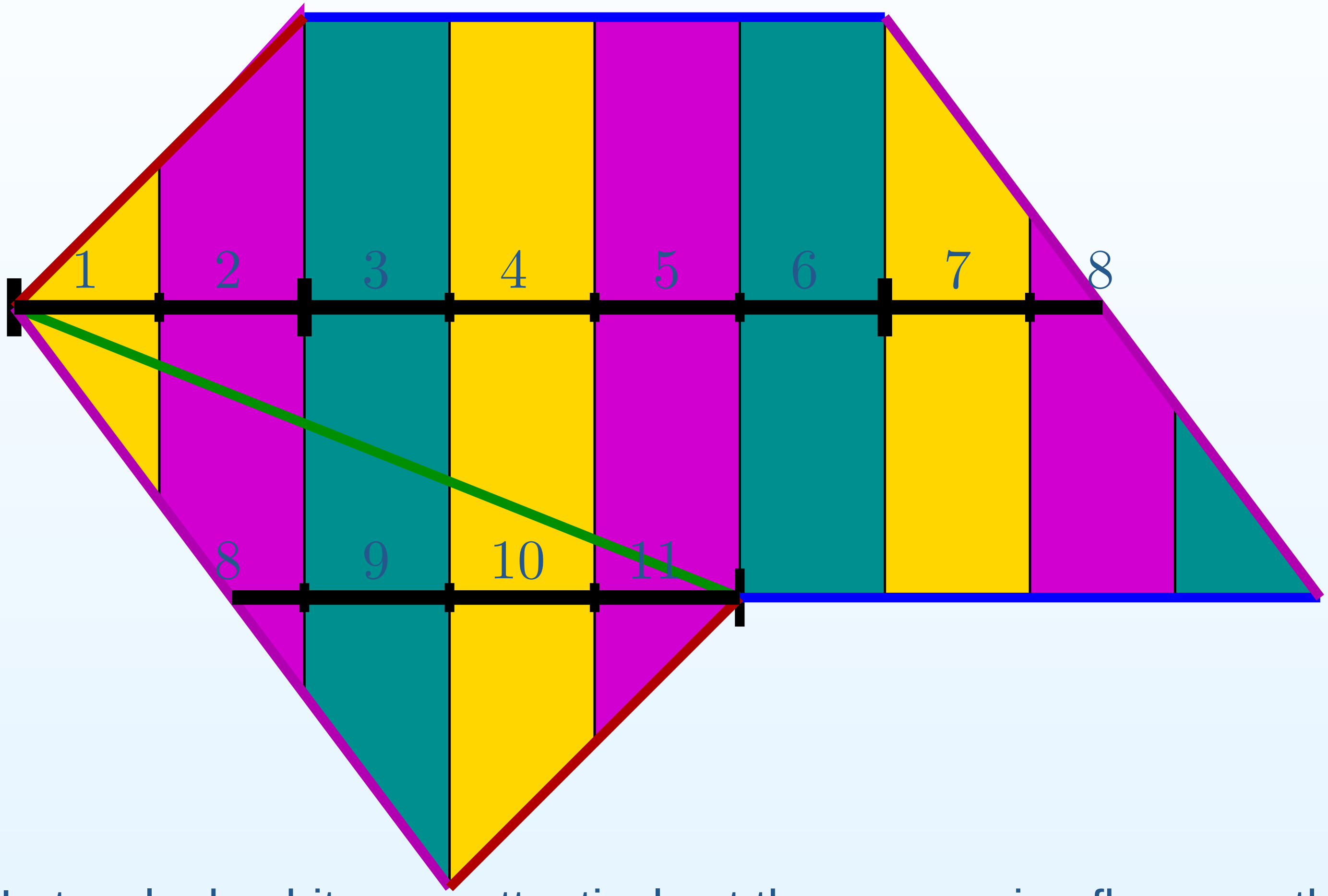
Let us look a bit more attentively at the suspension flow over the interval exchange transformation on the resulting flat surface.

Suspension flow on the flat torus



Let us look a bit more attentively at the suspension flow over the interval exchange transformation on the resulting flat surface. For this we modify the polygonal pattern of the surface by cutting a triangle on the right and placing it on the left.

Suspension flow on the flat torus



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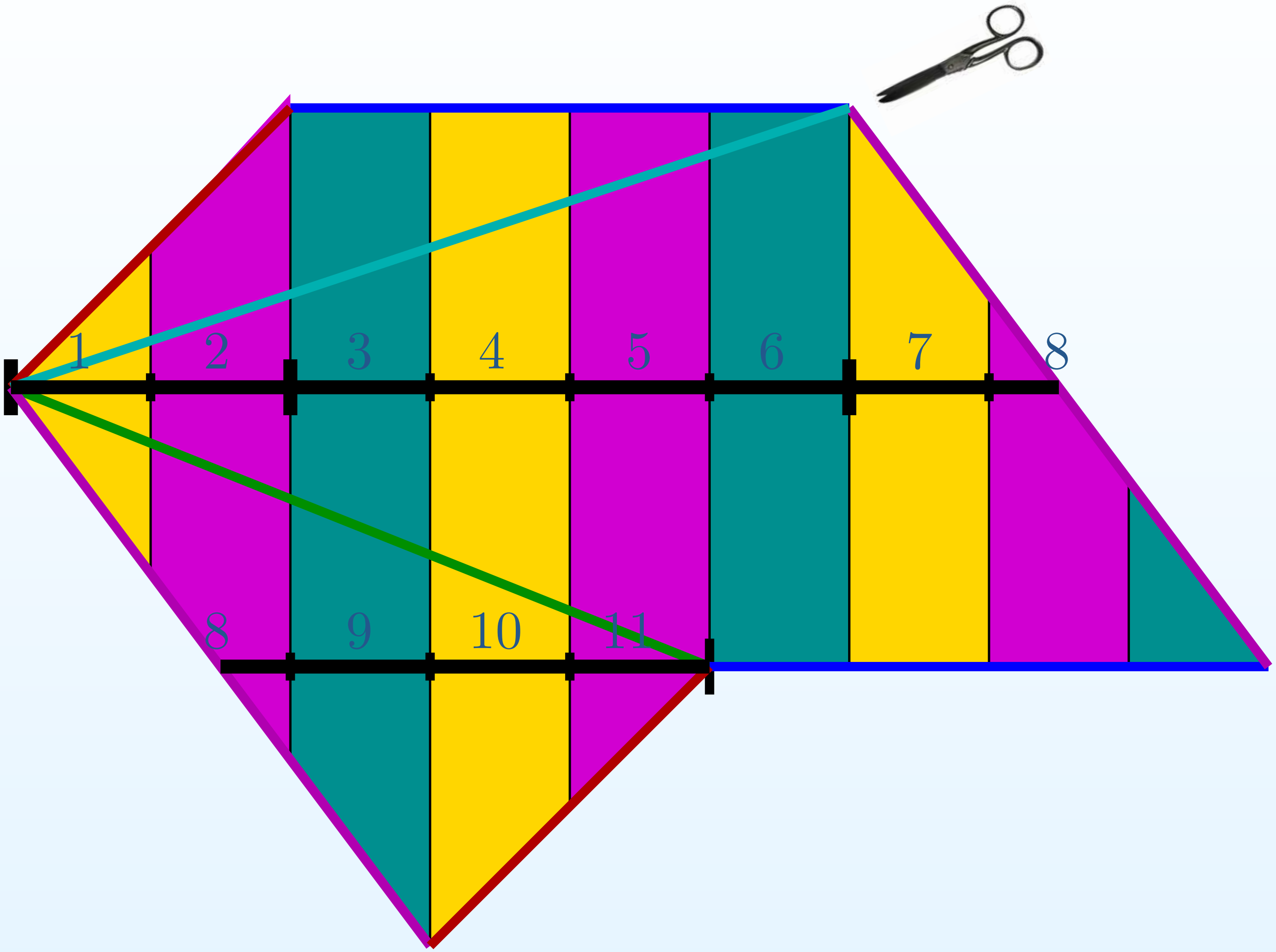

And then we chop and paste one more triangle

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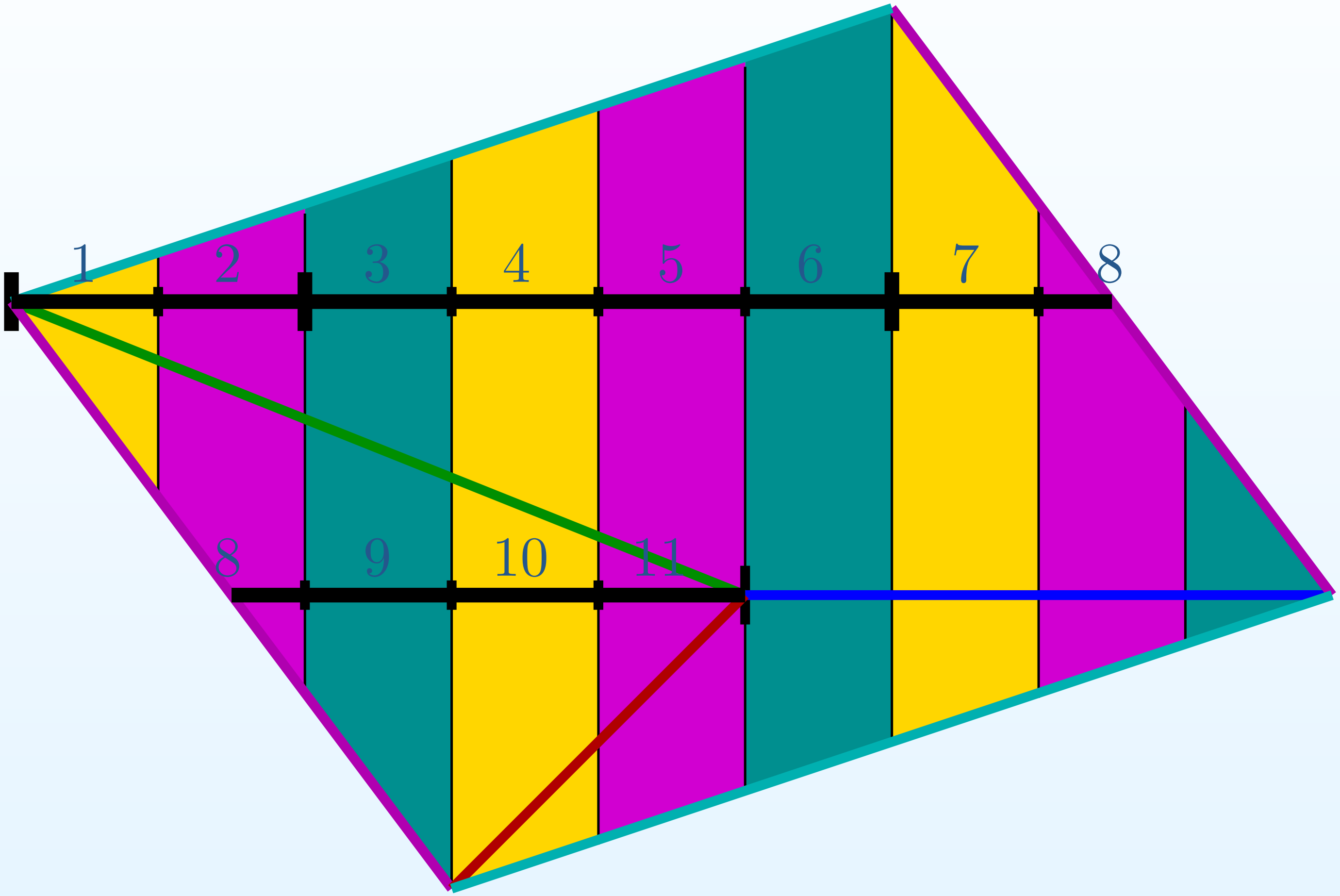

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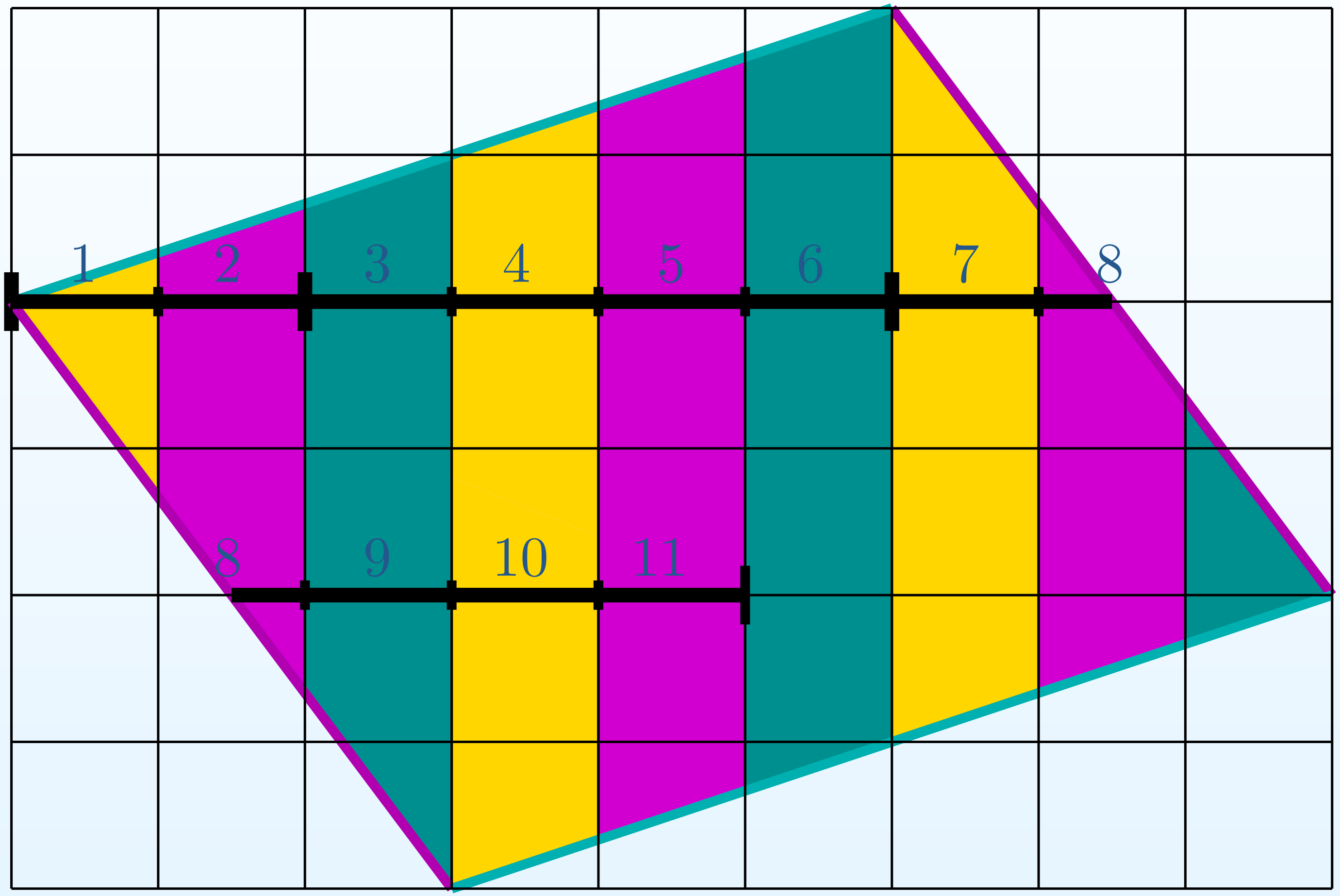


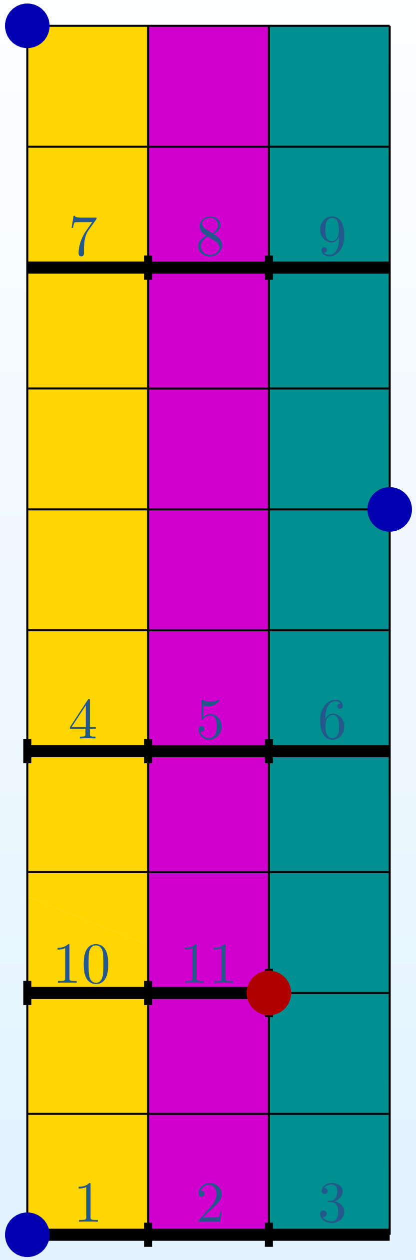
Placing the vertices of the suspension at the points of the lattice $\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$, we get the completely periodic vertical flow on a flat torus with integral sides.

Suspension flow on the flat torus

It is convenient to represent the torus as a cylinder filled with closed vertical trajectories with twisted identification of the vertical boundary circles. We recover once again the decomposition of our permutation into disjoint cycles:

$$(1, 10, 4, 7)(2, 11, 5, 8)(3, 6, 9)\,.$$

In this case it is non-cyclic, because the endpoints of the interval belong to distinct vertical leaves.

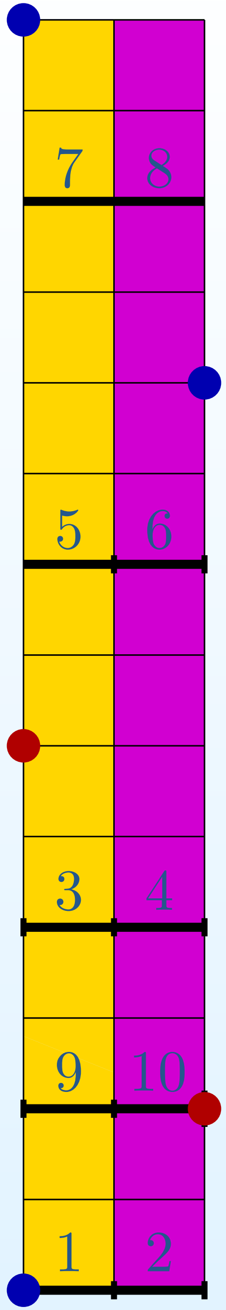


Suspension flow on the flat torus

The picture on the right illustrates the vertical flow on the torus corresponding to the second permutation, having the cyclic structure

$$(1,9,3,5,7)(2,10,4,6,8)\,.$$

In this case of cyclic but non-transitive permutation both endpoints of the interval belong to the same vertical leaf, but the torus is composed of several (in our case two) vertical bands of squares.

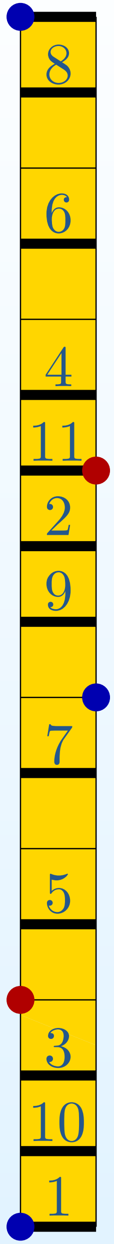


Suspension flow on the flat torus

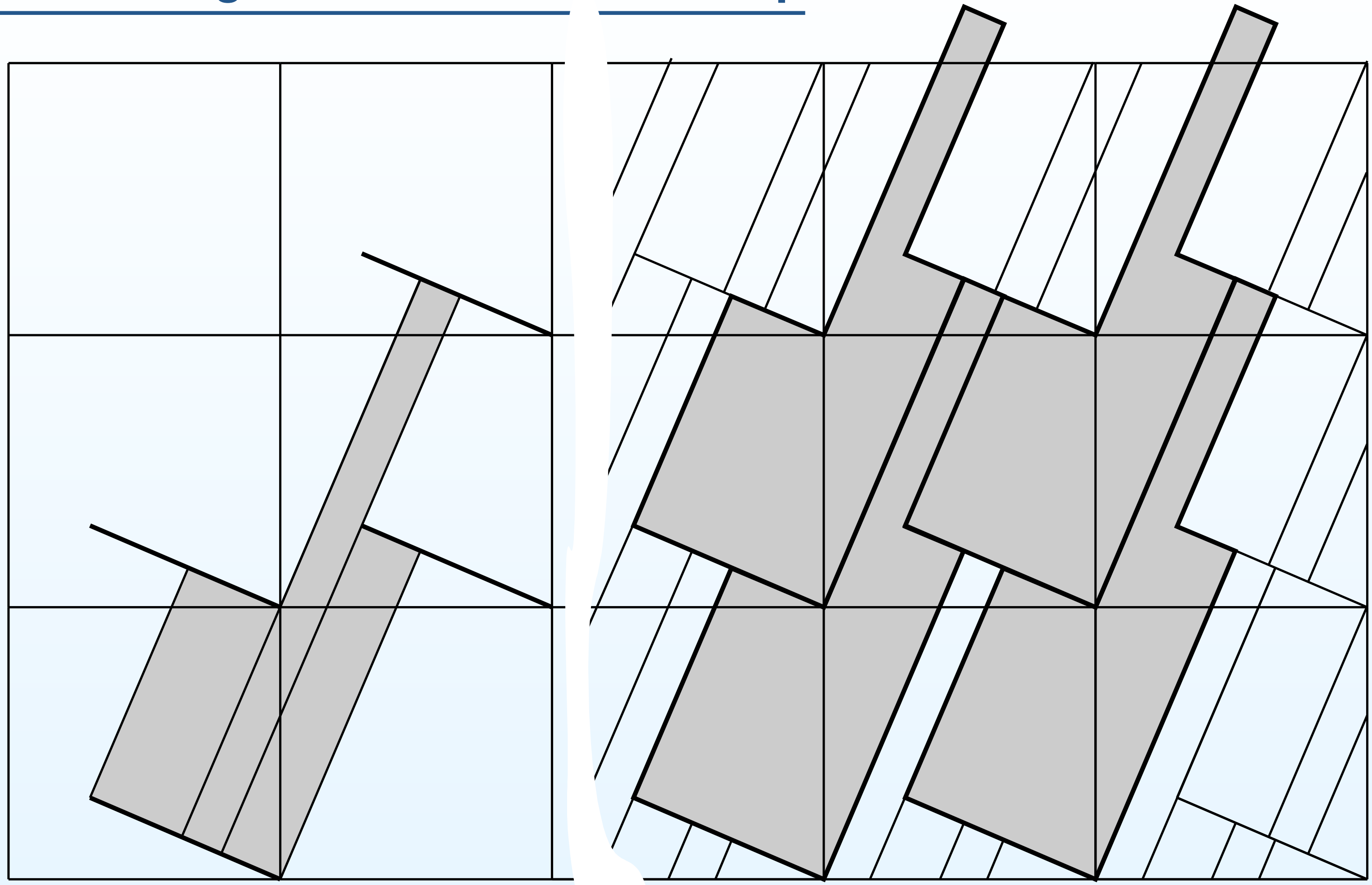
The last picture illustrates the vertical flow on the torus corresponding to the transitive permutation, having the cyclic structure

$$(1, 10, 3, 5, 7, 9, 2, 11, 4, 6, 8)\,.$$

In this case the torus is composed of a single vertical band of squares. Both endpoints of the interval under exchange necessarily belong to the same vertical leaf.



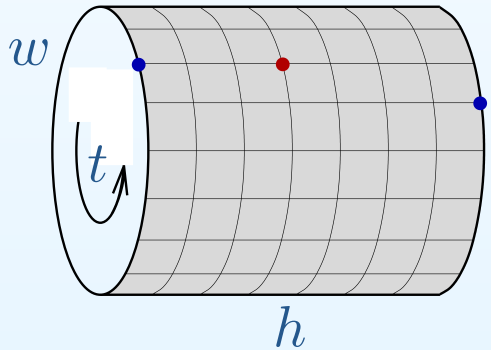
Interval exchange as the first return map

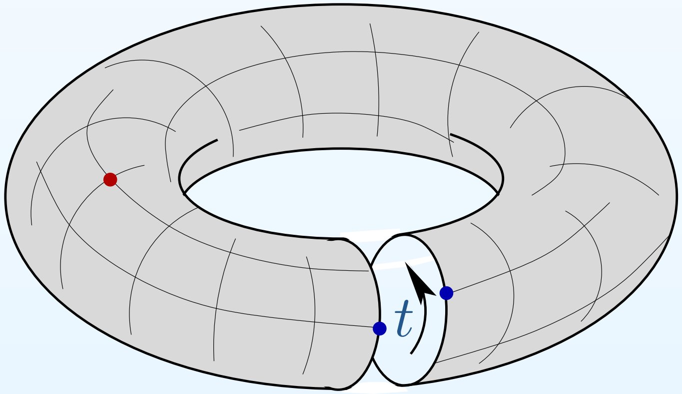


Directional flow on a torus. The first return map of a segment to itself is an interval exchange transformation of three subintervals unless the endpoints of the interval accidentally belong to the same trajectory, in which case we get an interval exchange transformation of two subintervals.

Count of square-tiled tori with two marked points

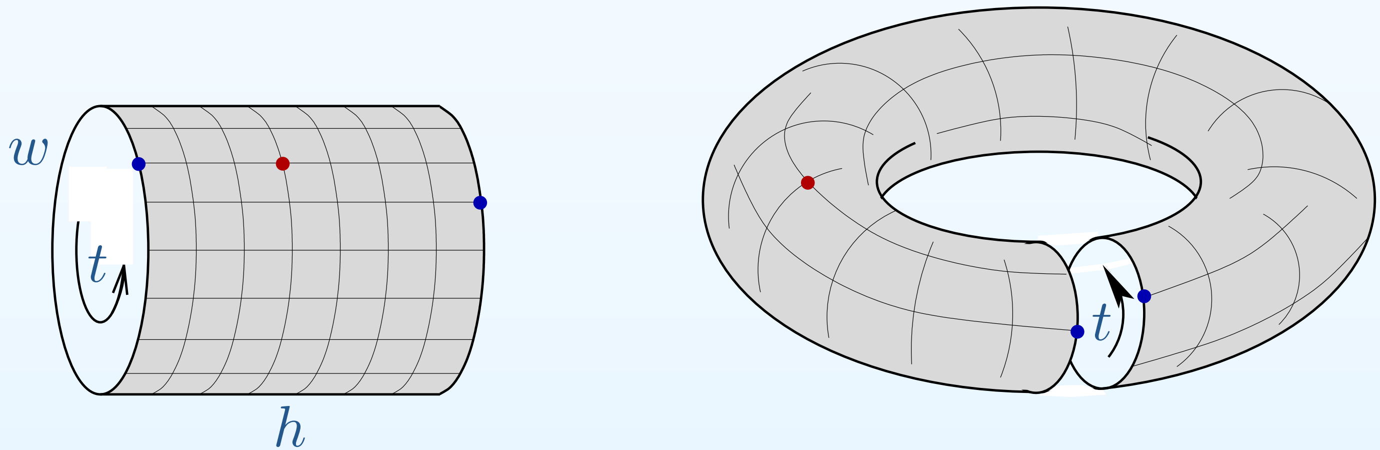
Let us count the number of square-tiled tori with two labeled marked points located at a pair of corners of the squares assuming that the tori are tiled by at most $N \gg 1$ squares. Cutting our flat torus by a vertical waist curve we get a cylinder with a waist curve of length $w \in \mathbb{N}$ and a distance between boundaries $h \in \mathbb{N}$. The number of squares in the tiling equals $w \cdot h$.





Count of square-tiled tori with two marked points

We can glue a torus from a cylinder with some integer twist t. Making an appropriate Dehn twist along the waist curve we can reduce the value of the twist t to one of the values $0, 1, \ldots, w-1$. Fixing the integer perimeter w and height h of a cylinder we get w square-tiled tori. There are $(w \cdot h - 1)$ ways to place two labeled marked points at a pair of distinct corners of squares.

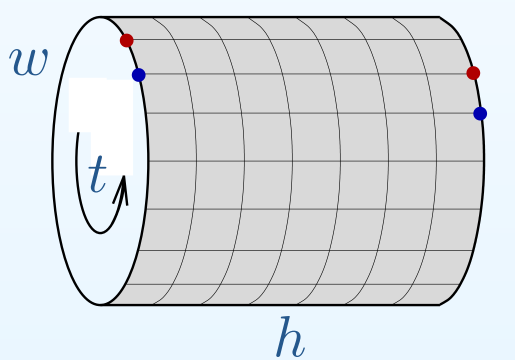


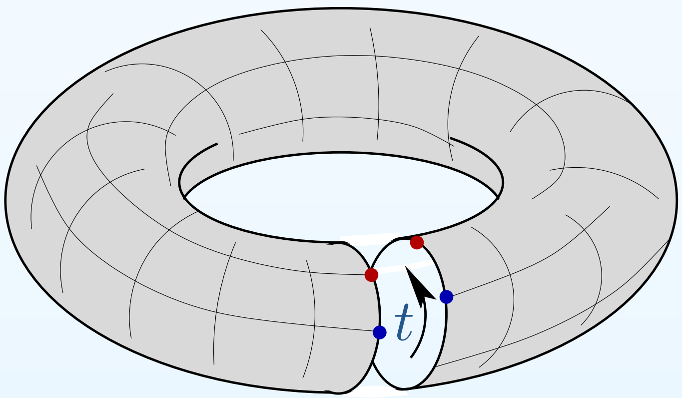
Number of such tori tiled with at most N squares $= \sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w(w\cdot h - 1) \approx$

$$\sum_{h\in\mathbb{N}} \sum_{\substack{w\in\mathbb{N}\\ w\le \frac{N}{h}}} w^2 h \approx \sum_{h\in\mathbb{N}} \frac{1}{3}\cdot\left(\frac{N}{h}\right)^3 \cdot h = \frac{N^3}{3}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^3}{3}\cdot\zeta(2) = \frac{N^3}{3}\cdot\frac{\pi^2}{6}.$$

Count when the two marked points are at the same leaf

We can perform a similar count in the case when both marked points belong to the same vertical leaf and when the number h of circular vertical bands of squares is fixed. Now, in addition to the choice of the twist parameter $t \in \{0, 1, \ldots, w-1\}$ there are $(w-1)$ ways to place two labeled marked points at the vertical leaf of length w.





The count of the number of tori as above tiled with at most N squares gives

$$\sum_{\substack{w\in\mathbb{N}\\ w\cdot h\le N}} w(w-1) \approx \sum_{\substack{w\in\mathbb{N}\\ w\le \frac{N}{h}}} w^2 \approx \frac{1}{3}\cdot\left(\frac{N}{h}\right)^3 = \frac{N^3}{3}\cdot\frac{1}{h^3}\,.$$

Asymptotic proportions

Unexpectedly, the restricted count gives the same order of magnitude N^3.

We conclude that the proportion $p_{1;h}(N)$ of square-tiled tori tiled with at most N squares satisfying the extra conditions:
— they have exactly h vertical circular bands;
— they have both marked points on the same vertical leaf

satisfies
$$\lim_{N\to\infty} p_{1;k}(N) = \frac{1}{h^3}\frac{1}{\zeta(2)} = \frac{1}{h^3}\frac{6}{\pi^2}\,.$$

In particular,
$$\lim_{N\to\infty} p_{1;1}(N) = \frac{6}{\pi^2}\,.$$

Denote by $p_{tr}(N)$ the proportion of transitive permutations among all (C,B,A)-permutations of at most N elements.

Theorem (I. Pak, A. Redlich, 2008)
$$\lim_{n\to+\infty} p_{tr}(N) = \frac{6}{\pi^2}\,.$$

In the next section we will see that the equality between the two limits is not a coincidence and we will give a complete answer to Arnold's problem.

Approach to Arnold's problem

Canonical suspension over an interval exchange

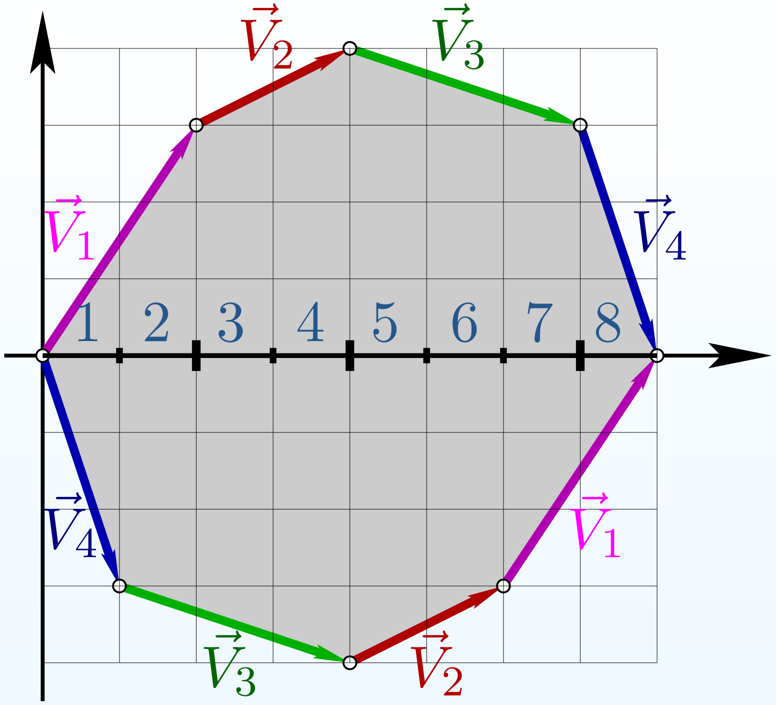
Consider an interval exchange transformation (iet) $T = (\pi, \lambda)$ of n subintervals, where T chops the interval $[0, \lambda_1 + \cdots + \lambda_n[$ into n consecutive subintervals of lengths $\lambda_1, \ldots, \lambda_n$ and places them on X preserving the orientation in the order $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ without gaps or overlaps.

We always assume that T does not send consecutive intervals to consecutive intervals, that is $\pi(j+1) \neq \pi(j) + 1$ for $j = 1, \ldots, n-1$. (This condition is slightly weaker than the standard *nondegeneracy* condition of an iet). We also assume that π does not have nontrivial invariant subsets of the form $\{1, ..., k\}$ (otherwise T acts independently on two disjoint intervals).

Canonical suspension over an interval exchange

Consider a broken line in the plane formed from vectors $\vec{V}_j = (\lambda_j, \pi(j) - j)$ and another broken line starting from the same point and composed from the same vectors now placed in the order $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ (as subintervals under exchange). Identifying the corresponding pairs of sides of the resulting polygon by parallel translations, we get a flat surface. The vertical flow on this surface realizes a suspension flow over the initial interval exchange. By convention, we mark the two points of the surface coming from the two vertices of the polygon corresponding to the endpoints of the broken lines.

Example of suspension



Suspension over an interval exchange transformation $T(\pi, \lambda)$ with parameters

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \lambda = (2, 2, 3, 1)\,.$$

The associated *interval exchange permutation* has the form

$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 3 & 4 & 1 & 2 \end{pmatrix}\,.$$

Vectors $\vec{V}_j = (\lambda_j, \pi(j) - j)$ of the canonical suspension have coordinates

$$\vec{V}_1 = (2, 3) \quad \vec{V}_2 = (2, 1) \quad \vec{V}_3 = (3, -1) \quad \vec{V}_4 = (1, -3)\,.$$

Bands of periodic trajectories

Definition. We say that cycles C_1 and C_2 of a permutation τ belong to the same *band*, if one can chose $k_1 \in C_1$ and $k_2 \in C_2$ such that

$$\tau^{(j)}(k_2) = \tau^{(j)}(k_1) + 1 \quad \text{or} \quad \tau^{(j)}(k_2) = \tau^{(j)}(k_1) - 1 \quad \text{for all } j \in \mathbb{Z}$$

and we consider the minimal equivalence relation induced by this property.

The permutation $(1, 10, 4, 7)(2, 11, 5, 8)(3, 6, 9)$ has two bands of cycles, where the cycles $(1, 10, 4, 7)$ and $(2, 11, 5, 8)$ belong to the same band. The permutation $(1, 9, 3, 5, 7)(2, 10, 4, 6, 8)$ has a single band of cycles.

We have seen that a (C, B, A)-permutation has a single band of cycles if and only if it is “cyclic” in the sense of Arnold; it has two band of cycles otherwise.

Important Observation. *Consider a permutation τ associated to an integer interval exchange transformation (π, λ), where $\lambda \in \mathbb{N}^n$. The number of bands of cycles of τ coincides with the number of maximal cylinders of the vertical suspension flow on the associated flat surface. (By convention, we mark the points on the surface (possibly a single point) corresponding to the endpoints of the interval if they are nonsingular points of the flat metric.)*

Why “bands of cycles” and not just cycles?

Fix a permutation π and consider statistics of the number of cycles of a random *interval exchange permutation* $\tau(\lambda,\pi)$ associated to an integer interval exchange transformation $T(\lambda,\pi)$ of the interval $[0,N[$ as $N\to\infty$. By “integer” interval exchange we call one with $\lambda\in\mathbb{N}^d$, where $d=\mathrm{Card}(\pi)$.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2022). *For any permutation π the mean value of the number of cycles of a random interval exchange permutation $\tau(\lambda,\pi)$ is infinite.*
For any stratum of Abelian differentials, the mean value of the number of vertical (horizontal) bands of squares of a random square-tiled surface in this stratum is infinite.

Remark. Note that for numerous separatrix diagrams, the corresponding mean value for square-tiled surfaces representing these particular diagrams is finite!

The above Theorem explains why an adequate interpretation of Arnold’s problem (the most general question about Young diagrams) suggests to consider *bands of cycles* and not cycles themselves.

Enhanced solution of Arnold's problem

Let π be a non degenerate irreducible permutation. Let $\mathcal{H}^{comp}(m_1, \dots, m_n)$ be a connected component of a stratum of Abelian differentials ambient for the canonical suspension over an interval exchange with a permutation π.

Let $d = \operatorname{Card} \pi$ be the number of elements in π.

Let $\operatorname{Vol} \mathcal{H}^{comp}(m_1, \dots, m_n)$ and $\operatorname{Vol}_k \mathcal{H}^{comp}(m_1, \dots, m_n)$ be respectively the Masur–Veech volume of the component and the contribution of k-cylinder square-tiled surfaces to this volume.

Let U be an open bounded set in $\mathbb{R}^d_+$. Denote by tU the set obtained from U by dilation with coefficient $t \in \mathbb{R}$. Denote by $IET(\pi, U, \varepsilon)$ and by $IET_k(\pi, U, \varepsilon)$ respectively the number of (π, λ)-integral interval exchange transformations such that $\lambda \in \mathbb{N}^d \cap \frac{1}{\varepsilon} U$ and the number of those of them, which have exactly k bands of periodic vertical trajectories.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) For any π, U, k as above one has

$$\lim_{\varepsilon \to +0} \frac{IET_k(\pi, I, \varepsilon)}{IET(\pi, I, \varepsilon)} = \frac{\operatorname{Vol}_k \mathcal{H}^{comp}(m_1, \dots, m_n)}{\operatorname{Vol} \mathcal{H}^{comp}(m_1, \dots, m_n)} \,.$$