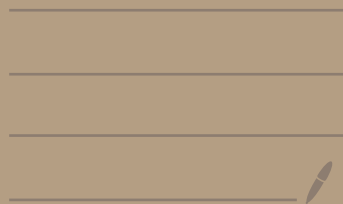


2020-9-18 Kähler geometry



e_1, \dots, e_r frame

(1)

$$\nabla e_j = \theta^i_j e_i$$

$$\tau e_j = a^i_j e_i$$

$$\nabla_x e_j = \theta^i_j(x) e_i$$

So if $s = s^1 e_1 + \dots + s^r e_r$

$$\begin{aligned} \nabla_x s &= (x s^1) e_1 + \dots + (x s^r) e_r \\ &\quad + s^1 \nabla_x e_1 + \dots + s^r \nabla_x e_r \end{aligned}$$

$$= (x s^i) e_i + s^j \theta^i_j(x) e_i$$

$$= (ds^i + \theta^i_j s^j)(x) e_i$$

$$\nabla \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix} \stackrel{\cdot}{=} d \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix} + \theta \begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix}$$

Def A Riemannian metric $g \in C^\infty(T^*M \otimes T^*M)$ on M is the one satisfying

$$g: T_p M \times T_p M \rightarrow \mathbb{R}$$

$$\downarrow \\ (x, T) \mapsto g(x, T)$$

is an inner product,
i.e. symmetric,
positive definite

Theorem - definition (homework)

(2)

Given a Riemannian metric on M
there is a unique connection of the
tangent bundle TM satisfying

$$\begin{cases} (1) & X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ (2) & \nabla_X Y - \nabla_Y X = [X, Y], \end{cases}$$

(1) metric compatible

(2) torsion-free, symmetric

{ (M, g) Riemannian mfd
 ∇ : Riemannian connection, Levi-Civita connection

When ∇ is defined on TM , it also defines
a connection on T^*M by duality:

$$\alpha \in C^0(T^*M), \quad X \in C^\infty(TM)$$

$$(*) \quad (\nabla_Y \alpha)(X) = Y(\alpha(X)) - \alpha(\nabla_Y X).$$

Further ∇ extends also to tensor products

$$\left(\otimes^p TM\right) \otimes \left(\otimes^q T^*M\right)$$

by derivation.

Remark $f \in C_a^\infty(M)$, we set $\nabla f := df$

$$\nabla_Y (\alpha \otimes X) = (\nabla_Y \alpha) \otimes X + \alpha \otimes \nabla_Y X$$

$c: TM \otimes T^*M \rightarrow C^\infty(M)$ evaluation
 "contraction"

$$(*) \Leftrightarrow \nabla_X (c(\alpha \otimes X)) = c(\nabla_X (\alpha \otimes X))$$

"the contraction and the covariant derivative commute."

local tensor calculus,

choose a local coordinate system (x^1, \dots, x^m) . Then

$S \in C^\infty((\otimes^p TM) \otimes (\otimes^q T^*M))$ can be expressed as

$$S = s^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

$$e_i = \frac{\partial}{\partial x^i}, \quad e^i = dx^i$$

$$\nabla_s = \nabla_R s^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

$$C^\infty(\left(\otimes^p TM\right) \otimes \left(\otimes^q T^*M\right) \otimes T^*M)$$

This is the definition of $\nabla_R s^{i_1 \dots i_p}_{j_1 \dots j_q}$.

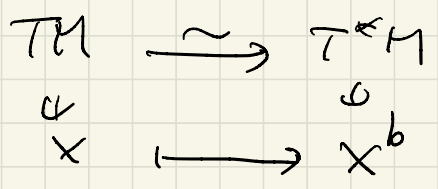
$$g = g_{ij} dx^i \otimes dx^j \quad \text{metric on } TM.$$

$$C^\infty(T^*M \otimes T^*M)$$

$$g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \quad (g^{ij}) = (g_{ij})^{-1}$$

metric on T^*M

Using Riemannian metric we can define an isomorphism



$$\text{by } (X^b, \gamma) \stackrel{\text{def}}{=} g(X, \gamma)$$

$$\left(X^b, \frac{\partial}{\partial x^i} \right) = g \left(X^j \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right) = g_{ij} X^j \quad (3)$$

$$\therefore X^b = g_{ij} X^j dx^i$$

$$\therefore (X^b)_i = g_{ij} X^j$$

$$X_i := g_{ij} X^j$$

b : musical isomorphism.

In the same way, for $\alpha = \alpha_i dx^i$

$$\alpha^\sharp = g^{ij} \alpha_i \frac{\partial}{\partial x^j} \in C^\infty(TM)$$

$$\alpha^\sharp := g^{ij} \alpha_j$$

One may also define

$$\nabla^i \alpha_j := g^{ik} \nabla_k \alpha_j$$

Then

$$\nabla^i (g_{jk} X^k) \stackrel{(1)}{=} \nabla^i X_j \stackrel{(2)}{=} g_{jk} (\nabla^i X^k)$$

two interpretations of $\nabla^i X_j$

But these two are compatible because

Lemma $\nabla g = 0$ "g is parallel" (6)

$$\begin{aligned} \textcircled{1} \quad (\nabla_X g)(Y, Z) &= X(g(Y, Z)) \\ &\quad - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= 0 \quad \text{by (1)} \end{aligned}$$

We define

$$\begin{aligned} R &\in C^\infty(T^*M \otimes T^*M \otimes T^*M \otimes T^*M) \\ &= R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \\ R_{ijkl} &= R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \\ &= g\left(\frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}\right) \end{aligned}$$

("Riemannian" curvature tensor)

If we set

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}}\right) \frac{\partial}{\partial x^k} = U^p \frac{\partial}{\partial x^p}$$

$$g\left(\frac{\partial}{\partial x^k}, U\right) = U^p g_{kp} = R_{ijkl}$$

$$\therefore \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = R_{ij}{}^k{}_l \frac{\partial}{\partial x^l}$$

$$\text{For } s = s^l \frac{\partial}{\partial x^l}$$

check yourself

(7)

$$\left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \right) s = s^l R_{ij}{}^k{}_{} \frac{\partial}{\partial x^k}$$

$$\left(= \frac{\partial}{\partial x^k} \cdot \underline{\underline{R_{ij}{}^k{}_{}}} \cdot s^l \right)$$

$$\left(\begin{array}{l} T(e_i - e_j) = (e_i - e_j) \cdot A \\ (x^i_j) = e_i \cdot a^i_j \cdot x^j \end{array} \right)$$

This means

$$\nabla_i \nabla_j s^k - \nabla_j \nabla_i s^k = R_{ij}{}^k{}_{} s^l$$

Ricci identity

equivalent to the definition of curvature

$$\left(\frac{\nabla_i \nabla_j \bigcirc}{=} = \frac{\nabla_j \nabla_i \bigcirc}{=} \right) \quad \underline{\underline{\neq \text{curvature}}}$$

In the same way

$$\begin{aligned} \nabla_i \nabla_j s_k - \nabla_j \nabla_i s_k &= (\nabla_i \nabla_j - \nabla_j \nabla_i) (g_{kp} s^p) \\ &= g_{kp} (\nabla_i \nabla_j - \nabla_j \nabla_i) s^p = g_{kp} R_{ij}{}^p{}_{} s^l \\ &= R_{ij}{}^k{}_{} s^l = R_{ij}{}^k{}^p s_p \end{aligned}$$

$$= -R_{ij}^k s_k \quad \leftarrow \text{Ricci identity in } 1\text{-form.} \quad (8)$$

By the following homework.

$$(1) R(X, Y, Z, W) = -R(Y, X, Z, W)$$

$$(2) R(X, Y, Z, W) = R(Z, W, X, Y)$$

$$(3) R(X, Y, Z, W) + R(Y, W, Z, X) + R(W, X, Z, Y)$$

$$= 0. \quad (\text{Jacobi identity})$$

the first Bianchi identity

$$\left\{ \begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\ R(X, Y)Z + R(Y, Z)X + R(Z, X)Y \\ &= [X, Y], Z + [Y, Z], X + [Z, X], Y \\ &= 0 \end{aligned} \right.$$

$$(4) (\nabla_X R)(Y, Z, W, V) + (\nabla_Y R)(Z, X, W, V)$$

$$+ (\nabla_Z R)(X, Y, W, V) = 0.$$

the second Bianchi identity

$$\underline{\text{Def}} \quad R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})Z$$

$$R(X, Y, Z, W) = g(Z, R(X, Y)W)$$

Both R are called the Riemannian curvature tensor.

Def $J \in C^\infty(E \rightarrow \Lambda(TM)) \cong C^\infty(TM \otimes T^*M)$ (9)
is called an almost complex structure if
 $J^2 = -id$.

Def (M, J) is called an almost complex manifold.

When is (M, J) a complex manifold?

Example: Let M be a complex manifold.

(z^1, \dots, z^m) local holo coordinates

$$z^i = x^i + \sqrt{-1} y^i$$

$(x^1, y^1, \dots, x^m, y^m)$ real local coordinates

