2020-9-18 Kähler geometry

①

$e_1, \dots, e_r$ frame

$$\nabla e_j = \theta^i{}_j\, e_i \qquad \boxed{T e_j = a^i{}_j\, e_i}$$

$$\nabla_X e_j = \theta^i{}_j(X)\, e_i$$

So if $s = s^1 e_1 + \dots + s^r e_r$

$$\nabla_X s = (X s^1) e_1 + \dots + (X s^r) e_r + s^1 \nabla_X e_1 + \dots + s^r \nabla_X e_r$$

$$= (X s^i) e_i + s^j \theta^i{}_j (X) e_i$$

$$= (ds^i + \theta^i{}_j s^j)(X)\, e_i$$

$$\nabla \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix} \doteq d \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix} + \theta \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix}$$

<u>Def</u> A Riemannian metric $g \in C^\infty(T^*M \otimes T^*M)$ on M is the one satisfying

$$g : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(X, Y) \longmapsto g(X, Y)$$

is an inner product. i.e. symmetric, positive definite

Theorem-Definition (homework)

Given a Riemannian metric on M there is a unique connection of the tangent bundle TM satisfying

$$\begin{cases} (1) & X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ (2) & \nabla_X Y - \nabla_Y X = [X,Y]. \end{cases}$$

(1) metric compatible

(2) torsion-free, symmetric

$\begin{cases} (M,g) & \text{Riemannian mfd} \\ \nabla : & \text{Riemannian connection, Levi-Civita connection} \end{cases}$

When ∇ is defined on TM, it also defines a connection on T^*M by duality:

$$\alpha \in C^\infty(T^*M), \quad X \in C^\infty(TM)$$

$$(*) \quad (\nabla_Y \alpha)(X) = Y(\alpha(X)) - \alpha(\nabla_Y X).$$

Further ∇ extends also to tensor products

$$\left(\overset{p}{\otimes} TM\right) \otimes \left(\overset{q}{\otimes} T^*M\right)$$

by derivation.

<u>Remark</u> $f \in C_a^\infty(M)$, we set

$$\nabla f := df$$

$$\nabla_Y(\alpha \otimes X) = (\nabla_Y \alpha) \otimes X + \alpha \otimes \nabla_Y X$$

$c : TM \otimes T^*M \to C^\infty(M)$ evaluation

"contraction"

$$(*) \Leftrightarrow \nabla_X(c(\alpha \otimes X)) = c(\nabla_Y(\alpha \otimes X))$$

" the contraction and the covariant derivative commute."

<u>local tensor calculus.</u>

choose a local coordinate system $(x^1, \dots, x^n)$. Then

$s \in C^\infty((\overset{p}{\otimes} TM) \otimes (\overset{q}{\otimes} T^*M))$ can be expressed as

$$s = s^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$$

$$e_i = \frac{\partial}{\partial x^i}, \quad e^i = dx^i$$

$$\nabla S = \nabla_k S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q} \otimes dx^k$$

$$\cap$$

$$C^\infty((\overset{p}{\otimes} TM) \otimes (\overset{q}{\otimes} T^*M) \otimes T^*M)$$

This is the definition of $\nabla_k S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$.

$$g = g_{ij}\, dx^i \otimes dx^j \quad \text{metric } TM.$$

$$\cap$$

$$C^\infty(T^*M \otimes T^*M)$$

$$g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \qquad (g^{ij}) = (g_{ij})^{-1}$$

metric on T^*M

Using Riemannian metric one can define an isomorphism

$$\begin{array}{ccc} TM & \xrightarrow{\sim} & T^*M \\ \cup & & \cup \\ X & \longmapsto & X^b \end{array}$$

by $(X^b, Y) \overset{\text{def}}{=} g(X, Y)$.

$$(X^\flat, \frac{\partial}{\partial x^i}) = g(X^j \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}) = g_{ij} X^j$$

$$\therefore X^\flat = g_{ij} X^j dx^i$$

$$\therefore (X^\flat)_i = g_{ij} X^j$$

$$X_i := g_{ij} X^j$$

$\flat$: musical isomorphism.

In the same way, for $\alpha = \alpha_i dx^i$

$$\alpha^\sharp = g^{ij} \alpha_i \frac{\partial}{\partial x^j} \in C^\infty(TM).$$

$$\alpha^j := g^{ij} \alpha_j$$

One may also define

$$\nabla^i \alpha_j := g^{ik} \nabla_k \alpha_j .$$

Then

$$\nabla^i (g_{jl} X^l) \overset{①}{=} \nabla^i X_j \overset{②}{=} g_{jk} (\nabla^i X^k)$$

two interpretations of $\nabla^i X_j$

But these two are compatible because

Lemma $\nabla g = 0$ "g is parallel"

$$(\nabla_X g)(Y, Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \quad \text{by (1)}$$

We define

$$R \in C^\infty(T^*M \otimes T^*M \otimes T^*M \otimes T^*M) = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

by

$$R_{ijkl} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = g\left(\frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}\right)$$

("Riemannian" curvature tensor)

If we set

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}}\right) \frac{\partial}{\partial x^l} = U^p \frac{\partial}{\partial x^p}$$

$$g\left(\frac{\partial}{\partial x^k}, U\right) = U^p g_{kp} = R_{ijkl}$$

$$\therefore \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} = R_{ij}{}^{k}{}_{l} \frac{\partial}{\partial x^k}$$

For $s = s^l \frac{\partial}{\partial x^l}$

check yourself

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \right) s = s^l R_{ij}{}^k{}_l \frac{\partial}{\partial x^k}$$

$$\left(= \frac{\partial}{\partial x^k} \cdot \underline{R_{ij}{}^k{}_l \, s^l} \right.$$

$$\left(T(e_1 \cdots e_r) = (e_1 \cdots e_r) \underline{A} \qquad T(x^j e_j) = e_i \, a^i{}_j \, x^j \right)$$

This means

$$\nabla_i \nabla_j s^k - \nabla_j \nabla_i s^k = R_{ij}{}^k{}_l \, s^l$$

<u>Ricci identity</u>

equivalent to the definition of curvature

$$\left(\underline{\nabla_i \nabla_j} \bigcirc = \underline{\nabla_j \nabla_i} \bigcirc + \underline{\text{curvature}} \right)$$

In the same way

$$\nabla_i \nabla_j s_k - \nabla_j \nabla_i s_k = (\nabla_i \nabla_j - \nabla_j \nabla_i)(g_{kp} s^p)$$

$$= g_{kp} (\nabla_i \nabla_j - \nabla_j \nabla_i) s^p = g_{kp} R_{ij}{}^p{}_l \, s^l$$

$$= R_{ijkl} s^l = R_{ijk}{}^p s_p$$

$= - R_{ij}{}^{k}{}_{l}\, s_k$ ← Ricci identity for 1-forms.

By the following homework.

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W)$

(2) $R(X, Y, Z, W) = R(Z, W, X, Y)$

(3) $R(X, Y, Z, W) + R(Y, W, Z, X) + R(W, X, Z, Y) = 0.$ (Jacobi identity)

the first Bianchi identity

$$\left\{ \begin{aligned} & R(X,Y)Z = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} Z \\ & R(X,Y)Z + R(Y,Z)X + R(Z,X,Y) \\ & \quad = [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \\ & \quad = 0 \end{aligned} \right.$$

(4) $(\nabla_X R)(Y, Z, W, V) + (\nabla_Y R)(Z, X, W, V) + (\nabla_Z R)(X, Y, W, V) = 0.$

the second Bianchi identity

Def $R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$

$R(X, Y, Z, W) = g(Z, R(X,Y)W)$

Both R are called the Riemannian curvature tensor.

Def $J \in C^\infty(\mathrm{End}(TM)) \cong C^\infty(TM \otimes T^*M)$
is called an almost complex structure if
$J^2 = -\mathrm{id}$.

Def (M, J) is called an almost complex manifold.

When is (M, J) a complex manifold ?

Example : Let M be a complex manifold.
$(z^1, \ldots, z^m)$ local holo coordinates
$$z^i = x^i + \sqrt{-1}\, y^i$$
$(x^1, y^1, \ldots, x^m, y^m)$ real local coordinates