Lecture No 11 April 12, 2022 (Tue)

Hilbert space of conti. square-integrable martingales

Fix $^\forall T > 0$ and define

$$\mathcal{M}^2_{c,T} := \{M = (M_t)_{t\in[0,T]};\ \text{square-integrable continuous } (\mathcal{F}_t)\text{-martingales s.t. } M_0 = 0\ a.s.\}.$$

For $M, N \in \mathcal{M}^2_{c,T}$, set

$$\langle M, N\rangle_t := \frac{1}{4}\big(\langle M+N\rangle_t - \langle M-N\rangle_t\big)$$

$$\underset{\text{Fact}}{=} \lim_{|\Delta|\to 0} \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}) \quad \text{in probability.}$$

Then, $\langle M, N\rangle_t$ is of bounded variation in t (as a difference of two increasing functions) $a.s.$ and $(M_t N_t - \langle M, N\rangle_t)$ is a continuous martingale. P: Check these.

We call $\langle M, N\rangle_t$ a cross variation of M and N.

Define an equivalence relation:

$$M \sim N \underset{\text{def}}{\Longleftrightarrow} M_t = N_t \ {}^{\forall} t \in [0, T] \text{ a.s.}$$
$$\Longleftrightarrow P(M_t = N_t \text{ for } {}^{\forall} t \in [0, T]) = 1$$

(M and N are called indistinguishable.)

We identify equivalent elements in $\mathcal{M}^2_{c,T}$.

[Proposition 10.8] $(\mathcal{M}^2_{c,T}, E[\langle \cdot, \cdot \rangle_T])$ is a real Hilbert space. □

[Proof] [Step 1] It is easy to see that $(M, N) := E[\langle M, N \rangle_T]$ $= E[M_T N_T]$ determines an inner product.

⚆ We only note $(M, M) = 0 \Longrightarrow M \sim 0$. Indeed, if $(M, M) = E[M_T^2] = 0$, then $M_T = 0$ a.s. Thus, $M_t = E[M_T | \mathcal{F}_t] = 0$ a.s. ${}^{\forall} t \in [0, T]$. However, since M is continuous, one can show that $M_t = 0$, ${}^{\forall} t \in [0, T]$ a.s. (Note that $[0, T]$ is an uncountable set.) □

[Step 2] Next, we show the completeness of the space $\mathcal{M}^2_{c,T}$. Let $M^n = (M^n_t)_{t\in[0,T]}$, $n = 1, 2, \ldots$ be a Cauchy sequence i.e.

$$E[(M^m_T - M^n_T)^2] \to 0 \quad (m, n \to \infty).$$

Then, by Doob's inequality,

$$E[\sup_{0\le t\le T}(M^m_t - M^n_t)^2] \le 4E[(M^m_T - M^n_T)^2] \to 0.$$

Thus, by taking a subsequence if necessary (see the next page), we see

$$M_t = {}^{\exists}\lim_{n'\to\infty} M^{n'}_t \quad \text{(uniform convergence in } t\text{)}\ a.s.$$

[Explanation for taking a subsequence] More precisely, for every $k = 1, 2, \ldots$, by Chebyshev's inequality,

$$P(\sup_{0 \le t \le T} |M_t^m - M_t^n| \ge 2^{-k}) \le 2^{2k} E[\sup_{0 \le t \le T} (M_t^m - M_t^n)^2] \to 0 \quad (n, m \to \infty),$$

so that one can find an increasing subsequence $\{n_k \nearrow \infty\}$ s.t.

$$P(\sup_{0 \le t \le T} |M_t^m - M_t^{n_k}| \ge 2^{-k}) \le 2^{-k}, \quad {}^\forall m \ge n_k$$

In particular, we obtain

$$P(\sup_{0 \le t \le T} |M_t^{n_{k+1}} - M_t^{n_k}| \ge 2^{-k}) \le 2^{-k}, \quad {}^\forall k = 1, 2, \ldots$$

This implies, by Borel-Cantelli's lemma noting $\sum_k$(RHS)$< \infty$, that

$$P(\sup_{0 \le t \le T} |M_t^{n_{k+1}} - M_t^{n_k}| < 2^{-k}, \text{e.v.}) = 1.$$

Thus, for a.s. ω, ${}^\exists K(\omega)$ s.t. for $k' > k \ge K(\omega)$, $\sup_{0 \le t \le T} |M_t^{n_{k'}} - M_t^{n_k}| < \sum_{i=k}^{k'-1} 2^{-i} \to 0$ as $k \to \infty$. This implies that, for a.s. ω, $M_t^{n_k}(\omega)$ is a Cauchy sequence in $(C([0, T]), \|\cdot\|_\infty)$. □

In particular, M is continuous. The square-integrability of M follows by Fatou's lemma:

$$E[M_t^2] \leq \varliminf_{n' \to \infty} E[(M_t^{n'})^2] < \infty.$$

and the martingale property of M follows from

$$E[M_t^{n'}, A] = E[M_s^{n'}, A], \quad 0 \leq s \leq t, A \in \mathcal{F}_s,$$

by taking the limit $n' \to \infty$ noting that $M_t^{n'}$ converges to M_t in L^2 sense. The proof of the completeness of $\mathcal{M}_{c,T}^2$ is concluded by the following. □

P: Finally show that M^n (itself without taking subsequence) converges to M constructed as above as $n \to \infty$ in the space $\mathcal{M}_{c,T}^2$.

10.6 Burkholder-Davis-Gundy's inequality

[Theorem 10.9] For ${}^{\forall}p > 0$, ${}^{\exists}c_p, C_p > 0$ such that if $(M_t) \in \mathcal{M}^2_{c,T}$ satisfies $E[\langle M \rangle_T^{\frac{p}{2}}] < \infty$, then we have

$$c_p E[\langle M \rangle_T^{\frac{p}{2}}] \leq E\left[\sup_{0 \leq t \leq T} |M_t|^p\right] \leq C_p E[\langle M \rangle_T^{\frac{p}{2}}]. \qquad \square$$

[Remark] This is shown for ${}^{\forall}p > 0$ and not necessarily $p \geq 1$.

[Proof] This can be shown by applying Itô's formula for stochastic integrals with respect to martingales;
see Ikeda-Watanabe, Chapter 3, §3 or Karatzas-Shreve, Chapter 3, Theorem 3.28. Here, we omit the proof. □

• Simple application of martingale property

One can determine the probability of escape at a or b for a martingale starting inside of the interval (a, b).

[Proposition 10.10] If a continuous martingale (M_t) satisfies $M_0 = x \in (a, b)$ and $\sigma = \sigma_{[a,b]^c} < \infty$ a.s. (i.e. σ is the first hitting time of M to the set $[a, b]^c$), then we have

$$P(M_\sigma = a) = \frac{b-x}{b-a}, \quad P(M_\sigma = b) = \frac{x-a}{b-a}. \qquad \square$$

[Proof] By Doob's optional sampling theorem, $x = E[M_{\sigma \wedge t}]$ holds for $^\forall t > 0$. But, noting $\sigma < \infty$ $a.s.$ by our assumption, letting $t \to \infty$ by Lebesgue's convergence theorem, we obtain

$$x = E[M_\sigma] = aP(M_\sigma = a) + bP(M_\sigma = b).$$

On the other hand, $\sigma < \infty$ $a.s.$ also implies

$$1 = P(M_\sigma = a) + P(M_\sigma = b).$$

Regarding these two identities as equations for $P(M_\sigma = a)$ and $P(M_\sigma = b)$ and solving them, we obtain the conclusion. $\square$

Summary of discussion on continuous time martingales and submartingales:

$(\Omega, \mathcal{F}, P)$: Probability space
$(\mathcal{F}_t)_{t\geq 0}$: right continuous filtration (or reference family)
$X = (X_t)_{t\geq 0}$: $(\mathcal{F}_t)$-adapted stochastic process, càdlàg

10.1 Definition
10.2 Markov time
10.3 Doob's inequality
  Submartingale convergence theorem
10.4 Doob's optional sampling theorem
10.5 Doob-Meyer decomposition
  Application: quadratic variation $\langle M \rangle_t$,
  cross variation $\langle M, N \rangle_t$, Hilbert space of continuous square-integrable martingales
10.6 Burkholder-Davis-Gundy's inequality

§11 Brown motion

11.1 Definition

$(\Omega, \mathcal{F}, P)$: Probability space

[Definition 11.1] A real-valued process $X = (X_t)_{t\geq 0}$ is called a Lévy process, if it is càdlàg, continuous in probability (i.e. $P(|X_{t+h} - X_t| \geq \varepsilon) \underset{h\to 0}{\to} 0, {}^\forall\varepsilon > 0$) and has independent and stationary increments, i.e.

- $0 = t_0 < t_1 < \cdots < t_n$, $n = 2, 3, \ldots \Longrightarrow (X_{t_i} - X_{t_{i-1}})_{i=1,2,\ldots,n}$ $\perp\!\!\!\perp$
- For ${}^\forall h > 0$, the distribution of $X_{t+h} - X_t$ does not depend on t □

Typical examples of Lévy processes are Brown motion (continuous case) and Poisson process (pure jump case).

[Definition 11.2] A real-valued process $B = (B_t)_{t \geq 0}$ is called a Brownian motion if

(1) $B_0 = 0$ a.s.

(2) B_t is continuous i.e. $t \mapsto B_t(\omega)$ is continuous (a.s. ω)

(3) B is a Lévy process and, for $0 \leq s < t$, the distribution of the increment $B_t - B_s$ is Gaussian distribution $N(0, t - s)$ with mean 0 and variance $t - s$. □

[Remark] For continuous Lévy processes, by the central limit theorem, the distribution of $B_t - B_s$ is automatically Gaussian. Thus, the Gaussian property in the condition (3) is actually unnecessary.

- The condition (3) implies

$$\begin{aligned} P(B_{t_i} - B_{t_{i-1}} \in A_i, 1 \le i \le n) \\ &= \prod_{1 \le i \le n} P(B_{t_i} - B_{t_{i-1}} \in A_i) \\ &= \int_{A_1} dx_1 \int_{A_2} dx_2 \cdots \int_{A_n} dx_n \prod_{i=1}^{n} p(t_i - t_{i-1}, x_i) \end{aligned}$$

for ${}^\forall n = 1, 2, 3, \ldots$, $0 = t_0 <^\forall t_1 < \cdots < t_n$ and ${}^\forall A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. Here,

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad t > 0, \ x \in \mathbb{R}.$$

Introduce a change of variables: $x_i = y_i - y_{i-1}, 1 \leq i \leq n$ (with $y_0 = 0$) in the above formula. Since Jacobian is given by

$$\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \cdot & \cdots & \cdot \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & 1 \end{vmatrix} = 1,$$

we obtain for ${}^{\forall}A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathbb{R})$,

$$P(B_{t_i} \in A_i, 1 \leq i \leq n)$$
$$= \int_{A_1} dy_1 \int_{A_2} dy_2 \cdots \int_{A_n} dy_n \prod_{i=1}^{n} p(t_i - t_{i-1}, y_{i-1}, y_i).$$

This formula determines a joint distribution of Brownian motion (not its increments) at n different times, or, as we will see, the probability of cylinder sets. Here,

$$p(t,x,y) = p(t, x-y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \quad t > 0,\ x, y \in \mathbb{R}$$

is the heat kernel (fundamental solution of heat equation) and called the transition probability of Brownian motion.

The above formula can be rewritten in the form of expectation:

$$E[f(B_{t_1}, \dots, B_{t_n})] = \int_{\mathbb{R}^n} f(y_1, \dots, y_n) \prod_{i=1}^{n} p(t_i - t_{i-1}, y_{i-1}, y_i) dy_i$$

for $^\forall f = f(y_1, \dots, y_n) \in C_b(\mathbb{R}^n)$. Indeed, for $f = \prod_{i=1}^{n} 1_{A_i}(y_i)$, this formula follows from that for probability stated above. Then, as usual, this formula is shown for step functions and for more general functions by taking the limit for an approximating sequence.

• $p(t,x,y)$ satisfies Chapman-Kolmogorov's identity (semi-group property):

$$p(t+s,x,z) = \int_{\mathbb{R}} p(t,x,y)p(s,y,z)dy, \quad t,s>0,\ x,z\in\mathbb{R}$$

☺ RHS is of the form of convolution. A simple proof is given by applying Fourier transform. □

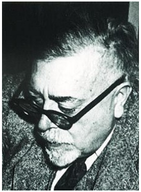
A probabilistic interpretation of this formula: the probability (density) that a particle starting at x arrives at z at time $t+s$ is given by the probability (integrated in the passing points y) that the particle goes first to y at time t and then from y to z after spending time s.

• We call Wiener measure the distribution of Brownian motion on the path spaces (see below for more details):

$$W := C([0,\infty),\mathbb{R}), \quad W_0 := \{w \in W; w_0 = 0\},$$

where $w \in W$ is denoted by $w = (w_t)_{t\geq 0}$ or $(w(t))_{t\geq 0}$.

These spaces, equipped with the topology of uniform convergence on each compact set of $[0,\infty)$, are Polish spaces (separable complete metric spaces). In particular, we can determine Borel fields (σ-fields generated by open sets) $\mathcal{B}(W), \mathcal{B}(W_0)$ on these spaces.


Wiener (from Wikipedia)

• We can also introduce Kolmogorov's σ-fields on these spaces generated by cylinder sets:

$$\mathcal{B}_K(W) := \sigma\left\{ C_{t_1,\ldots,t_n;A_n} \,;\, n \in \mathbb{N}, 0 \le t_1 < \cdots < t_n,\ A_n \in \mathcal{B}(\mathbb{R}^n) \right\},$$

where $C_{t_1,\ldots,t_n;A_n} := \{w \in W \,;\, (w(t_1), \ldots, w(t_n)) \in A_n\}$.

As we noted, Kolmogorov's σ-field is natural, since we want to define at least the probability that Brownian motion visits some sets A_n at given finitely many times $\{t_k\}_k$.

In fact, we have:

[Lemma 11.3] $\mathcal{B}(W) = \mathcal{B}_K(W)$ □

$\because$ We give only the outline.
($\supset$) is shown by the continuity of the map

$$W \ni w \mapsto (w(t_1), \dots, w(t_n)) \in \mathbb{R}^n.$$

($\subset$) For given $\bar{w} \in W, a > 0$, assuming $T \in \mathbb{Q}$,

$$\{w; \max_{0 \le t \le T} |w(t) - \bar{w}(t)| \le a\} = \bigcap_{r \in \mathbb{Q}, r \le T} \{w; |w(r) - \bar{w}(r)| \le a\}.$$

However, since RHS $\in \mathcal{B}_K(W)$, we see LHS$\in \mathcal{B}_K(W)$. This implies that bases of open sets $\in \mathcal{B}_K(W)$. □

[Definition 11.4] Take $\Omega = W_0, \mathcal{F} = \mathcal{B}(W_0)$ in Definition 11.2. A probability measure P on $(W_0, \mathcal{B}(W_0))$ is called Wiener measure if $B_t(w) = w(t), w \in W_0$ (called coordinate function) is a Brownian motion (under the measure P). □

• If a Brownian motion B exists on a certain probability space $(\Omega, \mathcal{F}, P)$ (we may assume $B(\omega) \in W_0$ for ${}^\forall\omega$ by removing a null set $\{\omega; B(\omega) \notin W_0\}$ from Ω), then the image measure of the map

$$\Omega \ni \omega \longmapsto B(\omega) \equiv (B_t(\omega))_{t \geq 0} \in W_0$$

is a Wiener measure. P: Show that this map is measurable.

[Hint] Denoting this map by Φ, first for a cylinder set C in W_0, show $\Phi^{-1}(C) \in \mathcal{F}$. Then, apply π-λ theorem to show that $\Phi^{-1}(A) \in \mathcal{F}$ holds also for $A \in \mathcal{B}_K(W_0^1)$.

• Wiener measure is unique, if it exists.

∵ Let $\mathcal{C}$ be the set of all cylinder sets. Then, the values of Wiener measure on $\mathcal{C}$ is determined and $\mathcal{C}$ is a π–system generating $\mathcal{B}_K(W)$. □