Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 9. Cocycles and subbundles related to the Kontsevich–Zorich cocycle

Anton Zorich

University Paris Cité

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Tautological line bundle and Hodge norm

Hodge bundle and tautological line bundle

Consider our favorite complex vector bundle $H^1_{\mathbb{C}}$ over the moduli space of pairs (complex curve X, holomorphic 1-form ω on X) having the cohomology $H^1(X,\mathbb{C})$ of X as a fiber over (X,ω). This $2g$-dimensional flat bundle $H^1_{\mathbb{C}}$ contains a g-dimensional non-flat subbundle having $H^{1,0}(X) \subset H^1(X,\mathbb{C})$ as a fiber over (X,ω). This subbundle $H^{1,0}$ is called the *Hodge bundle*.

The vector bundle $H^1_{\mathbb{C}}$ contains also a 1-dimension subbundle L. A fiber of L over a point (X,ω) is spanned by the cohomology class of ω. The line subbundle L is called the *tautological bundle* (unfortunately this term is strongly overcharged).

The Hodge bundle over the moduli space $\mathcal{H}_g$ of nonzero Abelian differentials can be pulled back from the Hodge bundle over the moduli space $\mathcal{M}_g$ of curves throught the natural projection $\mathcal{H}_g \to \mathcal{M}_g$ sending (X,ω) to X. The tautological bundle L over $\mathcal{H}_g$ can be pulled back from the projectivization $\mathrm{P}\mathcal{H}_g = \mathcal{H}_g/\mathbb{C}^*$, where the group $\mathbb{C}^*$ acts on $\mathcal{H}_g$ by multiplying the holomorphic 1-form ω in the pair (X,ω) by a constant complex factor.

Subbundles equivariant along the Teichmüller flow

Consider now the vector bundle $H^1_{\mathbb{R}}$ over $\mathcal{H}_g$ having the cohomology $H^1(X,\mathbb{R})$ of X as a fiber over (X,ω). This real $2g$-dimensional flat bundle $H^1_{\mathbb{R}}$ contains a real 2-dimensional vector bundle $L_{\mathbb{R}}$. A fiber of $L_{\mathbb{R}}$ over a point (X,ω) is spanned by the cohomology classes of $\omega_0 = \mathrm{Re}(\omega)$ and of $\omega_1 = \mathrm{Im}(\omega)$.

The Teichmüller flow g_t acts in period coordinates $H^1(X,\{P_1,\dots,P_n\};\mathbb{C})$ of the base $\mathcal{H}_g$ of the bundle $L_{\mathbb{R}}$ as $g_t([\omega_0 + i\omega_1]) = [e^t\omega_0 + ie^{-t}\omega_1]$. Thus, the vector bundle $L_{\mathbb{R}}$ is equivariant with respect to the parallel transport along trajectories of g_t, and even along the orbits of $\mathrm{GL}(2,\mathbb{R})$ in $\mathcal{H}_g$.

The splitting $L_{\mathbb{R}} = \mathrm{Vect}((\mathrm{Re}\,\omega)) \oplus \mathrm{Vect}(\mathrm{Im}(\omega))$ is equivariant with respect to the parallel transport along trajectories of g_t, but is not equivariant along the entire orbits of $\mathrm{GL}(2,\mathbb{R})$.

Hodge norm

The space of holomorphic 1-forms is endowed with a natural Hermitian metric:

$$\langle \alpha_1, \alpha_2 \rangle := \frac{i}{2} \int_X \alpha_1 \wedge \overline{\alpha_2}, \qquad \alpha_1, \alpha_2 \in H^{1,0} = \Gamma(X, \Omega^1_{hol}) \, .$$

This metric induces Euclidean metric in $H^1(X, \mathbb{R})$. Namely, for any two cohomology classes $c_1, c_2 \in H^1(X, \mathbb{R})$ one can find unique $\alpha_1, \alpha_2 \in H^{1,0}$ such that $c_1 = [\mathrm{Re}\, \alpha_1]$, $c_2 = [\mathrm{Re}\, \alpha_2]$. We define $\langle c_1, c_2 \rangle := \mathrm{Re} \langle \alpha_1, \alpha_2 \rangle$.

The Hodge $*$-operator on $H^1(X, \mathbb{R})$ is defined as $* : c \mapsto *c = [\mathrm{Im}\, \alpha(c)]$, where $\alpha(c)$ is the unique holomorphic 1-form such that $c = \mathrm{Re}[\alpha]$.

The corresponding norm

$$\|c\|^2 := \langle c, c \rangle = \frac{i}{2} \int_X \alpha(c) \wedge \overline{\alpha(c)} = \int_X \mathrm{Re}\, \alpha(c) \wedge \mathrm{Im}\, \alpha(c) = \int_X c \wedge *c$$

is called the *Hodge norm*; it *is not* preserved by the Gauss—Manin connection.

Variation of the Hodge norm along the Teichmüller flow

Lemma. *Variations of the Hodge norms of vectors* $[\omega_0], [\omega_1] \in L_{\mathbb{R}}$ *along the Teichmüller flow* g_t *satisfy the following relations:*

$$\frac{d}{dt} \log \|\omega_0\|_{g_t} = -1; \qquad \frac{d}{dt} \log \|\omega_1\|_{g_t} = 1\,.$$

Proof. Let $(X(t), \omega(t))$ be the point of the trajectory of g_t after a time t. The Teichmüller flow g_t acts in period coordinates $H^1(X, \{P_1, \ldots, P_n\}; \mathbb{C})$ as $g_t([\omega_0 + i\omega_1]) = [e^t\omega_0 + ie^{-t}\omega_1]$. Hence, the cohomology class of the *holomorphic* 1-form $\omega(t)$ on $X(t)$ is $[\omega(t)] = [e^t\omega_0 + ie^{-t}\omega_1]$. This implies that the following relations for the cohomology classes of the holomorphic forms $e^{-t}\omega(t)$ and $-ie^t\omega(t)$ on $X(t)$:

$$[e^{-t}\omega(t)] = [\omega_0 + ie^{-2t}\omega_1]\,; \qquad [-ie^t\omega(t)] = [\omega_1 - ie^{2t}\omega_0]\,.$$

Hence, we get the following relations which imply the Lemma:

$$\|\omega_0\|^2_{g_t} = \int_{X(t)} \omega_0 \wedge e^{-2t}\omega_1\,; \qquad \|\omega_1\|^2_{g_t} = \int_{X(t)} \omega_1 \wedge (-e^{2t}\omega_0)\,.$$

□

Lyapunov exponents of $L_{\mathbb{R}}$ along the Teichmüller flow

Corollary. *The Lyapunov exponents of the linear subbundles* $\mathrm{Vect}(\omega_0)$ *and* $\mathrm{Vect}(\omega_1)$ *of* $H^1_{\mathbb{R}}$ *equivariant along the Teichmüller flow* g_t *are equal to* -1 *and to* 1 *respectively.*

Proof. Lyapunov exponents of $\mathrm{Vect}(\omega_j)$, where $j = 0, 1$, are computed as

$$\lambda(\mathrm{Vect}(\omega_j)) := \lim_{t\to+\infty} \frac{\log \|\omega_j\|_{g_t}}{t} = \lim_{t\to+\infty} \frac{1}{t} \int_0^t \frac{d}{dt} \log \|\omega_j\|_{g_t}\, dt = \pm 1\,.$$

Lemma (G. Forni, 2002) *For any flat surface* S *in any stratum in any genus:*

$$\max_{c\in H^1(S,\mathbb{R})} \left| \frac{d \log \|c\|}{dt} \right| \le 1\,.$$

The above Lemma implies that all the Lyapunov exponents of $H^1_{\mathbb{R}}$ are bounded from below by -1 and from above by 1. It is not difficult to show that the exponents -1 and 1 found above for $\mathrm{Vect}(\omega_0)$ and $\mathrm{Vect}(\omega_1)$ are simple.

Trivial part of the Lyapunov spectrum of the Teichmüller flow

Geodesic flow on a compact hyperbolic surface

Let S be a compact hyperbolic surface. When we speak of a *geodesic flow on* S we, actually, mean the flow g_t acting on the 3-dimensional unit tangent bundle $M^3 = T^1 S$. Given a pair $(x, \vec{v})$ we launch a geodesic g_t from x in direction of the unit vector $\vec{v}$ and we follow the geodesic with a unit speed for a time t. We get a new point $(x_1, \vec{v}_1) = g_t(x, \vec{v}) \in T^1 S$ which is a pair: the point x_1 of S to which gets our geodesic after the time t and a new unit tangent vector $\vec{v}_1$ to our geodesic at x_1.

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We can extend the flow g_t to the total space $N^4 = TS \setminus \{0\}$ of the tangent bundle to S with removed zero section. Having a pair $(x, \vec{w}) \in N^4$ normalize the length of $\vec{w}$ to one: $\vec{v} = \vec{w}/\|\vec{w}\|$. A pair $(x, v))$ is already in M^3, so we apply g_t to it and then transform the resulting pair $(x_1, \vec{v}_1) = g_t(x, \vec{v}) \in T^1S$ into $(x_1, \vec{w}_1)$, where $\vec{w}_1 = \|w\| \cdot \vec{v}_1$. By construction the resulting flow g_t on N^4 preserves the norm of tangent vectors: $\|\vec{w}\| = \|\vec{w}_1\|$. Considering the flow k_s on N^4 acting as $k_s(x, \vec{v}) = (x, s\vec{v})$ we see that the flows g_t and k_s on N^4 commute. The flows g_t and k_s on N^4 do not have fixed points, and thus define 1-dimensional subbundles $G, K \subset TN$ in the tangent bundle to N^4.

Cocycle over the geodesic flow. Equivariant line bundles

The smooth flow g_t on N^4 defines a cocycle on the tangent bundle TN to N, namely $Dg_t : T_yN \to T_{g_t(y)}N$. We can restrict the vector bundle TN to the hypersurface $M^3 \subset N^4$ and get a 4-dimensional vector bundle $V = TN|_{M^3}$ over M^3. Clearly, V splits into a direct sum $V = TM^3 \oplus K|_{M^3}$, where the 1-dimensional bundle $K \subset TN$ is tangent to the flow k_s. It can be seen as a normal bundle to the hypersurface M_3 in N^4.

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The flow g_t preserves M^3 and acts on it ergodically with respect to the natural finite measure. We get an induced cocycle over g_t acting in the 4-dimensional vector bundle $V = TN|_{M^3}$. Since g_t and k_s commute, the splitting $V = TM^3 \oplus K|_{M^3}$ is equivariant with respect to the action. Since g_t preserves the natural norm in the fibers of $K|_{M^3}$, the Lyapunov exponent corrsponding to the equivariant line subbundle $K|_{M^3}$ is equal to zero.

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The summand TM^3 also has a 1-dimensional eqiuivariant subbundle G spanned by tangent vectors to the flow g_t on M^3. Since we move along geodesics with the constant speed 1, the natural norm in G is preserved by g_t and the single Lyapunov exponent of G is also equal to zero.

Back to the Teichmüller flow

The moduli space $\mathcal{H}_g$ of Abelian differentials can be seen as a foliated generalization of $N^4 = TS$. The real 4-dimensional orbits of $\mathrm{GL}(2,\mathbb{R})$ project to 2-dimensional Teichmüller disks in $\mathcal{M}_g$; the metric on the disks induced from the Teichmüller metric on $\mathcal{M}_g$ is the regular hyperbolic metric.

The real hypersurface $\mathcal{H}_g^1 \subset \mathcal{H}_g$ defined by the equation $\frac{i}{2}\int_X \omega \wedge \bar{\omega} = 1$ (meaning that the area of X in the flat metric defined by ω is equal to 1) is a generalization of the total space of the unit tangent bundle $M^3 = T^1 S$. The Teichmüller flow g_t is a leafwise geodesic flow g_t along the Teichmüller discs. The analog of the flow k_s is the flow (also denote by k_s), which acts as $k_s(X,\omega) = (X, s\cdot\omega)$.

Period coordiantes in a stratum $\mathcal{H}(m_1,\dots,m_n)$ provide local trivialization of the total space of the tangent bundle $T\mathcal{H}(m_1,\dots,m_n)$ (serving as a generalization of TN^4). The fiber of this bundle is $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. Let us find the vectors which span the real 1-dimensional subbundles G tangent to g_t and K tangent to k_s.

Back to the Teichmüller flow

Let $\omega = \omega_0 + i\,\omega_1$, where $\omega_0 = \mathrm{Re}(\omega)$ and $\omega_1 = \mathrm{Im}(\omega)$. The tangent vector to g_t in period coordinates has the form:

$$\frac{d}{dt}\Big|_{t=0}\left(e^t[\omega_0] + e^{-t} i\,[\omega_1]\right) = [\omega_0] - i\,[\omega_1] = [\bar{\omega}]\,.$$

and, hence, $G = \mathrm{Vect}([\omega_0] - i[\omega_1])$.

The flow g_t can be seen as the geodesic flow on $\mathcal{H}^1(m_1, \ldots, m_n)$ for the Teichmüller metric on the moduli space $\mathcal{M}_g$. It can be also seen as a leafwise geodesic flow along the Teichmüller discs for the hyperbolic metric on them. In any case, we follow geodesics with a unit speed in this metric, so the tangent vector to the flow g_t has unit norm.

Since $k_s(X, \omega) = (X, s \cdot \omega)$, the vector $[\omega]$ is tangent to the flow k_s:

$$\frac{d}{ds}\Big|_{s=1}(s[\omega]) = [\omega]\,.$$

Thus, $K = \mathrm{Vect}([\omega_0] + i[\omega_1])$ and $G \bigoplus_{\mathbb{R}} K = \mathrm{Vect}_{\mathbb{R}}([\omega_0], i[\omega_1])$.

Two cocycles over the Teichmüller flow

Consider the following cocycle over the Teichmuller flow g_t acting on $\mathcal{H}^1(m_1, \ldots, m_n)$: we consider the vector bundle $H^1_{rel,\mathbb{C}}$ over $\mathcal{H}^1(m_1, \ldots, m_n)$ and we pull the fibers of this bundle along trajectories of the flow g_t using the Gauss–Manin connection.

Consider now the second cocycle over the Teichmuller flow g_t on $\mathcal{H}^1(m_1, \ldots, m_n)$ which mimics the cocycle acting in the vector bundle $TN|_{M^3}$ considered above. As before we consider the action of g_t on $\mathcal{H}(m_1, \ldots, m_n)$, we consider the differential

$$Dg_t : T\mathcal{H}(m_1, \ldots, m_n) \to T\mathcal{H}(m_1, \ldots, m_n)$$

of the map g_t, and we restrict the bundle to the real hypersurface $\mathcal{H}^1(m_1, \ldots, m_n)$.

Two cocycles over the Teichmüller flow

The two cocycles are closely related. Namely, consider the splitting

$$H^1(X,\{P_1,\dots,P_n\})_{\mathbb{C}} = H^1(X,\{P_1,\dots,P_n\})_{\mathbb{R}} \oplus i H^1(X,\{P_1,\dots,P_n\})_{\mathbb{R}}$$

equivariant with respect to the flow g_t. Both cocycles act in the fibers of the resulting real vector bundle. The cocycle Dg_t acts as a composition of the first cocycle with uniform expansion of the first summand by e^t and uniform contraction of the second summand by e^{-t}.

This implies that the Lyapunov spectrum of the second cocycle is obtained from two copies of the spectrum of the first one by adding 1 to all exponents of the first copy and subtracting 1 from all exponents of the second copy.

The norms of the vectors in the equivariant subbundle $\mathrm{Vect}([\omega_0] \oplus \mathrm{Vect}([i\omega_1]$ do not change for the second cocycle, so the corresponding Lyapunov exponents of the second cocycle are equal to zero.

We have computed the Lyapunov exponents of $\mathrm{Vect}([\omega_0])$ and of $\mathrm{Vect}(i[\omega_1])$ for the first cocycle and obtained -1 and 1 respectively. Adding 1 to the first exponent and subtracting 1 from the second we rediscover zero Lyapunov exponents of $\mathrm{Vect}([\omega_0])$ and of $\mathrm{Vect}(i[\omega_1])$ for the second cocycle.

Exact sequences of a pair

Consider a pair of topological spaces $Y \subset X$, where X is our Riemann surface and $Y = \{P_1 \cup \cdots \cup P_n\}$ is the union of zeroes of the holomorphic 1-form ω. Consider the exact sequence of the pair (X, Y) in homology

$$\begin{array}{ccccccc} & H_1(Y) & \to & H_1(X) & \to & H_1(X,Y) & \stackrel{\partial}{\to} \\ \stackrel{\partial}{\to} & H_0(Y) & \to & H_0(X) & \to & H_0(X,Y) & \end{array}$$

And the dual one in cohomology:

$$\begin{array}{ccccccc} & H^1(Y) & \leftarrow & H^1(X) & \leftarrow & H^1(X,Y) & \stackrel{d}{\leftarrow} \\ \stackrel{d}{\leftarrow} & H^0(Y) & \leftarrow & H^0(X) & \leftarrow & H^0(X,Y) & \end{array}$$

The linear functions on the relative 1-cycles in the image of the map $d : H^0(Y) \to H^1(X, Y)$ are spanned by functions l_j, $j = 1, \ldots, n$, evaluated as follows: given a relative 1-cycle c we take its boundary ∂c and compute how many times (counted with sign) we see the point P_j in ∂c.

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And the dual one in cohomology:

$$\begin{array}{ccccccc} & 0 & \leftarrow & H^1(X) & \leftarrow & H^1(X,Y) & \stackrel{d}{\leftarrow} \\ \stackrel{d}{\leftarrow} & H^0(Y) & \leftarrow & H^0(X) & \leftarrow & 0 & \end{array}$$

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One more subbundle

By convention, all zeroes are labeled. Thus, an automorphisms of the translation surface (X, ω) induced by holonomy of the Gauss–Manin connection along a closed loop in the moduli space of Abelian differentials preserves this labeling. The induced automorphism of the space $H^1(X, Y)$ preserves the subspace $Im(d : H^0(Y) \to H^1(X, Y))$ and maps all linear functions l_j to themselves. Thus, it acts trivially on the subspace $Im(d)$.

We can consider now the associated fragment $H^0_{\mathbb{R}} \to H^1_{rel,\mathbb{R}} \to H^1_{\mathbb{R}}$ of the exact sequence of vector bundles over the stratum $\mathcal{H}(m_1, \ldots, m_n)$ and the natural cocycles induced by the holonomy of the Gauss–Manin connection along the flow g_t in these vector bundles.

The image $Im(d : H^0(Y) \to H^1(X, Y))$ of the map d gives rise to a real $(n-1)$-dimensional vector subbundle in the bundle $H^1_{rel,\mathbb{R}}$ on which the cocycle acts trivially. Thus, the spectrum of Lyapunov exponents of $H^1_{\mathbb{R}}(X, Y)$ contains $n-1$ zero.

Kontsevich–Zorich cocycle

General Lemma. *If we have an exact sequence of bundles* $V \to W \to W/V$ *then the Lyapunov exponents of* W *are the union of those in* V *and* W/V*.*

For a general proof see, for example, Remark 2.2.9(iii) in the notes https://math.uchicago.edu/∼sfilip/public_files/MET_lectures.pdf of S . Filip on Lyapunov exponents. It is easy to present a direct proof in our case. □

Conclusion. We see that the study of the Lyapunov spectra of all the above cocycles is reduced to the study of the Lyapunov exponents of the cocycle induced by pulling the fibers of the bundle $H^1_{\mathbb{R}}$ along trajectories of the Teichmüller flow on strata $\mathcal{H}(m_1, \dots, m_n)$ by means of the Gauss–Manin connection. This cocycle is often called the *Kontsevich–Zorich cocycle*. The fibers $H^1(X, \mathbb{R})$ of this cocycle have natural symplectic structure preserved by the cocycle, so its Lyapunov spectrum is symmetric with respect to the sign change. We have already computed the top and the bottom exponents $\pm\nu_1 = \pm 1$. Thus, the Lyapunov spectrum has the form

$$\nu_1 > \nu_2 \geq \dots \geq \nu_g \geq -\nu_g \geq \dots \geq -\nu_2 > -\nu_1 \,.$$