Combinatorics, Lecture 2, 2022/05/12

§1 Generating function (GF)

__Def__. The (ordinary) generating function for an infinite sequence $\{a_0, a_1, \cdots, a_n, \cdots\}$ is defined as a power series

$$f(x) = \sum_{i=0}^{+\infty} a_i \cdot x^i .$$

(i) When the power series $\sum_{i=0}^{+\infty} a_i x^i$ converges

(i.e., $\exists$ a radius $R>0$ of convergence),

there exists

we can view $f(x)$ as a function of x and we can apply operations of calculus.

For example, $a_n = \dfrac{f^{(n)}(0)}{n!}$

(ii) When we are not sure of the convergence, we can view GF as a formal series and take addition and multiplication.

Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{+\infty} b_n x^n$.

$\Rightarrow$ • $f(x) + g(x) = \sum_{n=0}^{+\infty} (a_n + b_n) x^n$

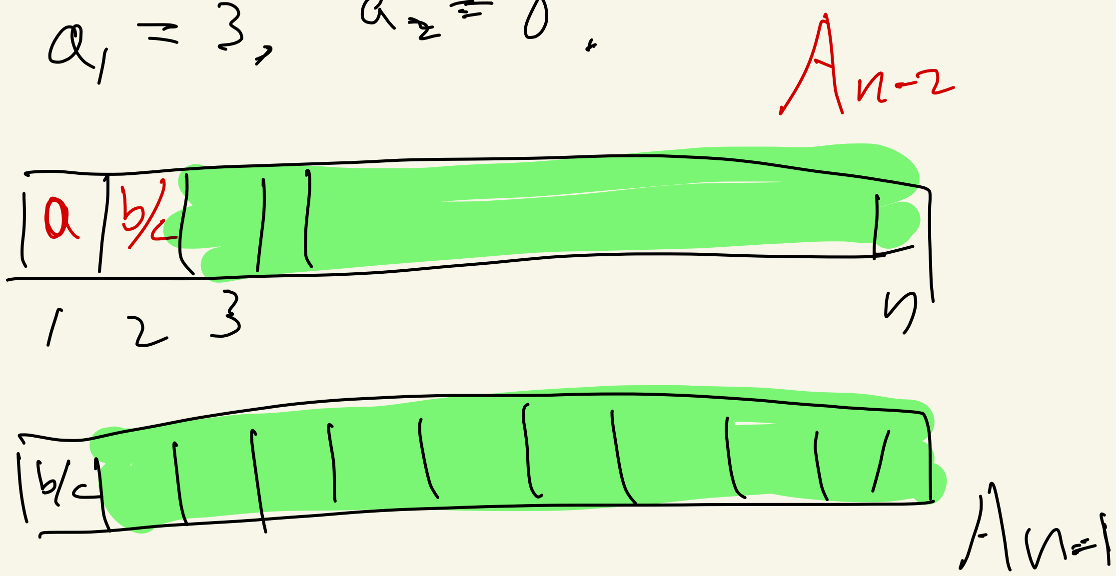
• $f(x) \cdot g(x) = \sum_{n=0}^{+\infty} c_n x^n$,

where $c_n = \sum_{\substack{i+j=n \\ i,j \geq 0}} a_i b_j$.

Example 1. Let A_n be the set of strings of length n with entries from the set $\{a, b, c\}$ with no "aa" occurring. Find $a_n = |A_n|$.

Sol: $a_1 = 3$, $a_2 = 8$.

$a_n =$



$$\Rightarrow \quad a_n = 2a_{n-2} + 2a_{n-1} \quad \text{for } n \geq 2$$

Set $a_0 = 1$. Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$.

Then
$$f(x) = 1 + 3x + \sum_{n\geq 2} (2a_{n-1} + 2a_{n-2}) x^n$$

$$= 1 + 3x + 2x \left(\sum_{n\geq 2} a_{n-1} x^{n-1} \right)$$
$$+ 2x^2 \left(\sum_{n\geq 2} a_{n-2} x^{n-2} \right)$$

$$\Rightarrow f(x) = 1 + 3x + 2x(f(x) - 1) + 2x^2 f(x)$$

$$\Rightarrow \quad \boxed{f(x) = \frac{1+x}{1-2x-2x^2}}$$

By partial fraction decomposing,

$$f(x) = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1+2x} + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1-2x}$$

$$\Rightarrow \quad a_n = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1} \left(\frac{-2}{\sqrt{3}+1} \right)^n$$
$$+ \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1} \right)^n. \quad ☒$$

Def. For any real r and a positive integer k,

let $$\binom{r}{k} := \frac{r(r-1)\cdots(r-k+1)}{k!}$$

Newton's Binomial Theorem For any real r

and $x \in (-1,1)$, $$(1+x)^r = \sum_{k=0}^{+\infty} \binom{r}{k} x^k$$

pf: By Taylor series $\boxtimes$

Example 2. For integers $n, k > 0$,

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}$$

$$= (-1)^k \binom{n+k-1}{k}$$

$$\Rightarrow \quad (1+x)^{-n} = \sum_{k=0}^{+\infty} (-1)^k \binom{n+k-1}{k} x^k$$

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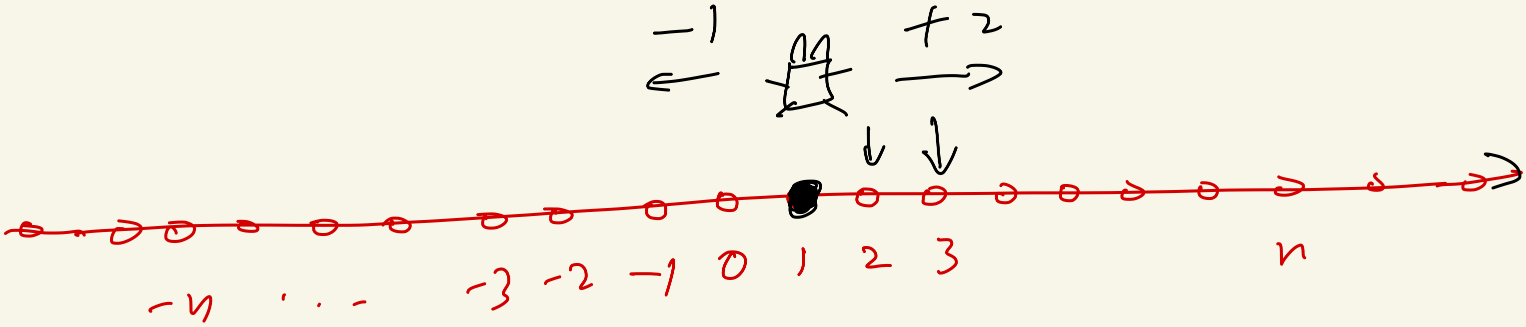
§2 Random Walk.



Consider a real axis with points $(0, \pm 1, \pm 2, \cdots)$.
A frog leaps among intege points according the following rules:

(1) At beginning, it sits at 1.

(2) In each coming step, the frog leaps either by distance 2 to the right (from i to $i+2$), or by distance 1 to the left (from i to $i-1$), each of which is randomly chosen with probability $\frac{1}{2}$, independently of each other.

What is the probability $\mathbb{P}(A)$ that the frog can reach 0 ?

$+ : i \to i+2$

$- : i \to i-1$

Solution: $(+, -)$

Let the probability space Ω be the set of infinite vectors, where each entry is in $\{+, -\}$.

Let A be the event that the frog can reach 0.

Let A_i be the event that the frog reaches 0 at the i^{th} step for the first time.

So $A = \bigcup_{i=1}^{+\infty} A_i$. A_i's disjoint

$$\Rightarrow \quad p(A) = \sum_{i=1}^{+\infty} p(A_i)$$

Let a_i be the number of vectors of the first i steps such that the frog starts at 1 and reaches 0 at the i^{th} step for the first time.

So $p(A_i) = \dfrac{a_i}{2^i}$

$$\Rightarrow \quad p(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i} = \boxed{f\left(\tfrac{1}{2}\right)}$$

where $\boxed{f(x) = \sum_{i=1}^{+\infty} a_i x^i}$

Let b_i . . . . . starts at 2 and reaches 0 at the i^{th} step for the first time.

Let c_i - - - - - - - starts at 3

and reaches ∞ - - - - - - - -

$$\Rightarrow \boxed{\begin{aligned} b_i &= \sum_{j=1}^{i-1} a_j a_{i-j} \\ c_i &= \sum_{j=1}^{i-1} a_j b_{i-j} \end{aligned}}$$

$$\Rightarrow \sum_{i=1}^{+\infty} b_i x^i = \left(\sum_{i=1}^{+\infty} a_i x^i\right)\left(\sum_{i=1}^{+\infty} a_i x^i\right)$$

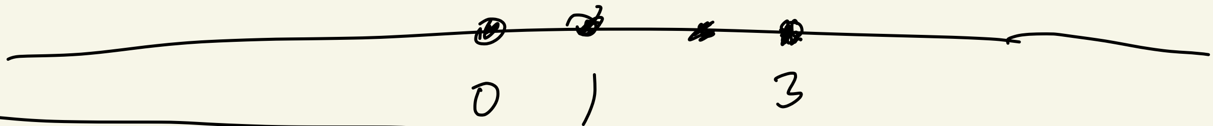
$$= f(x)^2$$

$$\sum_{i=1}^{+\infty} c_i x^i = \left(\sum a_i x^i\right)\left(\sum b_i x^i\right)$$

$$= f(x)^3$$

Then

$$f(x) = \sum_{i=1}^{+\infty} a_i x^i = x + \sum_{i\geq 2} a_i x^i$$



$$\boxed{i \geq 2, \quad a_i = c_{i-1}}$$

$$f(x) = x + \sum_{i \geq 2} c_{i-1} x^{i}$$

$$= x + x \left(\sum_{i \geq 2} c_{i-1} x^{i-1} \right)$$

$$= x + x \cdot f(x)^{3}$$

$$\Rightarrow f(x) = x + x \cdot f(x)^{3}$$

$$f\left(\tfrac{1}{2}\right) = \tfrac{1}{2} + \tfrac{1}{2} f\left(\tfrac{1}{2}\right)^{3}$$

$$\Rightarrow f\left(\tfrac{1}{2}\right) \in \left\{ 1, \frac{\sqrt{5}-1}{2}, \cancel{\frac{-\sqrt{5}-1}{2}} \right\}$$

$$\Rightarrow f\left(\tfrac{1}{2}\right) = \frac{\sqrt{5}-1}{2}$$

(why $f\left(\tfrac{1}{2}\right) \neq 1$? 练习) $\square$

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§3. Exponential generating function

Def. The exponential generating function for the sequence $\{a_0, a_1, a_2, \dots\}$ is the power series

$$f(x) = \sum_{n=0}^{+\infty} a_n \frac{x^n}{n!}$$

Problem 1. Let S_n = # selections of n letters from $\{a, b, c\}$ such that both of the numbers of a's and b's are even.

$$\Rightarrow \quad S_n = \sum_{\substack{x_1+x_2+x_3=n \\ x_1, x_2 \in \{0,2,4,\dots\} \\ x_3 \geq 0}} 1$$

By the previous fact, $S_n = [x^n] f(x)$,

where $f(x) = \left(\sum_{i \in \{0,2,4,6,\dots\}} x^i\right)^2 \left(\sum_{i \geq 0} x^i\right)$

Problem 2 Let T_n = # arrangements of n letters from $\{a, b, c\}$ such that both of the numbers of a's and b's are even.

Solution

$$T_n = \sum_{\substack{x_1+x_2+x_3=n \\ x_1, x_2 \in \{0,2,4,6,\dots\} \\ x_3 \geq 0}} \frac{n!}{x_1!\, x_2!\, x_3!}$$

Let $g(x) = \left(\sum_{i \in \{0,2,4,\dots\}} \frac{x^i}{i!} \right)^2 \left(\sum_{i \geq 0} \frac{x^i}{i!} \right)$

We claim $\quad [x^n]\, g = \frac{T_n}{n!}$

To see this, the term x^n in $g(x)$ becomes

$$\sum_{\substack{i+j+k=n \\ i,j \in \{0,2,4,6,\dots\} \\ k \geq 0}} \frac{x^i}{i!} \cdot \frac{x^j}{j!} \cdot \frac{x^k}{k!}$$

$$= \sum_{\dots} \frac{n!}{i!\,j!\,k!} \frac{x^n}{n!}$$

$= T_n \cdot \frac{x^n}{n!}$, proving the claim $\boxtimes$

[We will complete this problem next lecture]